ISR 2010

Introduction to Term Rewriting

lecture 8

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

exam, advanced topics

Outline

- Orthogonality
  - Definitions
  - Descendants
  - Parallel Moves Lemma
- Beyond Orthogonality
- Modularity
- Further Reading

Confluence

every two coinitial rewrite sequences can be joined

$$\begin{tikzcd}
& \cdot \arrow[dl, two heads] \arrow[dr, two heads] & \\
\cdot \arrow[dr, dotted, two heads] & \text{CR} & \cdot \arrow[dl, dotted, two heads] \\
& \cdot &
\end{tikzcd}$$

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- . . . yields uniqueness of normal forms
- . . . is decidable for terminating TRSs

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- . . . yields uniqueness of normal forms
- . . . is decidable for terminating TRSs
- . . . what about nonterminating TRSs ?

Examples (Non-Confluence)

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- no confluence because of critical pairs

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$$a \longrightarrow b$$

$$a \longrightarrow c$$

Examples (Non-Confluence)

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$$a \longrightarrow b \qquad\qquad b \longleftarrow a \longrightarrow c$$

$$a \longrightarrow c$$

Examples (Non-Confluence)

- no confluence because of critical pairs

$$\begin{aligned} a &\to b \\ a &\to c \end{aligned} \qquad\qquad b \leftarrow a \to c$$

- no critical pairs but no confluence (Klop 1978)

$$\begin{aligned} f(x,x) &\to a \\ g(x) &\to f(x, g(x)) \\ c &\to g(c) \end{aligned}$$

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$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{a} &\to \mathsf{c} \end{aligned} \qquad\qquad \mathsf{b} \leftarrow \mathsf{a} \to \mathsf{c}$$

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Exercises (Non-Confluence)

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2. $\mathsf{f}(\mathsf{g}(x), y) \to x \quad \mathsf{g}(\mathsf{a}) \to \mathsf{b}$

Exercises (Non-Confluence)

1. $f(f(x)) \to a$
2. $f(g(x), y) \to x \quad g(a) \to b$
3. $or(x, y) \to x \quad or(x, y) \to y$

Confluence via Critical Pairs

control interference of rewrite rules

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forbid interference of rewrite rules

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Confluence via Orthogonality

forbid interference of rewrite rules

- no critical pairs
- no equality checks

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- rewrite rule $\ell \to r$ is right-linear if r is linear

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Examples

- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
  left-linear but not right-linear

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- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
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- $\mathsf{f}(x, x) \to \mathsf{a}$
  right-linear but not left-linear

Definition

orthogonal TRS is left-linear and lacks criticial pairs

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Examples

$$\begin{aligned} \mathsf{I} \cdot x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

Definition

orthogonal TRS is left-linear and lacks criticial pairs

Examples

$$\mathsf{I} \cdot x \to x$$
$$(\mathsf{K} \cdot x) \cdot y \to x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

$$\mathsf{ack}(0, y) \to \mathsf{s}(y)$$
$$\mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$
$$\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

Theorem

orthogonal TRSs are confluent

$$\begin{tikzcd}
\forall s, t, u & & s \arrow[dl] \arrow[dr] & \\
& t \arrow[dr, dotted, two heads] & & u \arrow[dl, dotted, two heads] \\
\exists v & & v &
\end{tikzcd}$$

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{\color{red}\exists v} & & & {\color{red}v} &
\end{tikzcd}$$

Observation

for orthogonal TRSs there is canonical way to compute common reduct v

Lemma

orthogonal TRSs are locally confluent

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Proof Idea

distinguish two cases:

1. two disjoint redexes
2. two nested redexes

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Proof Idea

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Examples

1. $$\begin{aligned} \mathsf{a} &\longrightarrow \mathsf{c} \\ \mathsf{b} &\longrightarrow \mathsf{d} \end{aligned}$$

Lemma

orthogonal TRSs are locally confluent

Proof Idea

distinguish two cases:

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2. two nested redexes

Examples

1. $\mathsf{a} \rightarrow \mathsf{c}$ $\quad$ $\mathsf{f(c,b)} \leftarrow \mathsf{f(a,b)} \rightarrow \mathsf{f(a,d)}$

   $\mathsf{b} \rightarrow \mathsf{d}$

Lemma

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Proof Idea

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Examples

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2. $\mathsf{f}(x) \to \mathsf{g}(x, x)$

   $\mathsf{a} \to \mathsf{b}$

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   $\mathsf{b} \rightarrow \mathsf{d}$

2. $\mathsf{f}(x) \rightarrow \mathsf{g}(x,x)$ $\qquad$ $\mathsf{f(b)} \leftarrow \mathsf{f(a)} \rightarrow \mathsf{g(a,a)}$

   $\mathsf{a} \rightarrow \mathsf{b}$

Outline

- Orthogonality
  - Definitions
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- Modularity
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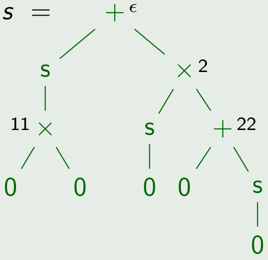
  - Results
- Further Reading

Example

$$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$



redex positions in s: $\epsilon \mid 11 \mid 2 \mid \quad 22 \quad \mid$

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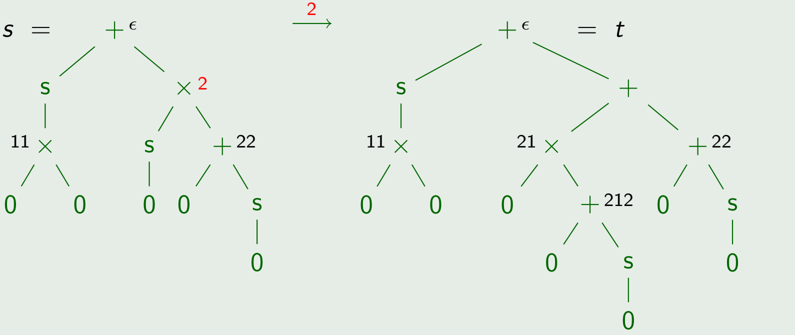

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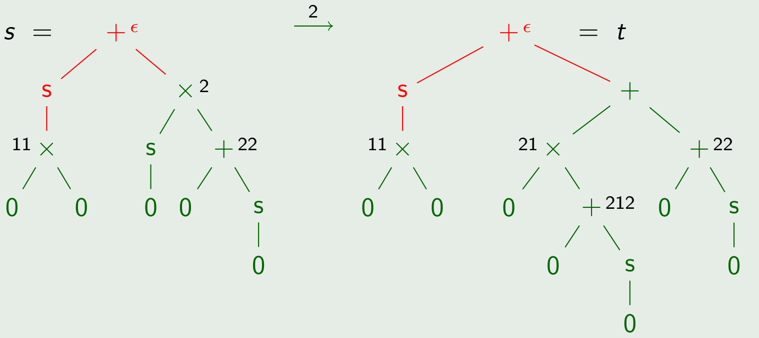

redex positions in s : ϵ | 11 | 2 | 22 |

redex positions in t : ϵ |

Example

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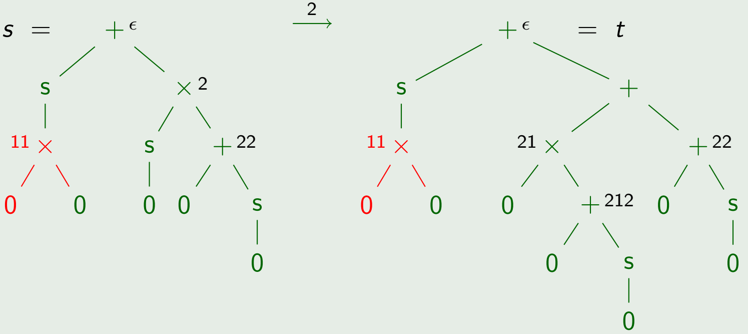

redex positions in s :	ϵ	11	2	22
redex positions in t :	ϵ	11		

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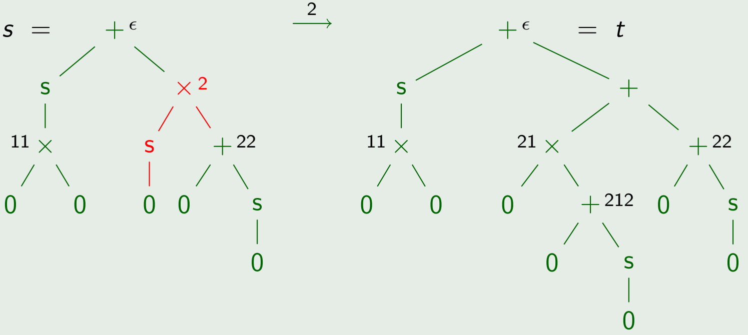

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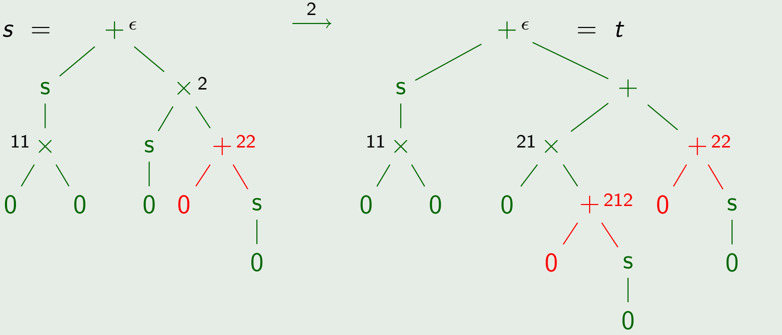

redex positions in s:	ϵ	11	2	22	
redex positions in t:	ϵ	11		212	22

redex at position 22 is duplicated

Example

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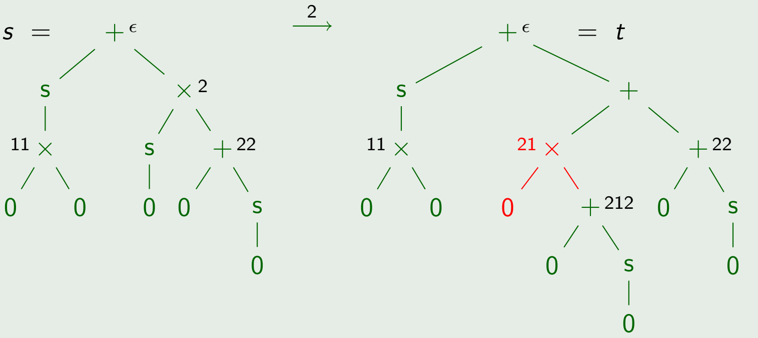


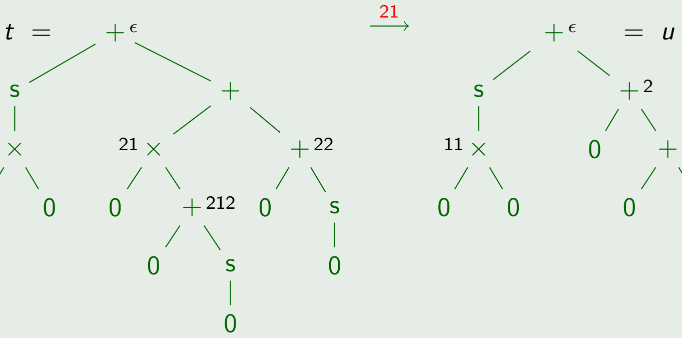
redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21

redex at position 22 is duplicated

redex at position 21 is created

Example

$0 + y \to y \quad s(x) + y \to s(x + y) \quad 0 \times y \to 0 \quad s(x) \times y \to (x \times y) + y$



redex positions in s :	ϵ	11	2	22		
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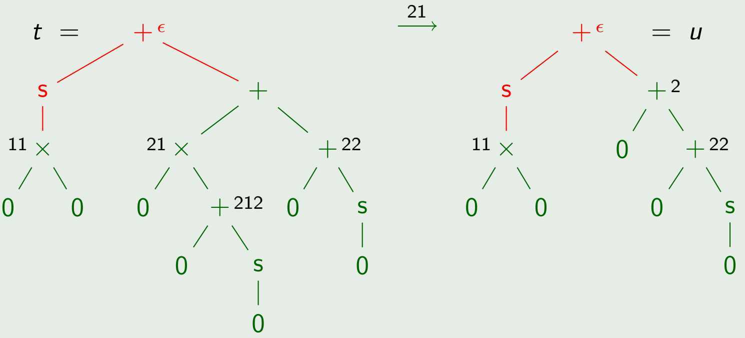

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21
redex positions in u:	ϵ					

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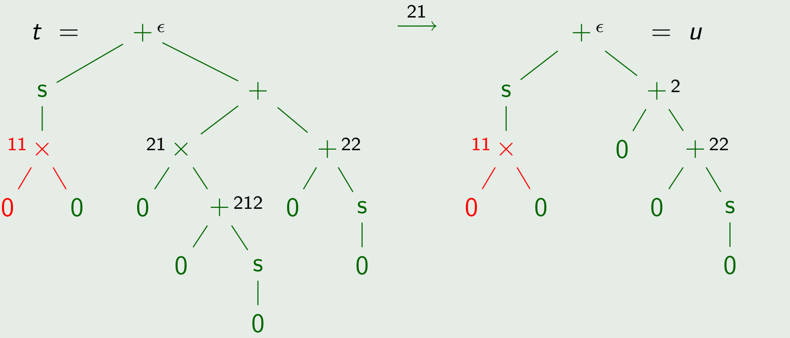

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21
redex positions in u:	ϵ	11				

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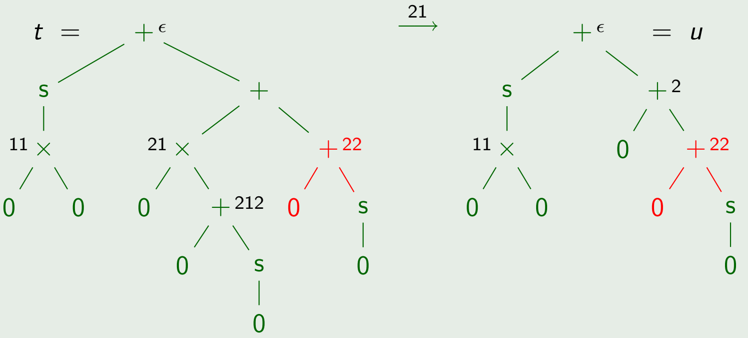

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21
redex positions in u:	ϵ	11			22	

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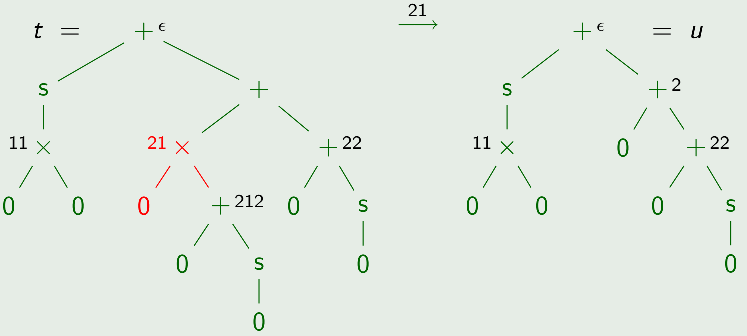

redex positions in s:	ϵ	11	2	22		
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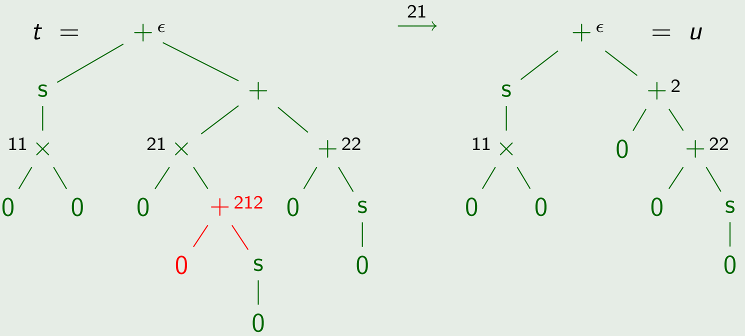

redex positions in s :	ϵ	11	2	22		
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redex positions in u :	ϵ	11			22	

redex at position 212 is erased

Example

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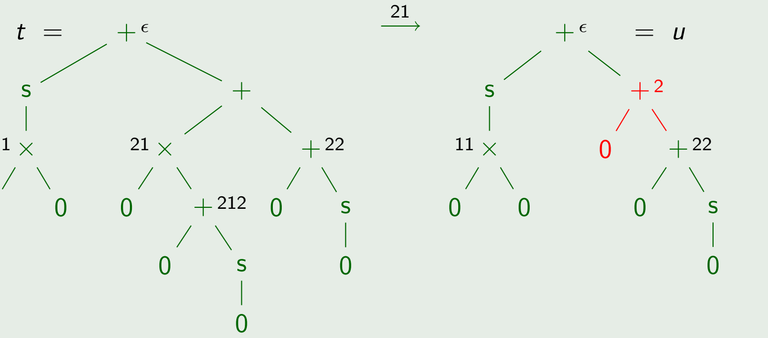

redex positions in s:	ϵ	11	2	22			
redex positions in t:	ϵ	11		212	22	21	
redex positions in u:	ϵ	11			22		2

redex at position 212 is erased

redex at position 2 is created

rewrite step A: $s \xrightarrow[\ell \to r]{p} t$ position $q \in \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Step)

- descendants of q in t

$$q \backslash A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}\mathrm{os}_{\mathcal{V}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

rewrite step A: $s \xrightarrow[\ell \to r]{p} t$ position $q \in \mathcal{P}\mathrm{os}(s)$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Step)

- descendants of q in t

$$q \backslash A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}\mathrm{os}_{\mathcal{V}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

- descendants of Q in t

$$Q \backslash A = \bigcup_{q \in Q} q \backslash A$$

Remark

- information about position is needed to determine descendants

$$\mathsf{f}(x) \to x \qquad\qquad \mathsf{f}(\mathsf{f}(x)) \xrightarrow[\mathsf{f}(x)\to x]{?} \mathsf{f}(x)$$

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$$\mathsf{f}(x) \to x \qquad\qquad \mathsf{f}(\mathsf{f}(x)) \xrightarrow[\mathsf{f}(x)\to x]{?} \mathsf{f}(x)$$

- information about rewrite rule is needed to determine descendants

$$\begin{array}{c} \mathsf{f}(x) \to \mathsf{f}(x) \\ \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \end{array} \qquad\qquad \mathsf{f}(\mathsf{a}) \xrightarrow[?]{\epsilon} \mathsf{f}(\mathsf{a})$$

rewrite sequence $A \colon s \rightarrow^* t$ set of positions $Q \subseteq \mathcal{P}os(s)$

Definition (Descendants after Rewrite Sequence)

descendants of Q in t

$$Q \backslash A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q \backslash A_1) \backslash A_2 & \text{If } A = A_1 ; A_2 \text{ with } A_1 \colon s \rightarrow u \text{ and } A_2 \colon u \rightarrow^* t \end{cases}$$

Lemma

- *for arbitrary TRSs:* *if* Q *is parallel* $(\forall\, p \neq q \in Q\colon p \parallel q)$ *then so is* $Q \backslash A$

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- *for orthogonal TRSs:* *if Q is set of redex positions then so is* $Q\backslash A$

Terminology

descendant of redex is called residual

Remark

- in non-left-linear TRS descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(x,x) &\to \mathsf{b} \end{aligned} \qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{f}(\mathsf{b},\mathsf{a})$$

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- in TRS with critical pairs descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(\mathsf{a}) &\to \mathsf{b} \end{aligned} \qquad \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b})$$

Outline

- Orthogonality
  - Definitions
  - Descendants
  - Parallel Moves Lemma
- Beyond Orthogonality
- Modularity
  - Definitions
  - Results
- Further Reading

Definition (Parallel Rewriting)

1. $= \subseteq ⇻$

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3. $f(s_1, \ldots, s_n) \mathrel{+\!\!\!\!\!\rightarrow} f(t_1, \ldots, t_n)$ if $s_i \mathrel{+\!\!\!\!\!\rightarrow} t_i$ for each $1 \leqslant i \leqslant n$

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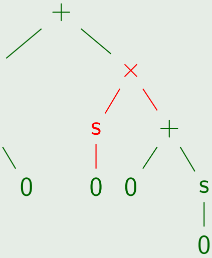
Example

$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$

$s =$



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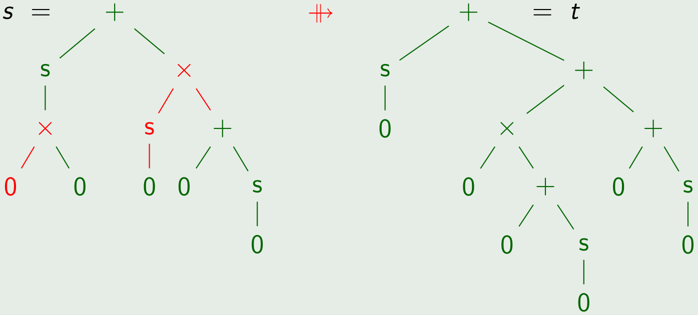

Lemma

$s \mathrel{\|\!\!\!\!\rightarrow} t \quad\iff\quad s \rightarrow^{*} t$ *by contracting redexes at pairwise parallel positions in* s

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!\rightarrow} t_1$ by contracting redexes at positions in P

B: $s \mathrel{+\!\!\!\!\!\rightarrow} t_2$ by contracting redexes at positions in Q

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Definition (Projection)

$A\colon\ s \mathrel{-\!\!\|\!\!\to} t_1$ by contracting redexes at positions in P

$B\colon\ s \mathrel{-\!\!\|\!\!\to} t_2$ by contracting redexes at positions in Q

- $B\backslash A\colon\ t_1 \mathrel{-\!\!\|\!\!\to} u_1$ by contracting redexes at positions in $Q\backslash A$
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- $A \sqcup B = A; B\backslash A$ and $B \sqcup A = B; A\backslash B$

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Parallel Moves Lemma

$$\begin{tikzcd}
 & s \arrow[dl, "A"', "\Vert" marking] \arrow[dr, "B", "\Vert" marking] & \\
t_1 \arrow[d, "B\backslash A"', "\Vert" marking] & & t_2 \arrow[d, "A\backslash B", "\Vert" marking] \\
u_1 & ? & u_2
\end{tikzcd}$$

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$A\colon\ s \mathrel{\parallel\!\!\!\!\!\!\!\rightarrow} t_1$ by contracting redexes at positions in P

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$A\colon\ s \mathrel{\|\!\!\!\to} t_1$ by contracting redexes at positions in P

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- $B\backslash A\colon\ t_1 \mathrel{\|\!\!\!\to} u_1$ by contracting redexes at positions in $Q\backslash A$
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- $A \sqcup B = A; B\backslash A$ and $B \sqcup A = B; A\backslash B$

Parallel Moves Lemma

$$\begin{tikzcd}
& s \arrow[dl, "A"', "\|" marking] \arrow[dr, "B", "\|" marking] \arrow[dd, bend right=90, "A \sqcup B"'] \arrow[dd, bend left=90, "B \sqcup A"] & \\
t_1 \arrow[dr, "B\backslash A"', "\|" marking] & & t_2 \arrow[dl, "A\backslash B", "\|" marking] \\
& u &
\end{tikzcd}$$

Definition

$A\colon s_1 \to^* t_1$ and $B\colon s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

1. $s_1 = s_2$
2. $t_1 = t_2$
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2. $t_1 = t_2$
3. $p \backslash A = p \backslash B$ for all redex positions p in s_1

Parallel Moves Lemma (with Permutation Equivalence)

$$\begin{tikzcd}
 & s \arrow[dl, "A"', "\parallel" marking] \arrow[dr, "B", "\parallel" marking] \arrow[dd, bend right=80, "A \sqcup B"'] \arrow[dd, bend left=80, "B \sqcup A"] & \\
t_1 \arrow[dr, "B \backslash A"', "\parallel" marking] & \simeq & t_2 \arrow[dl, "A \backslash B", "\parallel" marking] \\
 & u &
\end{tikzcd}$$

Definition

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \ \subseteq \ \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\ & d & \end{tikzcd}$$

$\exists d$

Lemma

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow \ \subseteq \ \rightarrow_{\diamond} \subseteq \ \rightarrow^*$ for some relation $\rightarrow_{\diamond}$ on A with diamond property

Corollary

orthogonal TRSs are confluent

Proof

$$\begin{tikzcd}
\cdot \arrow[r, two heads] \arrow[d, two heads] & \cdot \\
{} &
\end{tikzcd}$$

Corollary

orthogonal TRSs are confluent

Proof

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \\
\cdot \arrow[d] & & & & \\
\cdot \arrow[d] & & & & \\
\cdot & & & &
\end{tikzcd}$$

Corollary

orthogonal TRSs are confluent

Proof

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \\
\cdot \arrow[r, "\parallel" marking] \arrow[d] & \cdot & & & \\
\cdot \arrow[d] & & & & \\
\cdot & & & &
\end{tikzcd}$$

Corollary

orthogonal TRSs are confluent

Proof

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[r] & \cdot \\
\cdot \arrow[r, "\|" marking] \arrow[d] & \cdot \arrow[r, "\|" marking] & \cdot & & \\
\cdot \arrow[d] & & & & \\
\cdot & & & &
\end{tikzcd}$$

Corollary

orthogonal TRSs are confluent

Proof

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \\
\cdot \arrow[r, "\shortparallel" marking] \arrow[d] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \arrow[r, "\shortparallel" marking] & \cdot & \\
\cdot \arrow[d] & & & & \\
\cdot & & & &
\end{tikzcd}$$

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Proof

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[r] \arrow[d, "=" marking] & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[r, "\parallel" marking] \arrow[d] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[r, "\parallel" marking] \arrow[d] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \arrow[d, "=" marking] \\
\cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[r, "\parallel" marking] & \cdot
\end{tikzcd}$$

Outline

- Orthogonality
- Beyond Orthogonality
- Modularity
- Further Reading

Definitions

- critical pair $s \leftarrow\!\rtimes\!\rightarrow t$ is trivial if $s = t$

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Examples

$$\begin{aligned} x \vee \mathsf{T} &\to \mathsf{T} \\ \mathsf{T} \vee x &\to \mathsf{T} \\ \mathsf{F} \vee \mathsf{F} &\to \mathsf{F} \end{aligned}$$

Definitions

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$$\begin{aligned} \mathsf{p}(\mathsf{s}(x)) &\rightarrow x \\ \mathsf{s}(\mathsf{p}(x)) &\rightarrow x \end{aligned}$$

Definitions

- critical pair $s \leftarrow\rtimes\rightarrow t$ is trivial if $s = t$
- weakly orthogonal TRS is left-linear and has only trivial critical pairs
- overlay $s \leftarrow\bowtie\rightarrow t$ is critical pair originating from overlap $\langle l_1 \rightarrow r_1, \epsilon, l_2 \rightarrow r_2 \rangle$

Examples

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Definitions

- critical pair $s \leftarrow\!\rtimes\!\rightarrow t$ is trivial if $s = t$
- weakly orthogonal TRS is left-linear and has only trivial critical pairs
- overlay $s \leftarrow\!\bowtie\!\rightarrow t$ is critical pair originating from overlap $\langle l_1 \rightarrow r_1, \epsilon, l_2 \rightarrow r_2 \rangle$
- weakly orthogonal TRS is almost orthogonal if all critical pairs are overlays

Examples

$$\begin{aligned} x \vee \mathsf{T} &\rightarrow \mathsf{T} \\ \mathsf{T} \vee x &\rightarrow \mathsf{T} \\ \mathsf{F} \vee \mathsf{F} &\rightarrow \mathsf{F} \end{aligned}$$

Theorem

weakly orthogonal TRSs are confluent

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Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow} \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\parallel}$

$$\begin{tikzcd}
\cdot \arrow[r, "\parallel" marking] \arrow[d, "=" marking] & \cdot \\
\cdot &
\end{tikzcd}$$

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $\leftarrow\!\!\!\!\!\!\parallel \cdot \parallel\!\!\!\!\!\!\rightarrow \subseteq \parallel\!\!\!\!\!\!\rightarrow \cdot \leftarrow\!\!\!\!\!\!\parallel$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "\parallel" marking] & \cdot \\
\cdot \arrow[d, "=" marking] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\Vert} \cdot \mathrel{\Vert\!\!\!\!\!\rightarrow} \subseteq \mathrel{\Vert\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\Vert}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!\!/"] & \cdot \arrow[r, "\Vert" marking] & \cdot \\
\cdot \arrow[d, "=" marking] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\shortparallel} \cdot \mathrel{\shortparallel\!\!\!\!\!\rightarrow} \subseteq \mathrel{\shortparallel\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\shortparallel}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & \cdot \arrow[r, "\parallel" marking] & \cdot \\
\cdot \arrow[urr, "\parallel" marking] \arrow[d, "=" marking] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow} \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\parallel}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/", near start] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[dd, "=" marking] \\
\cdot \arrow[urr, "\parallel" marking] \arrow[d, "=" marking] & \text{IH} & \\
\cdot \arrow[rr, "\parallel" marking] & & \cdot
\end{tikzcd}$$

interesting case

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\!\!\rightarrow} \subseteq \mathrel{\parallel\!\!\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\!\!\parallel}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!\!/"] & \cdot \arrow[r, "\|" marking] & \cdot \arrow[ddd, "=" marking] \\
\cdot \arrow[urr, "\|" marking] \arrow[dd, "=" marking] & & \\
& \mathsf{IH} & \\
\cdot \arrow[rr, "\|" marking] & & \cdot
\end{tikzcd}$$

interesting case

- $\rightarrow \subseteq \mathrel{\parallel\!\!\!\!\!\!\!\rightarrow} \subseteq \rightarrow^*$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \; \subseteq \; \mathrel{+\!\!\!\!\!\!\rightarrow} \quad \Longrightarrow \quad CR$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; ⇻ \quad\Longrightarrow\quad$ *CR*

Proof Sketch

- $⇺ \cdot ⇻ \subseteq ⇻ \cdot ⇺$

$$\begin{tikzcd}
\cdot \arrow[r, "\shortparallel" marking] \arrow[d, "=" marking] & \cdot \\
\cdot &
\end{tikzcd}$$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \; \subseteq \; \mathrel{\,\|\!\!\!\!\rightarrow}$ $\implies$ CR

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\|\,} \cdot \mathrel{\,\|\!\!\!\!\rightarrow} \; \subseteq \; \mathrel{\,\|\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\|\,}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "\|" marking] & \cdot \\
\cdot \arrow[d, "=" marking] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\Vert\!\!\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\Vert} \cdot \mathrel{\Vert\!\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\Vert\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\Vert}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "\Vert" marking] & \cdot \\
\cdot \arrow[ur, "\Vert" marking] \arrow[d, "=" marking] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\|\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\|} \cdot \mathrel{\|\!\!\!\rightarrow} \;\subseteq\; \mathrel{\|\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\|}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "\|" marking] & \cdot \\
\cdot \arrow[ur, "\times" marking] \arrow[urr, "\|" marking] \arrow[d, "=" marking] & & \\
\cdot & &
\end{tikzcd}$$

interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\shortparallel\!\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\shortparallel} \cdot \mathrel{\shortparallel\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\shortparallel\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\shortparallel}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "\shortparallel" marking] & \cdot \arrow[dd, "=" marking] \\
\cdot \arrow[ur, "\times" marking] \arrow[urr, "\shortparallel" marking] \arrow[d, "=" marking] \arrow[dr, phantom, "\mathrm{IH}"] & & \\
\cdot \arrow[rr, "\shortparallel" marking] & & \cdot
\end{tikzcd}$$

interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; ⇻ \quad\Longrightarrow\quad CR$

Proof Sketch

- $⇺ \cdot ⇻ \;\subseteq\; ⇻ \cdot ⇺$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "\parallel" marking] & \cdot \arrow[dd, "=" marking] \\
\cdot \arrow[ur, "\times" marking] \arrow[urr, "\parallel" marking] \arrow[d, "=" marking] & \text{IH} & \\
\cdot \arrow[rr, "\parallel" marking] & & \cdot
\end{tikzcd}$$

interesting case

- $\rightarrow \;\subseteq\; ⇻ \;\subseteq\; \rightarrow^*$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\!\parallel}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "\parallel" description] & \cdot \arrow[dd, "=" description] \\
\cdot \arrow[ur, "\times" description] \arrow[urr, "\parallel" description] \arrow[d, "=" description] & & \\
\cdot \arrow[rr, "\parallel" description] & & \cdot
\arrow[from=2-1, to=3-3, phantom, "\mathrm{IH}"]
\end{tikzcd}$$

interesting case

- $\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\!\rightarrow} \;\subseteq\; \rightarrow^*$

Open Problem

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\leftarrow\!\!\!\!\!\!\parallel} \quad\Longrightarrow\quad CR\ ?$

Example

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x), \mathsf{b}) &\longrightarrow \mathsf{f}(\mathsf{g}(x), \mathsf{c}) \\ \mathsf{g}(\mathsf{c}) &\longrightarrow \mathsf{g}(\mathsf{a}) \\ \mathsf{a} &\longrightarrow \mathsf{c} \\ \mathsf{b} &\longrightarrow \mathsf{d} \end{aligned}$$

Theorem (Huet 1980)

linearity & $\leftarrow\ltimes\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad \mathit{CR}$

Theorem (Huet 1980)

$$\textit{linearity} \quad \& \quad \leftarrow\rtimes\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad \Longrightarrow \quad CR$$

Notation

$$\color{red}{\leftarrow\cdot\!\rtimes\rightarrow} \;=\; \leftarrow\rtimes\rightarrow \setminus \leftarrow\bowtie\rightarrow$$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\!\leftarrow) \quad\Longrightarrow\quad CR$

Notation

$\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;=\; \leftarrow\!\rtimes\!\rightarrow \setminus \leftarrow\!\bowtie\!\rightarrow$

Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow \quad\Longrightarrow\quad CR$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad CR$

Notation

$$\leftarrow\!\cdot\!\rtimes\!\rightarrow \;=\; \leftarrow\!\rtimes\!\rightarrow \setminus \leftarrow\!\bowtie\!\rightarrow$$

Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\cdot\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Theorem (van Oostrom 1996)

left-linearity & $\leftarrow\!\cdot\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{-\!\!\!\!\circ\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{-\!\!\!\!\circ\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Outline

- Orthogonality
- Beyond Orthogonality
- Modularity
  - Definitions
  - Results
- Further Reading

Definition

property of TRSs is modular if it is preserved under union

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Remark

without further restrictions 'no' property of TRSs is modular

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termination $a \longrightarrow b$ $b \longrightarrow a$

Definition

property of TRSs is modular if it is preserved under union

Remark

without further restrictions 'no' property of TRSs is modular

termination	$a \to b$	$b \to a$
confluence	$a \to b$	$a \to c$

Definition

property P is preserved under signature extension if

$$(\mathcal{F}, \mathcal{R}) \vDash P \quad \Longrightarrow \quad (\mathcal{F} \cup \mathcal{G}, \mathcal{R}) \vDash P$$

for all TRSs $(\mathcal{F}, \mathcal{R})$ and signatures $\mathcal{G}$ with $\mathcal{F} \cap \mathcal{G} \neq \varnothing$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- defined symbols $\mathcal{F}_{\mathcal{D}} = \{ \mathsf{root}(\ell) \mid \ell \to r \in \mathcal{R} \}$

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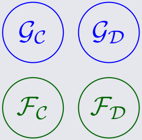
More Interesting Combinations



$(\mathcal{G}, \mathcal{S})$ $\mathcal{G}_\mathcal{C}$ $\mathcal{G}_\mathcal{D}$

$(\mathcal{F}, \mathcal{R})$ $\mathcal{F}_\mathcal{C}$ $\mathcal{F}_\mathcal{D}$

two TRSs disjoint

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- defined symbols $\mathcal{F}_{\mathcal{D}} \;=\; \{\, \mathrm{root}(\ell) \mid \ell \to r \in \mathcal{R} \,\}$
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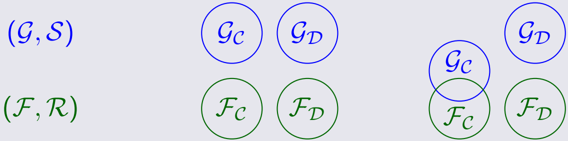
More Interesting Combinations



two TRSs disjoint constructor-sharing

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- defined symbols $\mathcal{F}_{\mathcal{D}} = \{\, \mathrm{root}(\ell) \mid \ell \to r \in \mathcal{R} \,\}$
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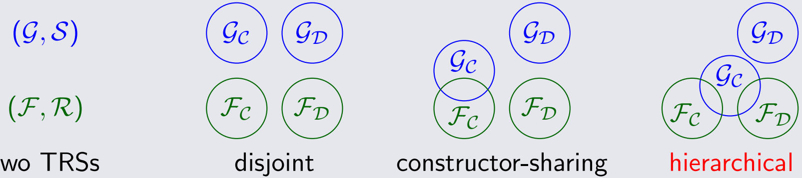
More Interesting Combinations



Example

$$
\begin{aligned}
0 + y &\to y \\
\mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\
0 - y &\to 0 \\
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
\mathsf{nil} +\!\!\!+\, x &\to x \\
(x : y) +\!\!\!+\, z &\to x : (y +\!\!\!+\, z) \\
\mathsf{true} \wedge \mathsf{false} &\to \mathsf{false} \\
\mathsf{false} \wedge \mathsf{true} &\to \mathsf{false} \\
x \wedge x &\to x \\
\mathsf{sum}(\mathsf{nil}) &\to 0 \\
\mathsf{sum}(x : y) &\to x + \mathsf{sum}(y)
\end{aligned}
$$

$$
\begin{aligned}
0 \times y &\to 0 \\
\mathsf{s}(x) \times y &\to x \times y + y \\
\mathsf{fib}(0) &\to \mathsf{s}(0) \\
\mathsf{fib}(\mathsf{s}(0)) &\to \mathsf{s}(0) \\
\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x) \\
0 \div \mathsf{s}(y) &\to 0 \\
\mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y)) \\
x < 0 &\to \mathsf{false} \\
0 < \mathsf{s}(y) &\to \mathsf{true} \\
\mathsf{s}(x) < \mathsf{s}(y) &\to x < y \\
\mathsf{length}(\mathsf{nil}) &\to 0 \\
\mathsf{length}(x : y) &\to \mathsf{s}(\mathsf{length}(y))
\end{aligned}
$$

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} +\!\!+ x \to x$ $(x : y) +\!\!+ z \to x : (y +\!\!+ z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
⑨	$\mathsf{sum}(\mathsf{nil}) \to 0$ $\mathsf{sum}(x : y) \to x + \mathsf{sum}(y)$	$\mathsf{length}(\mathsf{nil}) \to 0$ $\mathsf{length}(x : y) \to \mathsf{s}(\mathsf{length}(y))$	⑩

① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} +\!\!+ x \to x$ $(x : y) +\!\!+ z \to x : (y +\!\!+ z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
⑨	$\mathsf{sum}(\mathsf{nil}) \to 0$ $\mathsf{sum}(x : y) \to x + \mathsf{sum}(y)$	$\mathsf{length}(\mathsf{nil}) \to 0$ $\mathsf{length}(x : y) \to \mathsf{s}(\mathsf{length}(y))$	⑩

① $\underset{\mathsf{h}}{+}$ ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} \mathbin{+\!\!+} x \to x$ $(x : y) \mathbin{+\!\!+} z \to x : (y \mathbin{+\!\!+} z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
⑨	$\mathsf{sum}(\mathsf{nil}) \to 0$ $\mathsf{sum}(x : y) \to x + \mathsf{sum}(y)$	$\mathsf{length}(\mathsf{nil}) \to 0$ $\mathsf{length}(x : y) \to \mathsf{s}(\mathsf{length}(y))$	⑩

① $\underset{\mathsf{h}}{+}$ ② $\underset{\mathsf{cs}}{+}$ ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

Example

①	$0 + y \rightarrow y$ $\mathrm{s}(x) + y \rightarrow \mathrm{s}(x + y)$	$0 \times y \rightarrow 0$ $\mathrm{s}(x) \times y \rightarrow x \times y + y$	②
③	$0 - y \rightarrow 0$ $x - 0 \rightarrow x$ $\mathrm{s}(x) - \mathrm{s}(y) \rightarrow x - y$	$\mathrm{fib}(0) \rightarrow \mathrm{s}(0)$ $\mathrm{fib}(\mathrm{s}(0)) \rightarrow \mathrm{s}(0)$ $\mathrm{fib}(\mathrm{s}(\mathrm{s}(x))) \rightarrow \mathrm{fib}(\mathrm{s}(x)) + \mathrm{fib}(x)$	④
⑤	$\mathrm{nil} \mathbin{+\!\!+} x \rightarrow x$ $(x : y) \mathbin{+\!\!+} z \rightarrow x : (y \mathbin{+\!\!+} z)$	$0 \div \mathrm{s}(y) \rightarrow 0$ $\mathrm{s}(x) \div \mathrm{s}(y) \rightarrow \mathrm{s}((x - y) \div \mathrm{s}(y))$	⑥
⑦	$\mathrm{true} \wedge \mathrm{false} \rightarrow \mathrm{false}$ $\mathrm{false} \wedge \mathrm{true} \rightarrow \mathrm{false}$ $x \wedge x \rightarrow x$	$x < 0 \rightarrow \mathrm{false}$ $0 < \mathrm{s}(y) \rightarrow \mathrm{true}$ $\mathrm{s}(x) < \mathrm{s}(y) \rightarrow x < y$	⑧
⑨	$\mathrm{sum}(\mathrm{nil}) \rightarrow 0$ $\mathrm{sum}(x : y) \rightarrow x + \mathrm{sum}(y)$	$\mathrm{length}(\mathrm{nil}) \rightarrow 0$ $\mathrm{length}(x : y) \rightarrow \mathrm{s}(\mathrm{length}(y))$	⑩

① $\underset{\text{h}}{+}$ ② $\underset{\text{cs}}{+}$ ③ $\underset{\text{h}}{+}$ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} +\!\!+\; x \to x$ $(x : y) +\!\!+\; z \to x : (y +\!\!+\; z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
⑨	$\mathsf{sum}(\mathsf{nil}) \to 0$ $\mathsf{sum}(x : y) \to x + \mathsf{sum}(y)$	$\mathsf{length}(\mathsf{nil}) \to 0$ $\mathsf{length}(x : y) \to \mathsf{s}(\mathsf{length}(y))$	⑩

① $\underset{\mathsf{h}}{+}$ ② $\underset{\mathsf{cs}}{+}$ ③ $\underset{\mathsf{h}}{+}$ ④ $\underset{\mathsf{d}}{+}$ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} \mathbin{+\!\!+} x \to x$ $(x : y) \mathbin{+\!\!+} z \to x : (y \mathbin{+\!\!+} z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
⑨	$\mathsf{sum}(\mathsf{nil}) \to 0$ $\mathsf{sum}(x : y) \to x + \mathsf{sum}(y)$	$\mathsf{length}(\mathsf{nil}) \to 0$ $\mathsf{length}(x : y) \to \mathsf{s}(\mathsf{length}(y))$	⑩

① $\underset{\mathsf{h}}{+}$ ② $\underset{\mathsf{cs}}{+}$ ③ $\underset{\mathsf{h}}{+}$ ④ $\underset{\mathsf{d}}{+}$ ⑤ $\underset{\mathsf{h}}{+}$ ⑥ ⑦ ⑧ ⑨ ⑩

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
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⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
⑨	$\mathsf{sum}(\mathsf{nil}) \to 0$ $\mathsf{sum}(x : y) \to x + \mathsf{sum}(y)$	$\mathsf{length}(\mathsf{nil}) \to 0$ $\mathsf{length}(x : y) \to \mathsf{s}(\mathsf{length}(y))$	⑩

$$\textcircled{1} \underset{\text{h}}{+} \textcircled{2} \underset{\text{cs}}{+} \textcircled{3} \underset{\text{h}}{+} \textcircled{4} \underset{\text{d}}{+} \textcircled{5} \underset{\text{h}}{+} \textcircled{6} \underset{\text{d}}{+} \textcircled{7} \quad \textcircled{8} \quad \textcircled{9} \quad \textcircled{10}$$

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} +\!\!+\, x \to x$ $(x : y) +\!\!+\, z \to x : (y +\!\!+\, z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
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① $\underset{\mathsf{h}}{+}$ ② $\underset{\mathsf{cs}}{+}$ ③ $\underset{\mathsf{h}}{+}$ ④ $\underset{\mathsf{d}}{+}$ ⑤ $\underset{\mathsf{h}}{+}$ ⑥ $\underset{\mathsf{d}}{+}$ ⑦ $\underset{\mathsf{cs}}{+}$ ⑧ ⑨ ⑩

Example

①	$0 + y \to y$ $s(x) + y \to s(x + y)$	$0 \times y \to 0$ $s(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $s(x) - s(y) \to x - y$	$\mathrm{fib}(0) \to s(0)$ $\mathrm{fib}(s(0)) \to s(0)$ $\mathrm{fib}(s(s(x))) \to \mathrm{fib}(s(x)) + \mathrm{fib}(x)$	④
⑤	$\mathrm{nil} \mathbin{+\!\!+} x \to x$ $(x : y) \mathbin{+\!\!+} z \to x : (y \mathbin{+\!\!+} z)$	$0 \div s(y) \to 0$ $s(x) \div s(y) \to s((x - y) \div s(y))$	⑥
⑦	$\mathrm{true} \wedge \mathrm{false} \to \mathrm{false}$ $\mathrm{false} \wedge \mathrm{true} \to \mathrm{false}$ $x \wedge x \to x$	$x < 0 \to \mathrm{false}$ $0 < s(y) \to \mathrm{true}$ $s(x) < s(y) \to x < y$	⑧
⑨	$\mathrm{sum}(\mathrm{nil}) \to 0$ $\mathrm{sum}(x : y) \to x + \mathrm{sum}(y)$	$\mathrm{length}(\mathrm{nil}) \to 0$ $\mathrm{length}(x : y) \to s(\mathrm{length}(y))$	⑩

①	✜	②	✜	③	✜	④	✜	⑤	✜	⑥	✜	⑦	✜	⑧	✜	⑨		⑩
	h		cs		h		d		h		d		cs		h			

Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} +\!\!+\, x \to x$ $(x : y) +\!\!+\, z \to x : (y +\!\!+\, z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
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$$① \underset{\text{h}}{\text{✛}} ② \underset{\text{cs}}{\text{✛}} ③ \underset{\text{h}}{\text{✛}} ④ \underset{\text{d}}{\text{✛}} ⑤ \underset{\text{h}}{\text{✛}} ⑥ \underset{\text{d}}{\text{✛}} ⑦ \underset{\text{cs}}{\text{✛}} ⑧ \underset{\text{h}}{\text{✛}} ⑨ \underset{\text{cs}}{\text{✛}} ⑩$$

Outline

- Orthogonality
  - Definitions
  - Descendants
  - Parallel Moves Lemma
- Beyond Orthogonality
- Modularity
  - Definitions
  - Results
- Further Reading

Theorem (Toyama's Theorem)

confluence is modular for disjoint TRSs

Theorem (Toyama's Theorem)

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Remark

confluence is not modular for constructor-sharing TRSs

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confluence is not modular for constructor-sharing TRSs

Example

$$\begin{array}{rcl} \mathsf{f}(x,x) & \to & \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) & \to & \mathsf{b} \end{array} \qquad\qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

Theorem (Toyama's Theorem)

confluence is modular for disjoint TRSs

Remark

confluence is not modular for constructor-sharing TRSs

Example

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

Theorem (Toyama's Theorem)

confluence is modular for disjoint TRSs

Remark

confluence is not modular for constructor-sharing TRSs

Example

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

$$\mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{b}$$

Theorem

termination is not modular for disjoint TRSs

Theorem

termination is not modular for disjoint TRSs

Example

$$\mathrm{f}(\mathrm{a}, \mathrm{b}, x) \to \mathrm{f}(x, x, x)$$

$$\mathrm{g}(x, y) \to x$$
$$\mathrm{g}(x, y) \to y$$

Theorem

termination is not modular for disjoint TRSs

Example

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \rightarrow \mathsf{f}(x,x,x)$$

$$\begin{aligned} \mathsf{g}(x,y) &\rightarrow x \\ \mathsf{g}(x,y) &\rightarrow y \end{aligned}$$

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \rightarrow \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}))$$

Theorem

termination is not modular for disjoint TRSs

Example

$$\mathrm{f}(\mathrm{a},\mathrm{b},x) \to \mathrm{f}(x,x,x) \qquad \begin{aligned} \mathrm{g}(x,y) &\to x \\ \mathrm{g}(x,y) &\to y \end{aligned}$$

$$\begin{aligned} \mathrm{f}(\mathrm{a},\mathrm{b},\mathrm{g}(\mathrm{a},\mathrm{b})) &\to \mathrm{f}(\mathrm{g}(\mathrm{a},\mathrm{b}),\mathrm{g}(\mathrm{a},\mathrm{b}),\mathrm{g}(\mathrm{a},\mathrm{b})) \\ &\to \mathrm{f}(\mathrm{a},\mathrm{g}(\mathrm{a},\mathrm{b}),\mathrm{g}(\mathrm{a},\mathrm{b})) \end{aligned}$$

Theorem

termination is not modular for disjoint TRSs

Example

$$f(a, b, x) \to f(x, x, x) \qquad \begin{aligned} g(x, y) &\to x \\ g(x, y) &\to y \end{aligned}$$

$$\begin{aligned} f(a, b, g(a, b)) &\to f(g(a, b), g(a, b), g(a, b)) \\ &\to f(a, g(a, b), g(a, b)) \\ &\to f(a, b, g(a, b)) \end{aligned}$$

Theorem

termination is not modular for disjoint TRSs

Example

$$f(a,b,x) \to f(x,x,x)$$

$$g(x,y) \to x$$
$$g(x,y) \to y$$

duplicating

collapsing

$$\begin{aligned} f(a,b,g(a,b)) &\to f(g(a,b),g(a,b),g(a,b)) \\ &\to f(a,g(a,b),g(a,b)) \\ &\to f(a,b,g(a,b)) \end{aligned}$$

Theorem

disjoint union of terminating TRSs $\mathcal{R}$ and $\mathcal{S}$ is terminating if

- $\mathcal{R}$ *and* $\mathcal{S}$ *lack collapsing rules*
- $\mathcal{R}$ *and* $\mathcal{S}$ *lack duplicating rules*
- $\mathcal{R}$ *or* $\mathcal{S}$ *lacks both collapsing and duplicating rules*

Theorem

disjoint union of terminating TRSs $\mathcal{R}$ and $\mathcal{S}$ is terminating if

- *$\mathcal{R}$ and $\mathcal{S}$ lack collapsing rules*
- *$\mathcal{R}$ and $\mathcal{S}$ lack duplicating rules*
- *$\mathcal{R}$ or $\mathcal{S}$ lacks both collapsing and duplicating rules*

Corollary

termination is preserved under signature extension

Theorem

termination is not modular for disjoint TRSs

Example

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

$$\begin{aligned} \mathsf{g}(x,y) &\to x \\ \mathsf{g}(x,y) &\to y \end{aligned}$$

duplicating

not confluent

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \end{aligned}$$

Theorem

termination is not modular for disjoint confluent TRSs

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{llll} \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) & \mathsf{a} \to \mathsf{c} & & \mathsf{g}(x,y,y) \to x \\ \mathsf{f}(x,y,z) \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & & \mathsf{g}(y,y,x) \to x \end{array}$$

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{llll} \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) & \mathsf{a} \to \mathsf{c} & \qquad & \mathsf{g}(x,y,y) \to x \\ \mathsf{f}(x,y,z) \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & & \mathsf{g}(y,y,x) \to x \end{array}$$

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))$$

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{lll} f(a,b,x) \to f(x,x,x) & a \to c & g(x,y,y) \to x \\ f(x,y,z) \to c & b \to c & g(y,y,x) \to x \end{array}$$

$$\begin{aligned} f(a,b,g(a,b,b)) &\to f(g(a,b,b),g(a,b,b),g(a,b,b)) \\ &\to f(a,g(a,b,b),g(a,b,b)) \end{aligned}$$

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{lll} \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) & \mathsf{a} \to \mathsf{c} & \mathsf{g}(x,y,y) \to x \\ \mathsf{f}(x,y,z) \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{g}(y,y,x) \to x \end{array}$$

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \end{aligned}$$

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{llll} \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) & \mathsf{a} \to \mathsf{c} & & \mathsf{g}(x,y,y) \to x \\ \mathsf{f}(x,y,z) \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & & \mathsf{g}(y,y,x) \to x \end{array}$$

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{c},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \end{aligned}$$

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{lll} \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) & \mathsf{a} \to \mathsf{c} & \mathsf{g}(x,y,y) \to x \\ \mathsf{f}(x,y,z) \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{g}(y,y,x) \to x \end{array}$$

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{c},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \end{aligned}$$

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{lll} \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) & \mathsf{a} \to \mathsf{c} & \mathsf{g}(x,y,y) \to x \\ \mathsf{f}(x,y,z) \to \mathsf{c} & \mathsf{b} \to \mathsf{c} & \mathsf{g}(y,y,x) \to x \end{array}$$

not left-linear

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{c},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) \end{aligned}$$

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{aligned} f(a, b, x) &\to f(x, x, x) & a &\to c & g(x, y, y) &\to x \\ f(x, y, z) &\to c & b &\to c & g(y, y, x) &\to x \end{aligned}$$

no constructor system

not left-linear

$$\begin{aligned} f(a, b, g(a, b, b)) &\to f(g(a, b, b), g(a, b, b), g(a, b, b)) \\ &\to f(a, g(a, b, b), g(a, b, b)) \\ &\to f(a, g(c, b, b), g(a, b, b)) \\ &\to f(a, g(c, c, b), g(a, b, b)) \\ &\to f(a, b, g(a, b, b)) \end{aligned}$$

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is constructor system (CS) if $l_1, \ldots, l_n \in \mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$ for every left-hand side $f(l_1, \ldots, l_n)$ of rewrite rule in $\mathcal{R}$

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

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Theorem

- *weak normalization is modular for constructor-sharing TRSs*

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is constructor system (CS) if $l_1, \ldots, l_n \in \mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$ for every left-hand side $f(l_1, \ldots, l_n)$ of rewrite rule in $\mathcal{R}$

Theorem

- *weak normalization is modular for constructor-sharing TRSs*
- *local confluence is modular for constructor-sharing TRSs*

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is constructor system (CS) if $l_1, \ldots, l_n \in \mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$ for every left-hand side $f(l_1, \ldots, l_n)$ of rewrite rule in $\mathcal{R}$

Theorem

- *weak normalization is modular for constructor-sharing TRSs*
- *local confluence is modular for constructor-sharing TRSs*
- *semi-completeness is modular for constructor-sharing TRSs*

Outline

- Orthogonality
- Beyond Orthogonality
- Modularity
- Further Reading

Developing Developments
Vincent van Oostrom
TCS 175(1), pp. 159 – 181, 1997

Modular Termination of r-Consistent and Left-Linear Term Rewriting Systems
Manfred Schmidt-Schauß, Massimo Marchiori, and Sven Eric Panitz
TCS 149(2), pp. 361 – 374, 1995

Modularity of Confluence – Constructed
Vincent van Oostrom
Proc. 4th IJCAR, LNAI 5195, pp. 348 – 363, 2008

Confluence Tool

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