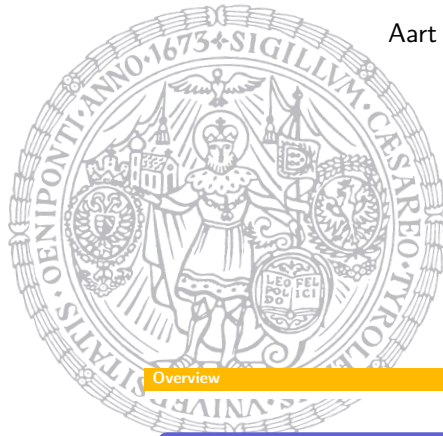




Introduction to Term Rewriting

lecture 8

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Overview

Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

Thursday

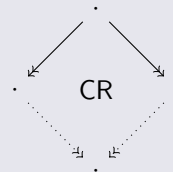
exam, advanced topics

Outline

- Orthogonality
 - Definitions
 - Descendants
 - Parallel Moves Lemma
- Beyond Orthogonality
- Modularity
- Further Reading

Confluence

every two coinitial rewrite sequences can be joined



- ... yields uniqueness of normal forms
- ... is decidable for terminating TRSs
- ... what about nonterminating TRSs ?

Examples (Non-Confluence)

- no confluence because of critical pairs

$$\begin{array}{l} a \rightarrow b \qquad b \leftarrow a \rightarrow c \\ a \rightarrow c \end{array}$$

- no critical pairs but no confluence (Klop 1978)

$$\begin{array}{l} f(x, x) \rightarrow a \qquad a^* \leftarrow c \rightarrow g(c) \rightarrow^* g(a) \\ g(x) \rightarrow f(x, g(x)) \\ c \rightarrow g(c) \end{array}$$

- no critical pairs but no confluence (Huet 1980)

$$\begin{array}{l} f(x, x) \rightarrow a \qquad a \leftarrow f(c, c) \rightarrow f(c, g(c)) \rightarrow b \\ f(x, g(x)) \rightarrow b \\ c \rightarrow g(c) \end{array}$$

Exercises (Non-Confluence)

- $f(f(x)) \rightarrow a$
- $f(g(x), y) \rightarrow x \quad g(a) \rightarrow b$
- $or(x, y) \rightarrow x \quad or(x, y) \rightarrow y$

Confluence via Critical Pairs

control interference of rewrite rules

- Critical Pair Lemma (lecture 5):
 $WCR \iff \leftarrow \times \rightarrow \subseteq \downarrow$ (all critical pairs are convergent)
- combine with Newman's Lemma (lecture 2):
 $SN \ \& \ \leftarrow \times \rightarrow \subseteq \downarrow \implies CR$
- observe from preceding examples:
 $\leftarrow \times \rightarrow = \emptyset \not\Rightarrow CR$

Confluence via Orthogonality

forbid interference of rewrite rules

- **no critical pairs**
- **no equality checks**

Definitions

- term t is **linear** if each variable in $\mathcal{V}\text{ar}(t)$ occurs exactly once in t
- rewrite rule $\ell \rightarrow r$ is **left-linear** if ℓ is linear
- TRS is left-linear if all rewrite rules are left-linear
- rewrite rule $\ell \rightarrow r$ is **right-linear** if r is linear
- rewrite rule $\ell \rightarrow r$ is **linear** if ℓ and r are linear
- TRS is (right-)linear if all rewrite rules are (right-)linear

Examples

- $g(x) \rightarrow f(x, g(x))$
left-linear but not right-linear
- $f(x, x) \rightarrow a$
right-linear but not left-linear

Definition

orthogonal TRS is left-linear and lacks critical pairs

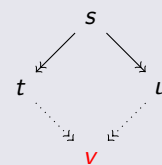
Examples

$$\begin{aligned} I \cdot x &\rightarrow x \\ (K \cdot x) \cdot y &\rightarrow x \\ ((S \cdot x) \cdot y) \cdot z &\rightarrow (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

$$\begin{aligned} \text{ack}(0, y) &\rightarrow s(y) \\ \text{ack}(s(x), 0) &\rightarrow \text{ack}(x, s(0)) \\ \text{ack}(s(x), s(y)) &\rightarrow \text{ack}(x, \text{ack}(s(x), y)) \end{aligned}$$

Theorem

orthogonal TRSs are confluent

 $\forall s, t, u$

 $\exists v$

Observation

for orthogonal TRSs there is canonical way to compute common reduct v

Lemma

orthogonal TRSs are locally confluent

Proof Idea

distinguish two cases:

- 1 two disjoint redexes
- 2 two nested redexes

Examples

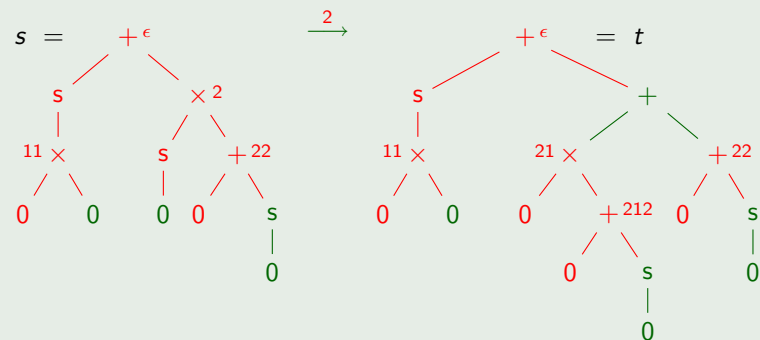
- | | | |
|---|---|---|
| 1 | $a \rightarrow c$
$b \rightarrow d$ | $f(c,b) \leftarrow f(a,b) \rightarrow f(a,d)$ |
| 2 | $f(x) \rightarrow g(x, x)$
$a \rightarrow b$ | $f(b) \leftarrow f(a) \rightarrow g(a,a)$ |

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Example

$0 + y \rightarrow y$ $s(x) + y \rightarrow s(x + y)$ $0 \times y \rightarrow 0$ $s(x) \times y \rightarrow (x \times y) + y$

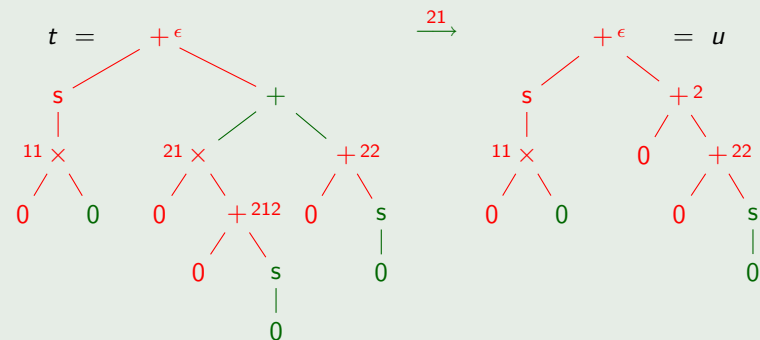


redex positions in s : $\epsilon \mid 11 \mid 2 \mid 22$
 redex positions in t : $\epsilon \mid 11 \mid 212 \mid 22 \mid 21$

redex at position 22 is **duplicated** redex at position 21 is **created**

Example

$0 + y \rightarrow y$ $s(x) + y \rightarrow s(x + y)$ $0 \times y \rightarrow 0$ $s(x) \times y \rightarrow (x \times y) + y$



redex positions in s : $\epsilon \mid 11 \mid 2 \mid 22$
 redex positions in t : $\epsilon \mid 11 \mid 212 \mid 22 \mid 21$
 redex positions in u : $\epsilon \mid 11 \mid 22$

redex at position 212 is **erased** redex at position 2 is **created**

rewrite step $A: s \xrightarrow[\ell \rightarrow r]{p} t$ position $q \in \mathcal{Pos}(s)$ set of positions $Q \subseteq \mathcal{Pos}(s)$

Definition (Descendants after Rewrite Step)

- **descendants** of q in t

$$q \setminus A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{Pos}_V(\ell) \\ \emptyset & \text{otherwise} \end{cases}$$

- descendants of Q in t

$$Q \setminus A = \bigcup_{q \in Q} q \setminus A$$

Remark

- information about **position** is needed to determine descendants

$$f(x) \rightarrow x \qquad f(f(x)) \xrightarrow[\text{f(x) \to x}]{?} f(x)$$

- information about **rewrite rule** is needed to determine descendants

$$\begin{array}{l} f(x) \rightarrow f(x) \\ f(a) \rightarrow f(a) \end{array} \qquad f(a) \xrightarrow[\text{?}]{\epsilon} f(a)$$

rewrite **sequence** $A: s \rightarrow^* t$ set of positions $Q \subseteq \mathcal{P}\text{os}(s)$

Definition (Descendants after Rewrite Sequence)

descendants of Q in t

$$Q \setminus A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q \setminus A_1) \setminus A_2 & \text{if } A = A_1; A_2 \text{ with } A_1: s \rightarrow u \text{ and } A_2: u \rightarrow^* t \end{cases}$$

Lemma

- for arbitrary TRSs: if Q is parallel ($\forall p \neq q \in Q: p \parallel q$) then so is $Q \setminus A$
- for orthogonal TRSs: if Q is set of redex positions then so is $Q \setminus A$

Terminology

descendant of redex is called **residual**

Remark

- in non-left-linear TRS descendant of redex is not necessarily redex

$$\begin{array}{l} a \rightarrow b \\ f(x, x) \rightarrow b \end{array} \quad \begin{array}{l} f(a, a) \rightarrow f(b, a) \end{array}$$

- in TRS with critical pairs descendant of redex is not necessarily redex

$$\begin{array}{l} a \rightarrow b \\ f(a) \rightarrow b \end{array} \quad \begin{array}{l} f(a) \rightarrow f(b) \end{array}$$

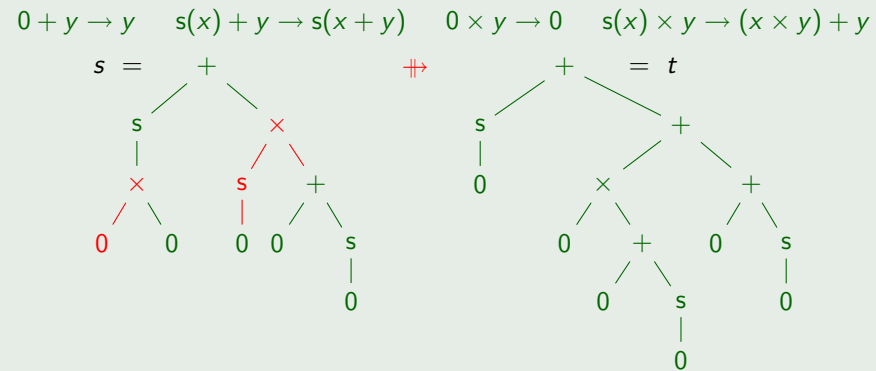
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Definition (Parallel Rewriting)

- 1 $= \subseteq \vdash$
- 2 $\overset{\epsilon}{\rightarrow} \subseteq \vdash$
- 3 $f(s_1, \dots, s_n) \vdash f(t_1, \dots, t_n)$ if $s_i \vdash t_i$ for each $1 \leq i \leq n$

Example



Lemma

$s \vdash t \iff s \rightarrow^* t$ by contracting redexes at pairwise parallel positions in s

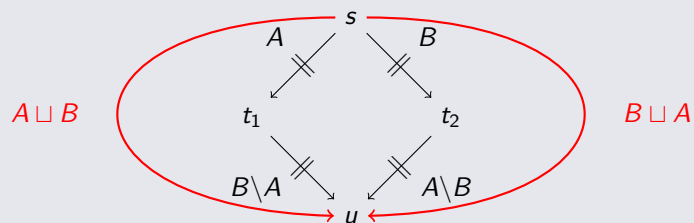
Definition (Projection)

$A: s \mapsto t_1$ by contracting redexes at positions in P

$B: s \mapsto t_2$ by contracting redexes at positions in Q

- $B \setminus A: t_1 \mapsto u_1$ by contracting redexes at positions in $Q \setminus A$
- $A \setminus B: t_2 \mapsto u_2$ by contracting redexes at positions in $P \setminus B$
- $A \sqcup B = A; B \setminus A$ and $B \sqcup A = B; A \setminus B$

Parallel Moves Lemma

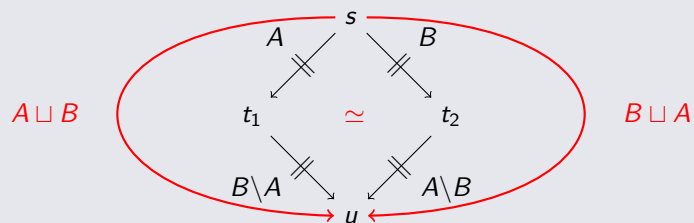


Definition

$A: s_1 \rightarrow^* t_1$ and $B: s_2 \rightarrow^* t_2$ are **permutation equivalent** ($A \simeq B$) if

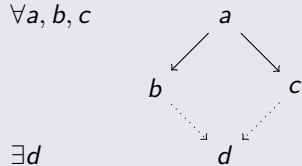
- 1 $s_1 = s_2$
- 2 $t_1 = t_2$
- 3 $p \setminus A = p \setminus B$ for all redex positions p in s_1

Parallel Moves Lemma (with Permutation Equivalence)



Definition

- **diamond property** \diamond
- $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$
- $\forall a, b, c$



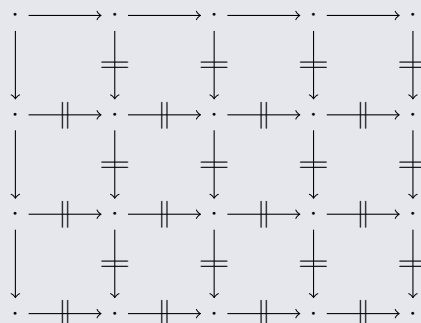
Lemma

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow \subseteq \rightarrow_{\diamond} \subseteq \rightarrow^*$ for some relation \rightarrow_{\diamond} on A with diamond property

Corollary

orthogonal TRSs are confluent

Proof



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Definitions

- critical pair $s \leftarrow \lambda \rightarrow t$ is **trivial** if $s = t$
- **weakly orthogonal** TRS is left-linear and has only trivial critical pairs
- **overlay** $s \leftarrow \lambda \rightarrow t$ is critical pair originating from overlap $\langle l_1 \rightarrow r_1, \epsilon, l_2 \rightarrow r_2 \rangle$
- weakly orthogonal TRS is **almost orthogonal** if all critical pairs are overlays

Examples

$$\begin{aligned} x \vee T &\rightarrow T \\ T \vee x &\rightarrow T \\ F \vee F &\rightarrow F \end{aligned}$$

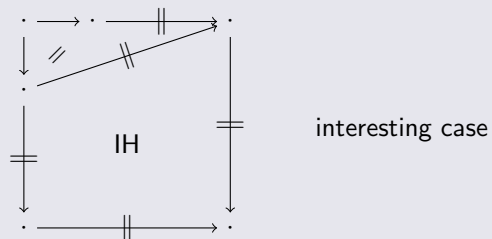
$$\begin{aligned} p(s(x)) &\rightarrow x \\ s(p(x)) &\rightarrow x \end{aligned}$$

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$



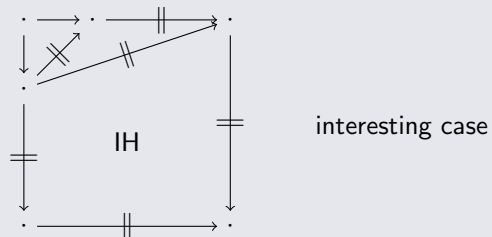
- $\rightarrow \subseteq \rightarrow \subseteq \rightarrow^*$

Theorem (Huet 1980)

left-linearity & $\leftarrow \times \rightarrow \subseteq \rightarrow \implies CR$

Proof Sketch

- $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$



- $\rightarrow \subseteq \rightarrow \subseteq \rightarrow^*$

Open Problem

left-linearity & $\leftarrow \times \rightarrow \subseteq \leftarrow \implies CR ?$

Example

$$\begin{array}{lcl}
 f(g(x), b) & \rightarrow & f(g(x), c) \\
 g(c) & \rightarrow & g(a) \\
 a & \rightarrow & c \\
 b & \rightarrow & d
 \end{array}$$

Theorem (Huet 1980)

$$\text{linearity} \ \& \ \leftarrow \bowtie \rightarrow \subseteq (\rightarrow^= \cdot^* \leftarrow) \cap (\rightarrow^* \cdot^= \leftarrow) \implies CR$$

Notation

$$\leftarrow \bowtie \rightarrow = \leftarrow \bowtie \rightarrow \setminus \leftarrow \bowtie \rightarrow$$

Theorem (Toyama 1988)

$$\text{left-linearity} \ \& \ \leftarrow \bowtie \rightarrow \subseteq \vdash \ \& \ \leftarrow \bowtie \rightarrow \subseteq \vdash \cdot^* \leftarrow \implies CR$$

Theorem (van Oostrom 1996)

$$\text{left-linearity} \ \& \ \leftarrow \bowtie \rightarrow \subseteq \dashrightarrow \ \& \ \leftarrow \bowtie \rightarrow \subseteq \dashrightarrow \cdot^* \leftarrow \implies CR$$

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Definition

property of TRSs is **modular** if it is preserved under union

Remark

without further restrictions 'no' property of TRSs is modular

<i>termination</i>	$a \rightarrow b$	$b \rightarrow a$
<i>confluence</i>	$a \rightarrow b$	$a \rightarrow c$

Definition

property P is **preserved under signature extension** if

$$(\mathcal{F}, \mathcal{R}) \models P \implies (\mathcal{F} \cup \mathcal{G}, \mathcal{R}) \models P$$

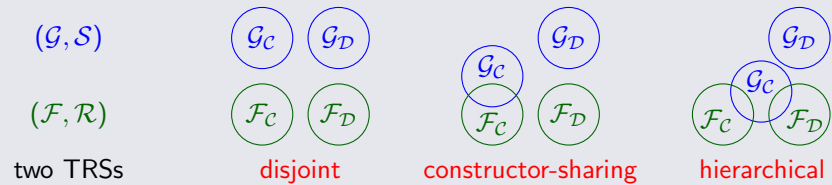
for all TRSs $(\mathcal{F}, \mathcal{R})$ and signatures \mathcal{G} with $\mathcal{F} \cap \mathcal{G} \neq \emptyset$

Definition

TRS \mathcal{R} over signature \mathcal{F}

- defined symbols $\mathcal{F}_D = \{ \text{root}(\ell) \mid \ell \rightarrow r \in \mathcal{R} \}$
- constructors $\mathcal{F}_C = \mathcal{F} \setminus \mathcal{F}_D$

More Interesting Combinations



Example

① $0 + y \rightarrow y$ $s(x) + y \rightarrow s(x + y)$	$0 \times y \rightarrow 0$ $s(x) \times y \rightarrow x \times y + y$ ②
③ $0 - y \rightarrow 0$ $x - 0 \rightarrow x$ $s(x) - s(y) \rightarrow x - y$	$\text{fib}(0) \rightarrow s(0)$ $\text{fib}(s(0)) \rightarrow s(0)$ ④ $\text{fib}(s(s(x))) \rightarrow \text{fib}(s(x)) + \text{fib}(x)$
⑤ $\text{nil} ++ x \rightarrow x$ $(x : y) ++ z \rightarrow x : (y ++ z)$	$0 \div s(y) \rightarrow 0$ ⑥ $s(x) \div s(y) \rightarrow s((x - y) \div s(y))$
$\text{true} \wedge \text{false} \rightarrow \text{false}$	$x < 0 \rightarrow \text{false}$
⑦ $\text{false} \wedge \text{true} \rightarrow \text{false}$ $x \wedge x \rightarrow x$	$0 < s(y) \rightarrow \text{true}$ ⑧ $s(x) < s(y) \rightarrow x < y$
⑨ $\text{sum}(\text{nil}) \rightarrow 0$ $\text{sum}(x : y) \rightarrow x + \text{sum}(y)$	$\text{length}(\text{nil}) \rightarrow 0$ $\text{length}(x : y) \rightarrow s(\text{length}(y))$ ⑩

① + ② + ③ + ④ + ⑤ + ⑥ + ⑦ + ⑧ + ⑨ + ⑩
 h cs h d h d cs h cs

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Theorem (Toyama's Theorem)

confluence is modular for disjoint TRSs

Remark

*confluence is **not** modular for constructor-sharing TRSs*

Example

$$\begin{array}{lcl}
 f(x, x) & \rightarrow & a \\
 f(x, g(x)) & \rightarrow & b \\
 c & \rightarrow & g(c) \\
 a & \leftarrow f(c, c) \rightarrow f(c, g(c)) & \rightarrow b
 \end{array}$$

Theorem

termination is *not* modular for disjoint TRSs

Example

$$\begin{array}{ll}
 f(a, b, x) \rightarrow f(x, x, x) & g(x, y) \rightarrow x \\
 & g(x, y) \rightarrow y \\
 \text{duplicating} & \text{not confluent} \\
 f(a, b, g(a, b)) \rightarrow f(g(a, b), g(a, b), g(a, b)) \\
 \rightarrow f(a, g(a, b), g(a, b)) \\
 \rightarrow f(a, b, g(a, b))
 \end{array}$$

Theorem

disjoint union of terminating TRSs \mathcal{R} and \mathcal{S} is terminating if

- \mathcal{R} and \mathcal{S} lack collapsing rules
- \mathcal{R} and \mathcal{S} lack duplicating rules
- \mathcal{R} or \mathcal{S} lacks both collapsing and duplicating rules

Corollary

termination is preserved under signature extension

Theorem

termination is *not* modular for disjoint confluent TRSs

Example

$$\begin{array}{lll} f(a, b, x) \rightarrow f(x, x, x) & a \rightarrow c & g(x, y, y) \rightarrow x \\ f(x, y, z) \rightarrow c & b \rightarrow c & g(y, y, x) \rightarrow x \end{array}$$

no constructor system

not left-linear

$$\begin{aligned} f(a, b, g(a, b, b)) &\rightarrow f(g(a, b, b), g(a, b, b), g(a, b, b)) \\ &\rightarrow f(a, g(a, b, b), g(a, b, b)) \\ &\rightarrow f(a, g(c, b, b), g(a, b, b)) \\ &\rightarrow f(a, g(c, c, b), g(a, b, b)) \\ &\rightarrow f(a, b, g(a, b, b)) \end{aligned}$$

Theorem

- termination is modular for disjoint *left-linear* confluent TRSs
- termination is modular for constructor-sharing confluent *CSs*

Definition

TRS \mathcal{R} over signature \mathcal{F} is **constructor system (CS)** if $l_1, \dots, l_n \in \mathcal{T}(\mathcal{F}_c, \mathcal{V})$ for every left-hand side $f(l_1, \dots, l_n)$ of rewrite rule in \mathcal{R}

Theorem

- weak normalization is modular for constructor-sharing TRSs
- local confluence is modular for constructor-sharing TRSs
- semi-completeness is modular for constructor-sharing TRSs

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- **Further Reading**

Further Reading



Developing Developments

Vincent van Oostrom

TCS 175(1), pp. 159 – 181, 1997



Modular Termination of r -Consistent and Left-Linear Term Rewriting Systems

Manfred Schmidt-Schauß, Massimo Marchiori, and Sven Eric Panitz

TCS 149(2), pp. 361 – 374, 1995



Modularity of Confluence – Constructed

Vincent van Oostrom

Proc. 4th IJCAR, LNAI 5195, pp. 348 – 363, 2008

Confluence Tool

- ACP