Introduction to Term Rewriting

lecture 7

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Knuth-Bendix Order
- Dependency Pairs
- Further Reading

Definition

rewrite system is terminating if there are no infinite rewrite sequences

Termination Methods

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, . . .

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- Knuth-Bendix Order
- Dependency Pairs
- Further Reading

Definitions

- weight function (w, w_0) consists of mapping $w\colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for all constants $c \in \mathcal{F}$
- weight of term t is

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- weight function (w, w_0) is admissible for precedence $>$ if

$$f > g \quad \forall\, g \in \mathcal{F} \setminus \{f\}$$

whenever f is unary function symbol in $\mathcal{F}$ with $w(f) = 0$

Example

- rewrite rules

$$\begin{aligned} \mathrm{e} \cdot x &\to x & x \cdot \mathrm{e} &\to x \\ x^- \cdot x &\to \mathrm{e} & x \cdot x^- &\to \mathrm{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\ \mathrm{e}^- &\to \mathrm{e} & (x \cdot y)^- &\to y^- \cdot x^- \\ x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y \end{aligned}$$

- weight function $w(\mathrm{e}) = w(\cdot) = w_0 = 1 \quad w(^-) = 0$

$$w(\mathrm{e} \cdot x) = 3 \qquad w(x) = 1 \qquad w((x \cdot y)^-) = 3$$

- precedence $^- > \cdot > \mathrm{e}$
- admissible because $^-$ is maximal in precedence

Definition

- relation $>_{\mathrm{kbo}}$ (Knuth-Bendix order) on terms:
  $s >_{\mathrm{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either
  1. $w(s) > w(t)$
  2. $w(s) = w(t)$ and either
     1. $\exists\, n > 0$ such that $s = f^n(t)$ and $t \in \mathcal{V}$
     2. $s = f(s_1, \dots, s_n)$ and $t = f(t_1, \dots, t_n)$ and $\exists\, i$
        $$\forall j < i \;\; s_j = t_j \qquad s_i >_{\mathrm{kbo}} t_i$$
     3. $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f > g$

Theorem

$>_{kbo}$ is reduction order if precedence $>$ is well-founded

Example (1)

- rewrite rules

$$
\begin{aligned}
e \cdot x &\to x & x \cdot e &\to x \\
x^- \cdot x &\to e & x \cdot x^- &\to e \\
(x \cdot y) \cdot z &\to x \cdot (y \cdot z) & x^{--} &\to x \\
e^- &\to e & (x \cdot y)^- &\to y^- \cdot x^- \\
x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y
\end{aligned}
$$

- weight function $w(e) = w(\cdot) = w_0 = 1 \quad w(^-) = 0$
- precedence $^- > \cdot > e$

$$e \cdot x >_{kbo} x \qquad x^{--} >_{kbo} x \qquad (x \cdot y)^- >_{kbo} y^- \cdot x^-$$

Example (2)

- rewrite rules

$$aa \to bbb \qquad bbbbb \to aaa$$

- weight function and precedence

$$
\begin{array}{lll}
w(a) = 3 & w(b) = 2 & a > b \\
w(a) = 5 & w(b) = 3 & b > a \\
w(a) = 13 & w(b) = 8 &
\end{array}
$$

Example (3)

- rewrite rules

$$\begin{array}{llll} 0+0\to 0 & 1+0\to 1 & \cdots & 9+0\to 9 \\ 0+1\to 1 & 1+1\to 2 & \cdots & 9+1\to 1:0 \\ \vdots & \vdots & & \vdots \\ 0+9\to 9 & 1+9\to 1:0 & \cdots & 9+9\to 1:8 \end{array}$$

$$x+(y:z)\to y:(x+z) \qquad 0:x\to x$$

$$(x:y)+z\to x:(y+z) \qquad x:(y:z)\to(x+y):z$$

- weight function

$$w(0)=w(1)=w(2)=w(3)=w(4)=w(+)=w_0=1$$

$$w(:)=2 \qquad w(5)=w(6)=w(7)=w(8)=w(9)=3$$

- precedence $+>:\quad +>5\quad +>6\quad +>7\quad +>8$

Example (4)

- rewrite rules

$$\begin{array}{lll} 11\to 43 & 33\to 56 & 55\to 62 \\ 12\to 21 & 34\to 11 & 56\to 12 \\ 22\to 111 & 44\to 3 & 66\to 21 \end{array}$$

- weight function and precedence

$$\begin{array}{lll} w(1)=31 & w(2)=47 & w(3)=41 \\ w(4)=21 & w(5)=43 & w(6)=3 \end{array}$$

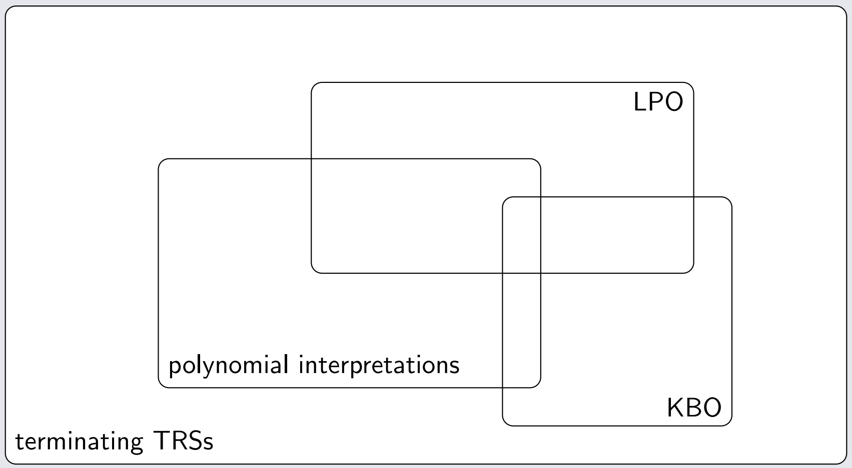
$$3>5>6>1>4 \qquad 1>2$$

Theorem

- *if* $> \subseteq \sqsupset$ *then* $>_{kbo} \subseteq \sqsupset_{kbo}$ (*incrementality*)
- *if* $>$ *is total then* $>_{kbo}$ *is total on ground terms*
- *following two problems are decidable:*
  1. *instance:* *terms* s, t *weight function* (w, w_0) *precedence* $>$
     question: $s >_{kbo} t$?
  2. *instance:* *terms* s, t
     question: $\exists$ *weight function* (w, w_0), $\exists$ *precedence* $>$ *such that* $s >_{kbo} t$?

Remark

KBO, LPO and polynomial interpretations are incomparable



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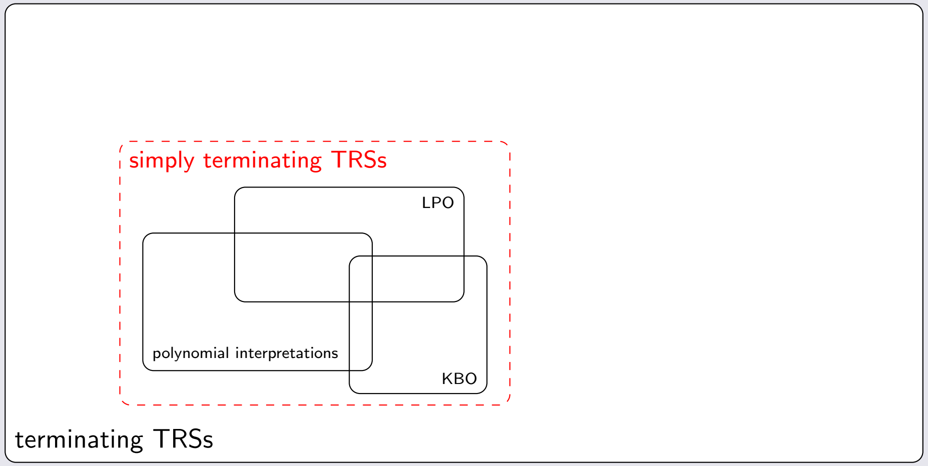
More Realistic View



dependency pairs make direct termination methods much more powerful

Definitions

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- $\mathcal{F}^\sharp = \mathcal{F} \cup \{f^\sharp \mid f \text{ is defined symbol of } \mathcal{R}\}$
- if $t = f(t_1, \ldots, t_n)$ with f defined then $t^\sharp = f^\sharp(t_1, \ldots, t_n)$
- dependency pair $\ell^\sharp \to u^\sharp$ of rewrite rule $\ell \to r$ satisfies
  - $u \trianglelefteq r$ and $u \ntriangleleft \ell$
  - $\mathrm{root}(u)$ is defined symbol
- $\mathrm{DP}(\mathcal{R})$ is set of all dependency pairs of $\mathcal{R}$

Theorem

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$t_1 \xrightarrow[\mathcal{R}]{*} t_2 \xrightarrow[DP(\mathcal{R})]{\epsilon} t_3 \xrightarrow[\mathcal{R}]{*} t_4 \xrightarrow[DP(\mathcal{R})]{\epsilon} \cdots$$

such that t_1 is terminating with respect to $\mathcal{R}$

Example

rewrite rule

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{f}(x)))) \qquad \mathcal{R}$$

infinite rewrite sequence in $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{g}(x))) \xrightarrow[\mathcal{R}]{} \mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{g}(x))))) \xrightarrow[\mathcal{R}]{} \mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{f}(x))))))) \xrightarrow[\mathcal{R}]{} \cdots$$

dependency pairs

$$\begin{aligned} \mathsf{f}^\sharp(\mathsf{g}(x)) &\to \mathsf{f}^\sharp(\mathsf{f}(x)) \\ \mathsf{f}^\sharp(\mathsf{g}(x)) &\to \mathsf{f}^\sharp(x) \end{aligned} \qquad \mathrm{DP}(\mathcal{R})$$

infinite sequence in $\mathcal{R} \cup \mathrm{DP}(\mathcal{R})$

$$\mathsf{f}^\sharp(\mathsf{g}(\mathsf{g}(x))) \xrightarrow[\mathrm{DP}(\mathcal{R})]{} \mathsf{f}^\sharp(\mathsf{f}(\mathsf{g}(x))) \xrightarrow[\mathcal{R}]{} \mathsf{f}^\sharp(\mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{f}(x))))) \xrightarrow[\mathrm{DP}(\mathcal{R})]{} \cdots$$

Example

rewrite rules

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

dependency pairs

$$\mathsf{s}(x) +^\sharp y \to x +^\sharp y$$
$$\mathsf{s}(x) \times^\sharp y \to (x \times y) +^\sharp y$$
$$\mathsf{s}(x) \times^\sharp y \to x \times^\sharp y$$

polynomial interpretation

$$0_{\mathbb{N}} = 0$$
$$\mathsf{s}_{\mathbb{N}}(x) = x + 1$$
$$+_{\mathbb{N}}(x, y) = x + y$$
$$\times_{\mathbb{N}}(x, y) = +^\sharp_{\mathbb{N}}(x, y) = \times^\sharp_{\mathbb{N}}(x, y) = x$$

Theorem

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$t_1 \gtrsim t_2 \quad > \quad t_3 \gtrsim t_4 \quad > \quad \cdots$$

such that t_1 is terminating with respect to $\mathcal{R}$

Definition

reduction pair $(>, \gtrsim)$ consists of well-founded order $>$ and preorder $\gtrsim$ such that

1. $>$ is closed under substitutions
2. $\gtrsim$ is closed under contexts and substitutions
3. $> \cdot \gtrsim \subseteq >$ or $\gtrsim \cdot > \subseteq >$

Theorem

TRS $\mathcal{R}$ is terminating $\iff$ $DP(\mathcal{R}) \subseteq >$ and $\mathcal{R} \subseteq \gtrsim$ for reduction pair $(>, \gtrsim)$

Example

rewrite rules

$$x - 0 \to 0 \qquad 0 \div \mathsf{s}(y) \to 0$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y \qquad \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$$

dependency pairs

$$\mathsf{s}(x) -^\sharp \mathsf{s}(y) \to x -^\sharp y$$
$$\mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to (x - y) \div^\sharp \mathsf{s}(y)$$
$$\mathsf{s}(x) \div^\sharp \mathsf{s}(y) \to x -^\sharp y$$

polynomial interpretation

$$0_{\mathbb{N}} = 0$$
$$\mathsf{s}_{\mathbb{N}}(x) = x + 1$$
$$-_{\mathbb{N}}(x, y) = -^\sharp_{\mathbb{N}}(x, y) = \div_{\mathbb{N}}(x, y) = \div^\sharp_{\mathbb{N}}(x, y) = x$$

Definitions

- well-founded weakly monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of nonempty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is weakly monotone in all coordinates:

  $$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$
- relation $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α
- relation $\geqslant_{\mathcal{A}}$ on terms: $s \geqslant_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) \geqslant [\alpha]_{\mathcal{A}}(t)$ for all assignments α

Lemma

$(>_{\mathcal{A}}, \geqslant_{\mathcal{A}})$ is reduction pair for every well-founded weakly monotone algebra $(\mathcal{A}, >)$

Example

rewrite rules

$$\mathsf{average}(0,0) \to 0 \qquad \mathsf{average}(\mathsf{s}(x),y) \to \mathsf{average}(x,\mathsf{s}(y))$$
$$\mathsf{average}(0,\mathsf{s}(0)) \to 0 \qquad \mathsf{average}(x,\mathsf{s}(\mathsf{s}(\mathsf{s}(y)))) \to \mathsf{s}(\mathsf{average}(\mathsf{s}(x),y))$$
$$\mathsf{average}(0,\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(0)$$

dependency pairs

$$\mathsf{average}^\sharp(\mathsf{s}(x),y) \to \mathsf{average}^\sharp(x,\mathsf{s}(y))$$
$$\mathsf{average}^\sharp(x,\mathsf{s}(\mathsf{s}(\mathsf{s}(y)))) \to \mathsf{average}^\sharp(\mathsf{s}(x),y)$$

polynomial interpretation

$$0_{\mathbb{N}} = 0 \qquad \mathsf{average}_{\mathbb{N}}(x,y) = x + y$$
$$\mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{average}^\sharp_{\mathbb{N}}(x,y) = 2x + y$$

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KBO Orientability
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