ISR 2010

Introduction to Term Rewriting

lecture 8

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, strategies

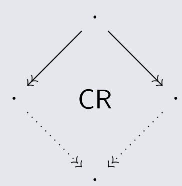
Thursday

exam, advanced topics

Outline

Confluence

every two coinitial rewrite sequences can be joined



- . . . yields uniqueness of normal forms
- . . . is decidable for terminating TRSs
- . . . what about nonterminating TRSs ?

Examples (Non-Confluence)

- no confluence because of critical pairs

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{b} \leftarrow \mathsf{a} \to \mathsf{c}$$
$$\mathsf{a} \to \mathsf{c}$$

- no critical pairs but no confluence (Klop 1978)

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{a} \;{}^*\!\!\leftarrow \mathsf{c} \to \mathsf{g}(\mathsf{c}) \to^* \mathsf{g}(\mathsf{a})$$
$$\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$$
$$\mathsf{c} \to \mathsf{g}(\mathsf{c})$$

- no critical pairs but no confluence (Huet 1980)

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{b}$$
$$\mathsf{f}(x, \mathsf{g}(x)) \to \mathsf{b}$$
$$\mathsf{c} \to \mathsf{g}(\mathsf{c})$$

Exercises (Non-Confluence)

1. $\mathsf{f}(\mathsf{f}(x)) \to \mathsf{a}$
2. $\mathsf{f}(\mathsf{g}(x), y) \to x \quad \mathsf{g}(\mathsf{a}) \to \mathsf{b}$
3. $\mathsf{or}(x, y) \to x \quad \mathsf{or}(x, y) \to y$

Confluence via Critical Pairs

control interference of rewrite rules

- Critical Pair Lemma (lecture 5):
  $\mathsf{WCR} \iff \leftarrow\!\rtimes\!\to \;\subseteq\; \downarrow$ (all critical pairs are convergent)
- combine with Newman's Lemma (lecture 2):
  $\mathsf{SN} \;\&\; \leftarrow\!\rtimes\!\to \;\subseteq\; \downarrow \implies \mathsf{CR}$
- observe from preceding examples:
  $\leftarrow\!\rtimes\!\to \;=\; \varnothing \;\not\Longrightarrow\; \mathsf{CR}$

Confluence via Orthogonality

forbid interference of rewrite rules

- no critical pairs
- no equality checks

Definitions

- term t is linear is each variable in $\mathcal{V}ar(t)$ occurs exactly once in t
- rewrite rule $\ell \to r$ is left-linear if ℓ is linear
- TRS is left-linear if all rewrite rules are left-linear
- rewrite rule $\ell \to r$ is right-linear if r is linear
- rewrite rule $\ell \to r$ is linear if ℓ and r are linear
- TRS is (right-)linear if all rewrite rules are (right-)linear

Examples

- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
  left-linear but not right-linear
- $\mathsf{f}(x, x) \to \mathsf{a}$
  right-linear but not left-linear

Definition

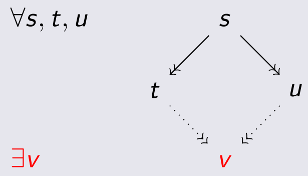
orthogonal TRS is left-linear and lacks criticial pairs

Examples

$$\mathsf{I} \cdot x \to x \qquad \mathsf{ack}(0, y) \to \mathsf{s}(y)$$
$$(\mathsf{K} \cdot x) \cdot y \to x \qquad \mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z) \qquad \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

Theorem

orthogonal TRSs are confluent



Observation

for orthogonal TRSs there is canonical way to compute common reduct v

Lemma

orthogonal TRSs are locally confluent

Proof Idea

distinguish two cases:

1. two disjoint redexes
2. two nested redexes

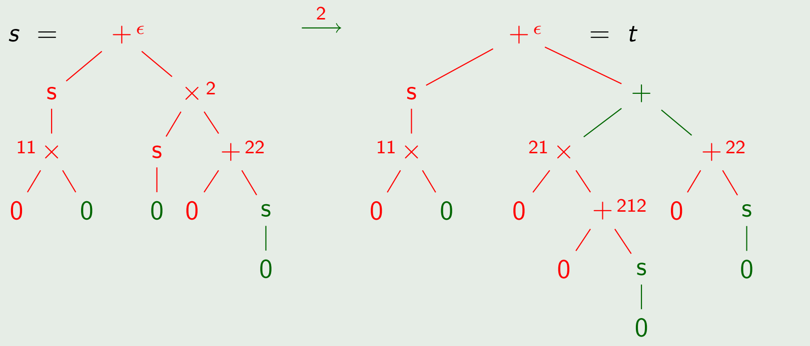
Examples

1. $\mathsf{a} \to \mathsf{c}$, $\mathsf{b} \to \mathsf{d}$ $\qquad \mathsf{f(c,b)} \leftarrow \mathsf{f(a,b)} \to \mathsf{f(a,d)}$
2. $\mathsf{f}(x) \to \mathsf{g}(x, x)$, $\mathsf{a} \to \mathsf{b}$ $\qquad \mathsf{f(b)} \leftarrow \mathsf{f(a)} \to \mathsf{g(a,a)}$

Outline

Example

$0 + y \to y \quad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \quad 0 \times y \to 0 \quad \mathsf{s}(x) \times y \to (x \times y) + y$



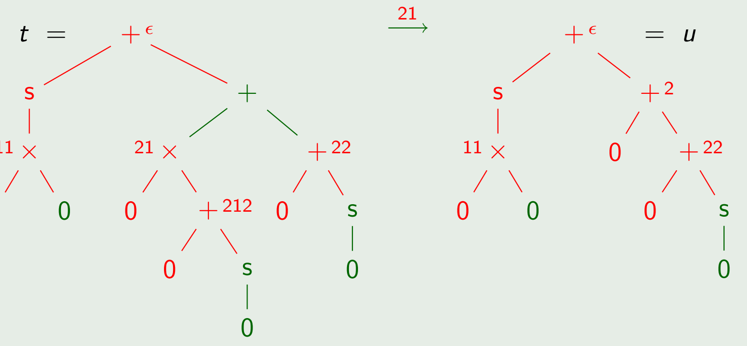
redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21

redex at position 22 is duplicated

redex at position 21 is created

Example

$0 + y \to y \quad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \quad 0 \times y \to 0 \quad \mathsf{s}(x) \times y \to (x \times y) + y$



redex positions in s:	ϵ	11	2	22			
redex positions in t:	ϵ	11		212	22	21	
redex positions in u:	ϵ	11			22		2

redex at position 212 is erased

redex at position 2 is created

rewrite step $A\colon s \xrightarrow[\ell \to r]{p} t$ position $q \in \mathcal{P}\mathrm{os}(s)$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Step)

- descendants of q in t

$$q \backslash A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{p p_3 p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = p p_1 p_2 \text{ with } p_1 \in \mathcal{P}\mathrm{os}_{\mathcal{V}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

- descendants of Q in t

$$Q \backslash A = \bigcup_{q \in Q} q \backslash A$$

Remark

- information about position is needed to determine descendants

$$\mathsf{f}(x) \to x \qquad \mathsf{f}(\mathsf{f}(x)) \xrightarrow[\mathsf{f}(x) \to x]{?} \mathsf{f}(x)$$

- information about rewrite rule is needed to determine descendants

$$\begin{array}{l} \mathsf{f}(x) \to \mathsf{f}(x) \\ \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \end{array} \qquad \mathsf{f}(\mathsf{a}) \xrightarrow[?]{\epsilon} \mathsf{f}(\mathsf{a})$$

rewrite sequence $A\colon s \to^* t$ set of positions $Q \subseteq \mathcal{P}os(s)$

Definition (Descendants after Rewrite Sequence)

descendants of Q in t

$$Q\backslash A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q\backslash A_1)\backslash A_2 & \text{If } A = A_1; A_2 \text{ with } A_1\colon s \to u \text{ and } A_2\colon u \to^* t \end{cases}$$

Lemma

- *for arbitrary TRSs:* *if Q is parallel ($\forall\, p \neq q \in Q\colon p \parallel q$) then so is $Q\backslash A$*
- *for orthogonal TRSs:* *if Q is set of redex positions then so is $Q\backslash A$*

Terminology

descendant of redex is called residual

Remark

- in non-left-linear TRS descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(x,x) &\to \mathsf{b} \end{aligned} \qquad \mathsf{f(a,a)} \to \mathsf{f(b,a)}$$

- in TRS with critical pairs descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f(a)} &\to \mathsf{b} \end{aligned} \qquad \mathsf{f(a)} \to \mathsf{f(b)}$$

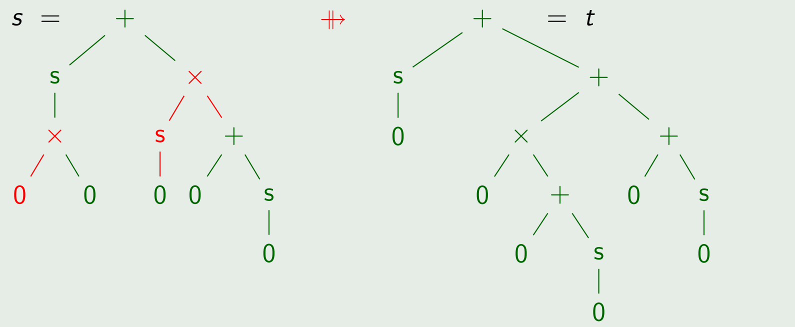
Outline

Definition (Parallel Rewriting)

1. $= \subseteq \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel$
2. $\xrightarrow{\epsilon} \subseteq \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel$
3. $f(s_1, \ldots, s_n) \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel\; f(t_1, \ldots, t_n)$ if $s_i \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel\; t_i$ for each $1 \leqslant i \leqslant n$

Example

$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$



Lemma

$s \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel\; t \iff s \to^* t$ by contracting redexes at pairwise parallel positions in s

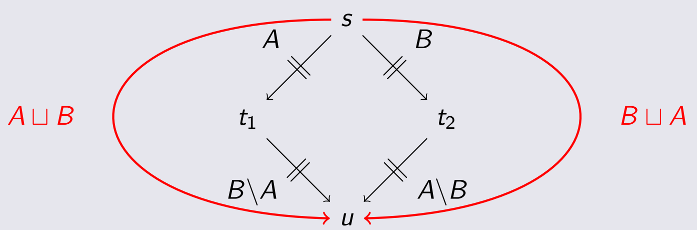
Definition (Projection)

A: $s \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel\; t_1$ by contracting redexes at positions in P

B: $s \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel\; t_2$ by contracting redexes at positions in Q

- $B \backslash A$: $t_1 \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel\; u_1$ by contracting redexes at positions in $Q \backslash A$
- $A \backslash B$: $t_2 \;\twoheadrightarrow\!\!\!\!\!\!\!\parallel\; u_2$ by contracting redexes at positions in $P \backslash B$
- $A \sqcup B = A; B \backslash A$ and $B \sqcup A = B; A \backslash B$

Parallel Moves Lemma

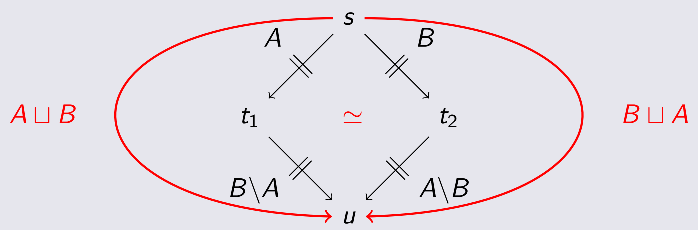


Definition

A: $s_1 \to^* t_1$ and B: $s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

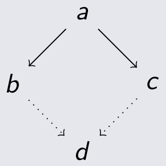
1. $s_1 = s_2$
2. $t_1 = t_2$
3. $p \backslash A = p \backslash B$ for all redex positions p in s_1

Parallel Moves Lemma (with Permutation Equivalence)



Definition

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \; \subseteq \; \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$ $\exists d$



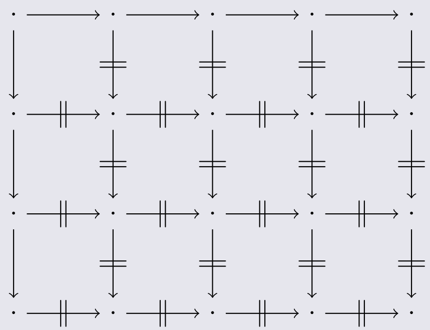
Lemma

ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow \; \subseteq \; \rightarrow_{\diamond} \; \subseteq \; \rightarrow^*$ for some relation $\rightarrow_{\diamond}$ on A with diamond property

Corollary

orthogonal TRSs are confluent

Proof



Outline

- Orthogonality
- **Beyond Orthogonality**
- Modularity
- Further Reading

Definitions

- critical pair $s \leftarrow\!\rtimes\!\rightarrow t$ is trivial if $s = t$
- weakly orthogonal TRS is left-linear and has only trivial critical pairs
- overlay $s \leftarrow\!\bowtie\!\rightarrow t$ is critical pair originating from overlap $\langle l_1 \rightarrow r_1, \epsilon, l_2 \rightarrow r_2 \rangle$
- weakly orthogonal TRS is almost orthogonal if all critical pairs are overlays

Examples

$$x \vee \mathsf{T} \rightarrow \mathsf{T}$$
$$\mathsf{T} \vee x \rightarrow \mathsf{T}$$
$$\mathsf{F} \vee \mathsf{F} \rightarrow \mathsf{F}$$

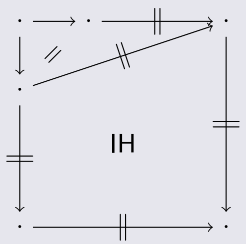
$$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$$
$$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$$

Theorem

weakly orthogonal TRSs are confluent

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\!\Vert} \cdot \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \subseteq \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\!\Vert}$



interesting case

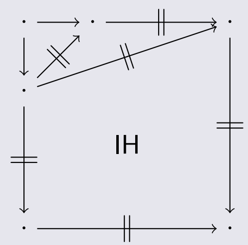
- $\rightarrow \subseteq \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \subseteq \rightarrow^*$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \subseteq \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \implies$ *CR*

Proof Sketch

- $\mathrel{\leftarrow\!\!\!\!\!\!\Vert} \cdot \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \subseteq \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\!\Vert}$



interesting case

- $\rightarrow \subseteq \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \subseteq \rightarrow^*$

Open Problem

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \subseteq \mathrel{\leftarrow\!\!\!\!\!\!\Vert} \implies$ *CR ?*

Example

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x),\mathsf{b}) &\rightarrow \mathsf{f}(\mathsf{g}(x),\mathsf{c}) \\ \mathsf{g}(\mathsf{c}) &\rightarrow \mathsf{g}(\mathsf{a}) \\ \mathsf{a} &\rightarrow \mathsf{c} \\ \mathsf{b} &\rightarrow \mathsf{d} \end{aligned}$$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \subseteq (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \implies$ *CR*

Notation

$\leftarrow\!\rtimes\!\rightarrow \; = \; \leftarrow\!\rtimes\!\rightarrow \setminus \leftarrow\!\bowtie\!\rightarrow$

Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \subseteq \mathrel{\Vert\!\!\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \subseteq \mathrel{\Vert\!\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \implies$ *CR*

Theorem (van Oostrom 1996)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \subseteq \mathrel{\circ\!\!\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \subseteq \mathrel{\circ\!\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \implies$ *CR*

Outline

- Orthogonality
- Beyond Orthogonality
- **Modularity**
  - Definitions
  - Results
- Further Reading

Definition

property of TRSs is **modular** if it is preserved under union

Remark

without further restrictions 'no' property of TRSs is modular

termination	$a \to b$	$b \to a$
confluence	$a \to b$	$a \to c$

Definition

property P is **preserved under signature extension** if

$$(\mathcal{F}, \mathcal{R}) \vDash P \quad \Longrightarrow \quad (\mathcal{F} \cup \mathcal{G}, \mathcal{R}) \vDash P$$

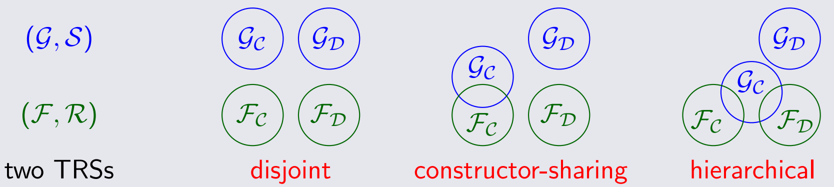
for all TRSs $(\mathcal{F}, \mathcal{R})$ and signatures $\mathcal{G}$ with $\mathcal{F} \cap \mathcal{G} \neq \varnothing$

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- defined symbols $\mathcal{F}_\mathcal{D} = \{ \mathsf{root}(\ell) \mid \ell \to r \in \mathcal{R} \}$
- constructors $\mathcal{F}_\mathcal{C} = \mathcal{F} \setminus \mathcal{F}_\mathcal{D}$

More Interesting Combinations



Example

①	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$0 \times y \to 0$ $\mathsf{s}(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	$\mathsf{fib}(0) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(0)) \to \mathsf{s}(0)$ $\mathsf{fib}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{fib}(\mathsf{s}(x)) + \mathsf{fib}(x)$	④
⑤	$\mathsf{nil} +\!\!+\, x \to x$ $(x : y) +\!\!+\, z \to x : (y +\!\!+\, z)$	$0 \div \mathsf{s}(y) \to 0$ $\mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$	⑥
⑦	$\mathsf{true} \wedge \mathsf{false} \to \mathsf{false}$ $\mathsf{false} \wedge \mathsf{true} \to \mathsf{false}$ $x \wedge x \to x$	$x < 0 \to \mathsf{false}$ $0 < \mathsf{s}(y) \to \mathsf{true}$ $\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	⑧
⑨	$\mathsf{sum}(\mathsf{nil}) \to 0$ $\mathsf{sum}(x : y) \to x + \mathsf{sum}(y)$	$\mathsf{length}(\mathsf{nil}) \to 0$ $\mathsf{length}(x : y) \to \mathsf{s}(\mathsf{length}(y))$	⑩

① $+_{\mathsf{h}}$ ② $+_{\mathsf{cs}}$ ③ $+_{\mathsf{h}}$ ④ $+_{\mathsf{d}}$ ⑤ $+_{\mathsf{h}}$ ⑥ $+_{\mathsf{d}}$ ⑦ $+_{\mathsf{cs}}$ ⑧ $+_{\mathsf{h}}$ ⑨ $+_{\mathsf{cs}}$ ⑩

Outline

Theorem (Toyama's Theorem)

confluence is modular for disjoint TRSs

Remark

confluence is not modular for constructor-sharing TRSs

Example

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) &\to \mathsf{b} \end{aligned} \qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

$$\mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{b}$$

Theorem

termination is not modular for disjoint TRSs

Example

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \begin{aligned} \mathsf{g}(x,y) &\to x \\ \mathsf{g}(x,y) &\to y \end{aligned}$$

duplicating — not confluent

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \end{aligned}$$

Theorem

disjoint union of terminating TRSs $\mathcal{R}$ and $\mathcal{S}$ is terminating if

- *$\mathcal{R}$ and $\mathcal{S}$ lack collapsing rules*
- *$\mathcal{R}$ and $\mathcal{S}$ lack duplicating rules*
- *$\mathcal{R}$ or $\mathcal{S}$ lacks both collapsing and duplicating rules*

Corollary

termination is preserved under signature extension

Theorem

termination is not modular for disjoint confluent TRSs

Example

$$\begin{array}{lll} f(a,b,x) \to f(x,x,x) & a \to c & g(x,y,y) \to x \\ f(x,y,z) \to c & b \to c & g(y,y,x) \to x \end{array}$$

no constructor system — not left-linear

$$\begin{aligned} f(a,b,g(a,b,b)) &\to f(g(a,b,b),g(a,b,b),g(a,b,b)) \\ &\to f(a,g(a,b,b),g(a,b,b)) \\ &\to f(a,g(c,b,b),g(a,b,b)) \\ &\to f(a,g(c,c,b),g(a,b,b)) \\ &\to f(a,b,g(a,b,b)) \end{aligned}$$

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS $\mathcal{R}$ over signature $\mathcal{F}$ is constructor system (CS) if $l_1, \dots, l_n \in \mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$ for every left-hand side $f(l_1, \dots, l_n)$ of rewrite rule in $\mathcal{R}$

Theorem

- *weak normalization is modular for constructor-sharing TRSs*
- *local confluence is modular for constructor-sharing TRSs*
- *semi-completeness is modular for constructor-sharing TRSs*

Outline

Developing Developments
Vincent van Oostrom
TCS 175(1), pp. 159 – 181, 1997

Modular Termination of r-Consistent and Left-Linear Term Rewriting Systems
Manfred Schmidt-Schauß, Massimo Marchiori, and Sven Eric Panitz
TCS 149(2), pp. 361 – 374, 1995

Modularity of Confluence – Constructed
Vincent van Oostrom
Proc. 4th IJCAR, LNAI 5195, pp. 348 – 363, 2008

Confluence Tool

- ACP