Introduction to Term Rewriting

lecture 8

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

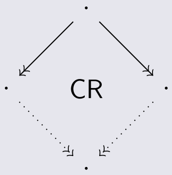
exam, advanced topics

Outline

- Orthogonality
  - Definitions
  - Descendants
  - Parallel Moves Lemma
- Beyond Orthogonality
- Modularity
- Further Reading

Confluence

every two coinitial rewrite sequences can be joined



- ... yields uniqueness of normal forms
- ... is decidable for terminating TRSs
- ... what about nonterminating TRSs ?

Examples (Non-Confluence)

- no confluence because of critical pairs

$$\begin{aligned} a &\to b \\ a &\to c \end{aligned} \qquad\qquad b \leftarrow a \to c$$

- no critical pairs but no confluence (Klop 1978)

$$\begin{aligned} f(x,x) &\to a \\ g(x) &\to f(x,g(x)) \\ c &\to g(c) \end{aligned} \qquad\qquad a \;{}^{*}\!\!\leftarrow c \to g(c) \to^{*} g(a)$$

- no critical pairs but no confluence (Huet 1980)

$$\begin{aligned} f(x,x) &\to a \\ f(x,g(x)) &\to b \\ c &\to g(c) \end{aligned} \qquad\qquad a \leftarrow f(c,c) \to f(c,g(c)) \to b$$

Exercises (Non-Confluence)

1. $\mathsf{f}(\mathsf{f}(x)) \to \mathsf{a}$
2. $\mathsf{f}(\mathsf{g}(x), y) \to x \quad \mathsf{g}(\mathsf{a}) \to \mathsf{b}$
3. $\mathsf{or}(x, y) \to x \quad \mathsf{or}(x, y) \to y$

Confluence via Critical Pairs

control interference of rewrite rules

- Critical Pair Lemma (lecture 5):

  $\mathsf{WCR} \iff \leftarrow\rtimes\rightarrow\ \subseteq\ \downarrow$ (all critical pairs are convergent)

- combine with Newman's Lemma (lecture 2):

  $\mathsf{SN}\ \&\ \leftarrow\rtimes\rightarrow\ \subseteq\ \downarrow \implies \mathsf{CR}$

- observe from preceding examples:

  $\leftarrow\rtimes\rightarrow\ =\ \varnothing \;\not\Longrightarrow\; \mathsf{CR}$

Confluence via Orthogonality

forbid interference of rewrite rules

- no critical pairs
- no equality checks

Definitions

- term t is linear is each variable in $\mathcal{V}\mathrm{ar}(t)$ occurs exactly once in t
- rewrite rule $\ell \to r$ is left-linear if ℓ is linear
- TRS is left-linear if all rewrite rules are left-linear
- rewrite rule $\ell \to r$ is right-linear if r is linear
- rewrite rule $\ell \to r$ is linear if ℓ and r are linear
- TRS is (right-)linear if all rewrite rules are (right-)linear

Examples

- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
  left-linear but not right-linear
- $\mathsf{f}(x, x) \to \mathsf{a}$
  right-linear but not left-linear

Definition

orthogonal TRS is left-linear and lacks criticial pairs

Examples

$$\begin{aligned} \mathsf{I} \cdot x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

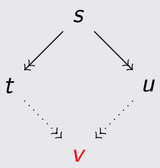
$$\begin{aligned} \mathsf{ack}(0, y) &\to \mathsf{s}(y) \\ \mathsf{ack}(\mathsf{s}(x), 0) &\to \mathsf{ack}(x, \mathsf{s}(0)) \\ \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)) \end{aligned}$$

Theorem

orthogonal TRSs are confluent

$\forall s, t, u$

$\exists v$



Observation

for orthogonal TRSs there is canonical way to compute common reduct v

Lemma

orthogonal TRSs are locally confluent

Proof Idea

distinguish two cases:

1. two disjoint redexes
2. two nested redexes

Examples

1. $\mathsf{a} \to \mathsf{c}$, $\mathsf{b} \to \mathsf{d}$ $\qquad$ $\mathsf{f(c,b)} \leftarrow \mathsf{f(a,b)} \to \mathsf{f(a,d)}$
2. $\mathsf{f}(x) \to \mathsf{g}(x,x)$, $\mathsf{a} \to \mathsf{b}$ $\qquad$ $\mathsf{f(b)} \leftarrow \mathsf{f(a)} \to \mathsf{g(a,a)}$

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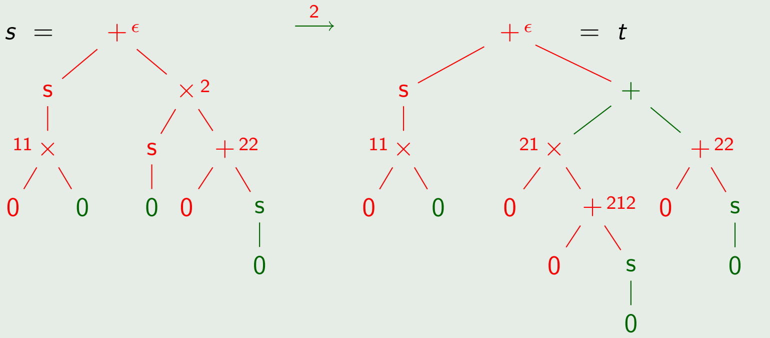
Example

$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$



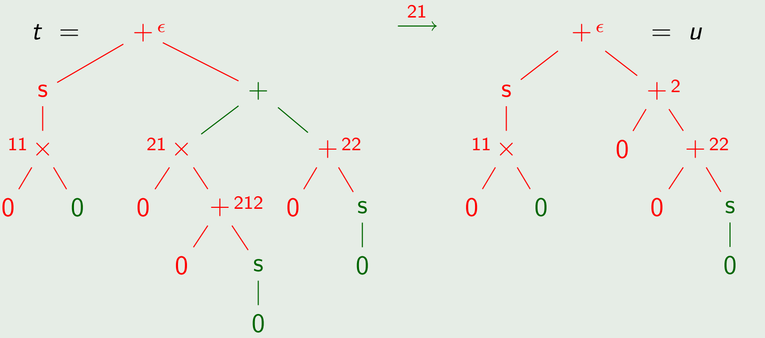
redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21

redex at position 22 is duplicated

redex at position 21 is created

Example

$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$



redex positions in s:	ϵ	11	2	22			
redex positions in t:	ϵ	11		212	22	21	
redex positions in u:	ϵ	11			22		2

redex at position 212 is erased redex at position 2 is created rewrite step A: $s \xrightarrow[\ell \to r]{p} t$ position $q \in \mathcal{P}os(s)$ set of positions $Q \subseteq \mathcal{P}os(s)$

Definition (Descendants after Rewrite Step)

- descendants of q in t

$$q\backslash A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}os_{\mathcal{V}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

- descendants of Q in t

$$Q\backslash A = \bigcup_{q \in Q} q\backslash A$$

Remark

- information about position is needed to determine descendants

$$\mathsf{f}(x) \to x \qquad\qquad \mathsf{f}(\mathsf{f}(x)) \xrightarrow[\mathsf{f}(x)\to x]{?} \mathsf{f}(x)$$

- information about rewrite rule is needed to determine descendants

$$\begin{array}{c} \mathsf{f}(x) \to \mathsf{f}(x) \\ \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \end{array} \qquad\qquad \mathsf{f}(\mathsf{a}) \xrightarrow[?]{\epsilon} \mathsf{f}(\mathsf{a})$$

rewrite sequence $A\colon s \to^* t$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Sequence)

descendants of Q in t

$$Q\backslash A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q\backslash A_1)\backslash A_2 & \text{If } A = A_1; A_2 \text{ with } A_1\colon s \to u \text{ and } A_2\colon u \to^* t \end{cases}$$

Lemma

- *for arbitrary TRSs:* *if* Q *is parallel* $(\forall\, p \neq q \in Q\colon p \parallel q)$ *then so is* $Q \backslash A$
- *for orthogonal TRSs:* *if* Q *is set of redex positions then so is* $Q \backslash A$

Terminology

descendant of redex is called residual

Remark

- in non-left-linear TRS descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(x,x) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{f}(\mathsf{b},\mathsf{a})$$

- in TRS with critical pairs descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(\mathsf{a}) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b})$$

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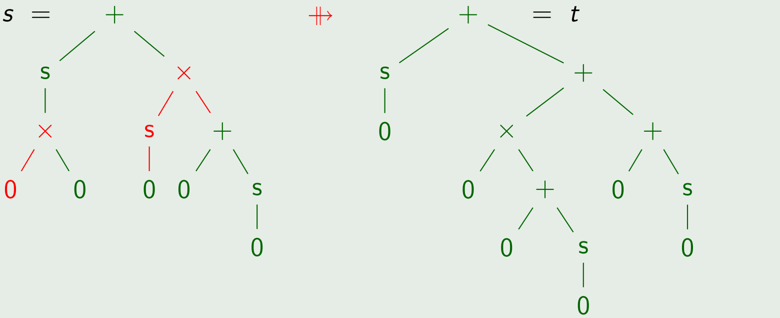
Definition (Parallel Rewriting)

1. $= \; \subseteq \; \mathrel{+\!\!\!\!\!\rightarrow}$
2. $\xrightarrow{\epsilon} \; \subseteq \; \mathrel{+\!\!\!\!\!\rightarrow}$
3. $f(s_1, \dots, s_n) \mathrel{+\!\!\!\!\!\rightarrow} f(t_1, \dots, t_n)$ if $s_i \mathrel{+\!\!\!\!\!\rightarrow} t_i$ for each $1 \leqslant i \leqslant n$

Example

$0 + y \rightarrow y \qquad \mathsf{s}(x) + y \rightarrow \mathsf{s}(x + y) \qquad 0 \times y \rightarrow 0 \qquad \mathsf{s}(x) \times y \rightarrow (x \times y) + y$



Lemma

$s \mathrel{+\!\!\!\!\!\!\!\;\;\longrightarrow} t \quad \Longleftrightarrow \quad s \rightarrow^{*} t$ *by contracting redexes at pairwise parallel positions in* s

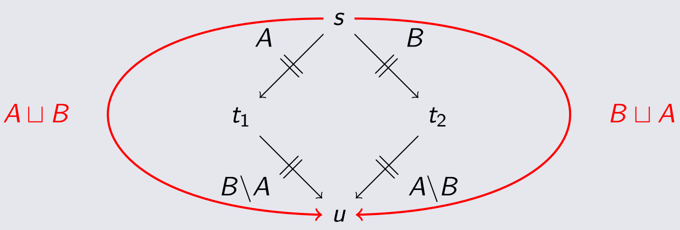
Definition (Projection)

$A\colon\; s \mathrel{+\!\!\!\!\!\rightarrow} t_1$ by contracting redexes at positions in P

$B\colon\; s \mathrel{+\!\!\!\!\!\rightarrow} t_2$ by contracting redexes at positions in Q

- $B \backslash A$: $t_1 \mathrel{+\!\!\!\!\!\rightarrow} u_1$ by contracting redexes at positions in $Q \backslash A$
- $A \backslash B$: $t_2 \mathrel{+\!\!\!\!\!\rightarrow} u_2$ by contracting redexes at positions in $P \backslash B$
- $A \sqcup B = A; B \backslash A$ and $B \sqcup A = B; A \backslash B$

Parallel Moves Lemma

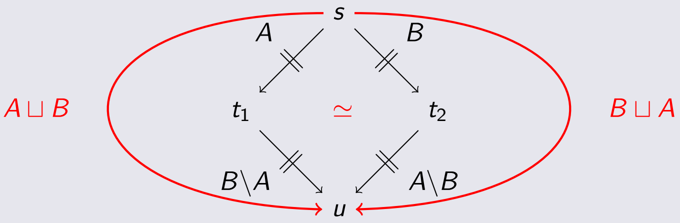


Definition

$A\colon s_1 \to^* t_1$ and $B\colon s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

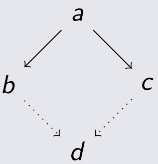
1. $s_1 = s_2$
2. $t_1 = t_2$
3. $p \backslash A = p \backslash B$ for all redex positions p in s_1

Parallel Moves Lemma (with Permutation Equivalence)



Definition

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$

    $\exists d$

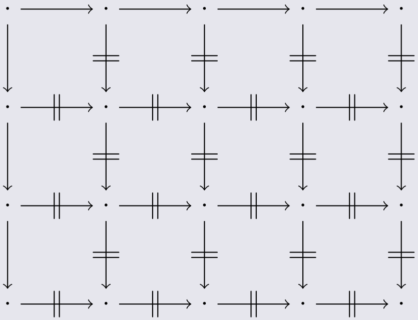
Lemma

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow \;\subseteq\; \rightarrow_\diamond \;\subseteq\; \rightarrow^*$ for some relation $\rightarrow_\diamond$ on A with diamond property

Corollary

orthogonal TRSs are confluent

Proof



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Definitions

- critical pair $s \leftarrow\rtimes\rightarrow t$ is trivial if $s = t$
- weakly orthogonal TRS is left-linear and has only trivial critical pairs
- overlay $s \leftarrow\bowtie\rightarrow t$ is critical pair originating from overlap $\langle l_1 \to r_1, \epsilon, l_2 \to r_2 \rangle$
- weakly orthogonal TRS is almost orthogonal if all critical pairs are overlays

Examples

$$x \vee \mathsf{T} \to \mathsf{T}$$
$$\mathsf{T} \vee x \to \mathsf{T}$$
$$\mathsf{F} \vee \mathsf{F} \to \mathsf{F}$$

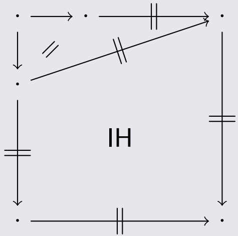
$$\mathsf{p}(\mathsf{s}(x)) \to x$$
$$\mathsf{s}(\mathsf{p}(x)) \to x$$

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $⇺ \cdot ⇻ \subseteq ⇻ \cdot ⇺$



interesting case

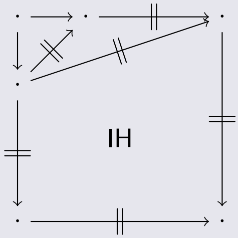
- $\rightarrow \subseteq ⇻ \subseteq \rightarrow^*$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \|\!\!\!\!\!\longrightarrow \quad\Longrightarrow\quad$ *CR*

Proof Sketch

- $\longleftarrow\!\!\!\!\!\| \cdot \|\!\!\!\!\!\longrightarrow \;\subseteq\; \|\!\!\!\!\!\longrightarrow \cdot \longleftarrow\!\!\!\!\!\|$



interesting case

- $\rightarrow \;\subseteq\; \|\!\!\!\!\!\longrightarrow \;\subseteq\; \rightarrow^*$

Open Problem

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \longleftarrow\!\!\!\!\!\| \quad\Longrightarrow\quad$ *CR ?*

Example

$$
\begin{aligned}
\mathsf{f}(\mathsf{g}(x), \mathsf{b}) &\rightarrow \mathsf{f}(\mathsf{g}(x), \mathsf{c}) \\
\mathsf{g}(\mathsf{c}) &\rightarrow \mathsf{g}(\mathsf{a}) \\
\mathsf{a} &\rightarrow \mathsf{c} \\
\mathsf{b} &\rightarrow \mathsf{d}
\end{aligned}
$$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad CR$

Notation

$\leftarrow\!\cdot\!\rtimes\!\rightarrow \;=\; \leftarrow\!\rtimes\!\rightarrow \setminus \leftarrow\!\bowtie\!\rightarrow$

Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\cdot\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Theorem (van Oostrom 1996)

left-linearity & $\leftarrow\!\cdot\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\circ\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{\circ\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

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Definition

property of TRSs is modular if it is preserved under union

Remark

without further restrictions 'no' property of TRSs is modular

termination	$a \to b$	$b \to a$
confluence	$a \to b$	$a \to c$

Definition

property P is preserved under signature extension if

$$(\mathcal{F}, \mathcal{R}) \vDash P \quad \Longrightarrow \quad (\mathcal{F} \cup \mathcal{G}, \mathcal{R}) \vDash P$$

for all TRSs $(\mathcal{F}, \mathcal{R})$ and signatures $\mathcal{G}$ with $\mathcal{F} \cap \mathcal{G} \neq \varnothing$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$

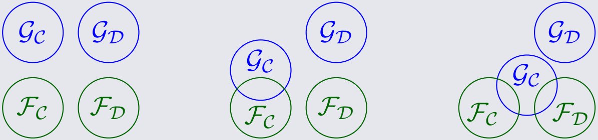
- defined symbols $\mathcal{F}_{\mathcal{D}} = \{ \operatorname{root}(\ell) \mid \ell \to r \in \mathcal{R} \}$
- constructors $\mathcal{F}_{\mathcal{C}} = \mathcal{F} \setminus \mathcal{F}_{\mathcal{D}}$

More Interesting Combinations

$(\mathcal{G}, \mathcal{S})$

$(\mathcal{F}, \mathcal{R})$

two TRSs



disjoint constructor-sharing hierarchical

Example

①	$0 + y \to y$ $s(x) + y \to s(x + y)$	$0 \times y \to 0$ $s(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $s(x) - s(y) \to x - y$	$fib(0) \to s(0)$ $fib(s(0)) \to s(0)$ $fib(s(s(x))) \to fib(s(x)) + fib(x)$	④
⑤	$nil ++ x \to x$ $(x : y) ++ z \to x : (y ++ z)$	$0 \div s(y) \to 0$ $s(x) \div s(y) \to s((x - y) \div s(y))$	⑥
⑦	$true \wedge false \to false$ $false \wedge true \to false$ $x \wedge x \to x$	$x < 0 \to false$ $0 < s(y) \to true$ $s(x) < s(y) \to x < y$	⑧
⑨	$sum(nil) \to 0$ $sum(x : y) \to x + sum(y)$	$length(nil) \to 0$ $length(x : y) \to s(length(y))$	⑩

① ✢ ② ✢ ③ ✢ ④ ✢ ⑤ ✢ ⑥ ✢ ⑦ ✢ ⑧ ✢ ⑨ ✢ ⑩

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Theorem (Toyama's Theorem)

confluence is modular for disjoint TRSs

Remark

confluence is not modular for constructor-sharing TRSs

Example

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

$$\mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{b}$$

Theorem

termination is not modular for disjoint TRSs

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \rightarrow \mathsf{f}(x, x, x)$$

$$\mathsf{g}(x, y) \rightarrow x \qquad \mathsf{g}(x, y) \rightarrow y$$

duplicating

not confluent

$$\begin{aligned} \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g}(\mathsf{a}, \mathsf{b})) &\rightarrow \mathsf{f}(\mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{g}(\mathsf{a}, \mathsf{b})) \\ &\rightarrow \mathsf{f}(\mathsf{a}, \mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{g}(\mathsf{a}, \mathsf{b})) \\ &\rightarrow \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g}(\mathsf{a}, \mathsf{b})) \end{aligned}$$

Theorem

disjoint union of terminating TRSs $\mathcal{R}$ and $\mathcal{S}$ is terminating if

- *$\mathcal{R}$ and $\mathcal{S}$ lack collapsing rules*
- *$\mathcal{R}$ and $\mathcal{S}$ lack duplicating rules*
- *$\mathcal{R}$ or $\mathcal{S}$ lacks both collapsing and duplicating rules*

Corollary

termination is preserved under signature extension

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad\qquad \mathsf{g}(x,y,y) \to x$$

$$\mathsf{f}(x,y,z) \to \mathsf{c} \qquad\qquad \mathsf{b} \to \mathsf{c} \qquad\qquad \mathsf{g}(y,y,x) \to x$$

no constructor system — not left-linear

$$\begin{aligned}
\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{c},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))
\end{aligned}$$

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is constructor system (CS) if $l_1, \ldots, l_n \in \mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$ for every left-hand side $f(l_1, \ldots, l_n)$ of rewrite rule in $\mathcal{R}$

Theorem

- *weak normalization is modular for constructor-sharing TRSs*
- *local confluence is modular for constructor-sharing TRSs*
- *semi-completeness is modular for constructor-sharing TRSs*

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Developing Developments
Vincent van Oostrom
TCS 175(1), pp. 159 – 181, 1997

Modular Termination of r-Consistent and Left-Linear Term Rewriting Systems
Manfred Schmidt-Schauß, Massimo Marchiori, and Sven Eric Panitz
TCS 149(2), pp. 361 – 374, 1995

Modularity of Confluence – Constructed
Vincent van Oostrom
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Confluence Tool

- ACP