



Introduction to Term Rewriting

lecture 9

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Sunday

introduction, examples, abstract rewriting, equational reasoning, term rewriting

Monday

termination, completion

Tuesday

completion, termination

Wednesday

confluence, modularity, **strategies**

Thursday

exam, advanced topics

Outline

- Strategies
 - Definitions
 - Results
 - Optimality
- Strategy Annotations
- Further Reading



Definition

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- rewrite strategy \mathcal{S} is perpetual if every maximal \mathcal{S} rewrite sequence starting from any non-terminating term is infinite

Lemma

for terminating TRSs every strategy is *normalizing* and *perpetual*

Example

- rewrite rules

$$\begin{array}{llll}
 0 + 0 \rightarrow 0 & 1 + 0 \rightarrow 1 & \dots & 9 + 0 \rightarrow 9 \\
 0 + 1 \rightarrow 1 & 1 + 1 \rightarrow 2 & \dots & 9 + 1 \rightarrow 1 : 0 \\
 0 + 2 \rightarrow 2 & 1 + 2 \rightarrow 3 & \dots & 9 + 2 \rightarrow 1 : 1 \\
 0 + 3 \rightarrow 3 & 1 + 3 \rightarrow 4 & \dots & 9 + 3 \rightarrow 1 : 2 \\
 0 + 4 \rightarrow 4 & 1 + 4 \rightarrow 5 & \dots & 9 + 4 \rightarrow 1 : 3 \\
 0 + 5 \rightarrow 5 & 1 + 5 \rightarrow 6 & \dots & 9 + 5 \rightarrow 1 : 4 \\
 0 + 6 \rightarrow 6 & 1 + 6 \rightarrow 7 & \dots & 9 + 6 \rightarrow 1 : 5 \\
 0 + 7 \rightarrow 7 & 1 + 7 \rightarrow 8 & \dots & 9 + 7 \rightarrow 1 : 6 \\
 0 + 8 \rightarrow 8 & 1 + 8 \rightarrow 9 & \dots & 9 + 8 \rightarrow 1 : 7 \\
 0 + 9 \rightarrow 9 & 1 + 9 \rightarrow 1 : 0 & \dots & 9 + 9 \rightarrow 1 : 8 \\
 x + (y : z) \rightarrow y : (x + z) & & & 0 : x \rightarrow x \\
 (x : y) + z \rightarrow x : (y + z) & & & x : (y : z) \rightarrow (x + y) : z
 \end{array}$$

- term

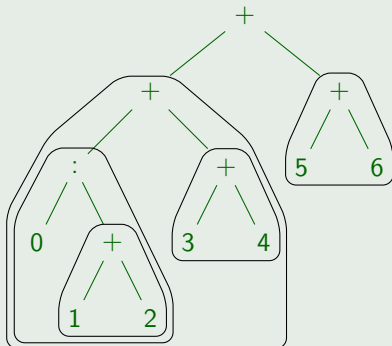
$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$

Example (cont'd)

term

$$0 : 1 + 2 + 3 + 4 + 5 + 6$$

tree representation

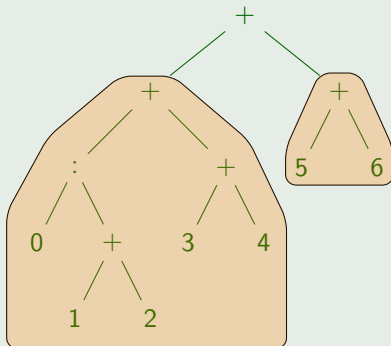


Example (cont'd)

term

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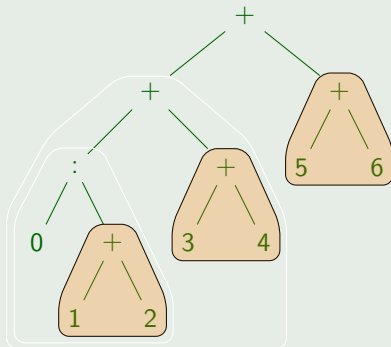
outermost redexes

Example (cont'd)

term

$$0 : 1 + 2 + 3 + 4 + 5 + 6$$

tree representation



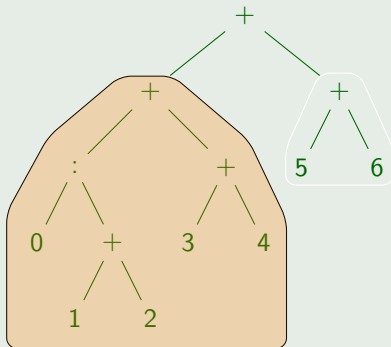
innermost redexes

Example (cont'd)

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$$0 : 1 + 2 + 3 + 4 + 5 + 6$$

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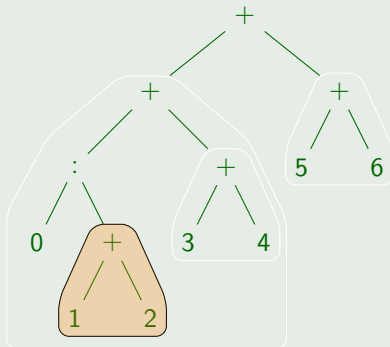
leftmost outermost strategy

Example (cont'd)

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$$0 : 1 + 2 + 3 + 4 + 5 + 6$$

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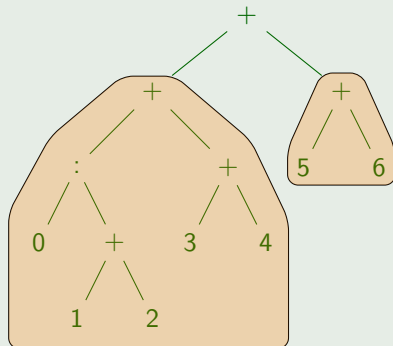
leftmost innermost strategy

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term

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tree representation



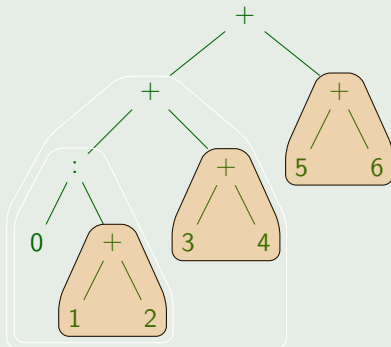
parallel outermost strategy

Example (cont'd)

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$$0 : 1 + 2 + 3 + 4 + 5 + 6$$

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parallel innermost strategy

Example (cont'd)

- **leftmost outermost** strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$

- **leftmost innermost** strategy

$$(0 : (1 + 2)) + (3 + 4) + (5 + 6)$$

Example (cont'd)

- **leftmost outermost** strategy

$$\begin{aligned} & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\ & \rightarrow (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \end{aligned}$$

- **leftmost innermost** strategy

$$\begin{aligned} & (0 : (1 + 2)) + (3 + 4) + (5 + 6) \\ & \rightarrow ((0 : 3) + (3 + 4)) + (5 + 6) \end{aligned}$$

Example (cont'd)

- **leftmost outermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \rightarrow (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \\
 & \rightarrow 0 : (((1 + 2) + (3 + 4)) + (5 + 6))
 \end{aligned}$$

- **leftmost innermost** strategy

$$\begin{aligned}
 & (0 : (1 + 2)) + (3 + 4) + (5 + 6) \\
 & \rightarrow ((0 : 3) + (3 + 4)) + (5 + 6) \rightarrow (3 + (3 + 4)) + (5 + 6)
 \end{aligned}$$

Example (cont'd)

- **leftmost outermost** strategy

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 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
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 \end{aligned}$$

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 & (0 : (1 + 2)) + (3 + 4) + (5 + 6) \\
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 & \rightarrow (3 + 7) + (5 + 6)
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Example (cont'd)

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Example (cont'd)

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Example (cont'd)

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 \end{aligned}$$

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 & \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \rightarrow (1 + 1) : (0 + 1)
 \end{aligned}$$

- **leftmost innermost** strategy 10 redexes

$$\begin{aligned}
 & (0 : (1 + 2)) + (3 + 4) + (5 + 6) \\
 & \rightarrow ((0 : 3) + (3 + 4)) + (5 + 6) \rightarrow (3 + (3 + 4)) + (5 + 6) \\
 & \rightarrow (3 + 7) + (5 + 6) \rightarrow (1 : 0) + (5 + 6) \rightarrow (1 : 0) + (1 : 1) \\
 & \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \rightarrow 1 : (1 : 1) \\
 & \rightarrow (1 + 1) : 1 \rightarrow 2 : 1
 \end{aligned}$$

Example (cont'd)

- **leftmost outermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \rightarrow (0 : ((1 + 2) + (3 + 4))) + (5 + 6) \\
 & \rightarrow 0 : (((1 + 2) + (3 + 4)) + (5 + 6)) \\
 & \rightarrow ((1 + 2) + (3 + 4)) + (5 + 6) \rightarrow (3 + (3 + 4)) + (5 + 6) \\
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 & \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \rightarrow (1 + 1) : (0 + 1) \\
 & \rightarrow 2 : (0 + 1)
 \end{aligned}$$

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 & \rightarrow (1 + 1) : 1 \rightarrow 2 : 1
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Example (cont'd)

- **leftmost outermost** strategy 12 redexes

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 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
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 & \rightarrow ((1 + 2) + (3 + 4)) + (5 + 6) \rightarrow (3 + (3 + 4)) + (5 + 6) \\
 & \rightarrow (3 + 7) + (5 + 6) \rightarrow (1 : 0) + (5 + 6) \rightarrow 1 : (0 + (5 + 6)) \\
 & \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \rightarrow (1 + 1) : (0 + 1) \\
 & \rightarrow 2 : (0 + 1) \rightarrow 2 : 1
 \end{aligned}$$

- **leftmost innermost** strategy 10 redexes

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 & (0 : (1 + 2)) + (3 + 4) + (5 + 6) \\
 & \rightarrow ((0 : 3) + (3 + 4)) + (5 + 6) \rightarrow (3 + (3 + 4)) + (5 + 6) \\
 & \rightarrow (3 + 7) + (5 + 6) \rightarrow (1 : 0) + (5 + 6) \rightarrow (1 : 0) + (1 : 1) \\
 & \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \rightarrow 1 : (1 : 1) \\
 & \rightarrow (1 + 1) : 1 \rightarrow 2 : 1
 \end{aligned}$$

Example (cont'd)

- **parallel outermost** strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$

- **parallel innermost** strategy

$$((0 : (1 + 2)) + (3 + 4)) + (5 + 6)$$

Example (cont'd)

- **parallel outermost** strategy

$$\begin{aligned} & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\ & \quad \mapsto (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \end{aligned}$$

- **parallel innermost** strategy

$$\begin{aligned} & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\ & \quad \mapsto ((0 : 3) + 7) + (1 : 1) \end{aligned}$$

Example (cont'd)

- **parallel outermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \dashv\vdash (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
 & \quad \rightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1))
 \end{aligned}$$

- **parallel innermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
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 \end{aligned}$$

Example (cont'd)

- **parallel outermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \Downarrow (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
 & \rightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
 & \rightarrow ((1 + 2) + (3 + 4)) + (1 : 1)
 \end{aligned}$$

- **parallel innermost** strategy

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 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \Downarrow ((0 : 3) + 7) + (1 : 1) \rightarrow (3 + 7) + (1 : 1) \\
 & \rightarrow (1 : 0) + (1 : 1)
 \end{aligned}$$

Example (cont'd)

- **parallel outermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \Rightarrow (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
 & \quad \rightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
 & \quad \rightarrow ((1 + 2) + (3 + 4)) + (1 : 1) \\
 & \quad \rightarrow 1 : (((1 + 2) + (3 + 4)) + 1)
 \end{aligned}$$

- **parallel innermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \Rightarrow ((0 : 3) + 7) + (1 : 1) \rightarrow (3 + 7) + (1 : 1) \\
 & \quad \rightarrow (1 : 0) + (1 : 1) \rightarrow 1 : (0 + (1 : 1))
 \end{aligned}$$

Example (cont'd)

- **parallel outermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \Downarrow (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
 & \rightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
 & \rightarrow ((1 + 2) + (3 + 4)) + (1 : 1) \\
 & \rightarrow 1 : (((1 + 2) + (3 + 4)) + 1) \\
 & \quad \Downarrow 1 : ((3 + 7) + 1)
 \end{aligned}$$

- **parallel innermost** strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \Downarrow ((0 : 3) + 7) + (1 : 1) \rightarrow (3 + 7) + (1 : 1) \\
 & \rightarrow (1 : 0) + (1 : 1) \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1))
 \end{aligned}$$

Example (cont'd)

- parallel outermost strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
 & \quad \rightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
 & \quad \rightarrow ((1 + 2) + (3 + 4)) + (1 : 1) \\
 & \quad \rightarrow 1 : (((1 + 2) + (3 + 4)) + 1) \\
 & \quad \mapsto 1 : ((3 + 7) + 1) \rightarrow 1 : ((1 : 0) + 1)
 \end{aligned}$$

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$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto ((0 : 3) + 7) + (1 : 1) \rightarrow (3 + 7) + (1 : 1) \\
 & \quad \rightarrow (1 : 0) + (1 : 1) \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \\
 & \quad \rightarrow 1 : (1 : 1)
 \end{aligned}$$

Example (cont'd)

- parallel outermost strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
 & \quad \rightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
 & \quad \rightarrow ((1 + 2) + (3 + 4)) + (1 : 1) \\
 & \quad \rightarrow 1 : (((1 + 2) + (3 + 4)) + 1) \\
 & \quad \mapsto 1 : ((3 + 7) + 1) \rightarrow 1 : ((1 : 0) + 1) \rightarrow 1 : (1 : (0 + 1))
 \end{aligned}$$

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$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto ((0 : 3) + 7) + (1 : 1) \rightarrow (3 + 7) + (1 : 1) \\
 & \quad \rightarrow (1 : 0) + (1 : 1) \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \\
 & \quad \rightarrow 1 : (1 : 1) \rightarrow (1 + 1) : 1
 \end{aligned}$$

Example (cont'd)

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$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
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 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
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 & \quad \rightarrow 1 : (1 : 1) \rightarrow (1 + 1) : 1 \rightarrow 2 : 1
 \end{aligned}$$

Example (cont'd)

- parallel outermost strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
 & \quad \rightarrow 0 : (((1 + 2) + (3 + 4)) + (1 : 1)) \\
 & \quad \rightarrow ((1 + 2) + (3 + 4)) + (1 : 1) \\
 & \quad \rightarrow 1 : (((1 + 2) + (3 + 4)) + 1) \\
 & \quad \mapsto 1 : ((3 + 7) + 1) \rightarrow 1 : ((1 : 0) + 1) \rightarrow 1 : (1 : (0 + 1)) \\
 & \quad \rightarrow (1 + 1) : (0 + 1) \mapsto 2 : 1
 \end{aligned}$$

- parallel innermost strategy

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto ((0 : 3) + 7) + (1 : 1) \rightarrow (3 + 7) + (1 : 1) \\
 & \quad \rightarrow (1 : 0) + (1 : 1) \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \\
 & \quad \rightarrow 1 : (1 : 1) \rightarrow (1 + 1) : 1 \rightarrow 2 : 1
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Example (cont'd)

- **parallel outermost** strategy 12 redexes in 9 steps

$$\begin{aligned}
 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto (0 : ((1 + 2) + (3 + 4))) + (1 : 1) \\
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- **parallel innermost** strategy 10 redexes in 8 steps

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 & ((0 : (1 + 2)) + (3 + 4)) + (5 + 6) \\
 & \quad \mapsto ((0 : 3) + 7) + (1 : 1) \rightarrow (3 + 7) + (1 : 1) \\
 & \quad \rightarrow (1 : 0) + (1 : 1) \rightarrow 1 : (0 + (1 : 1)) \rightarrow 1 : (1 : (0 + 1)) \\
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Definition

development of set of redex positions Q in term t is rewrite sequence starting from t in which all contracted redex positions descend from position in Q

Example

- rewrite rules

$$\begin{array}{ll} 0 + y \rightarrow y & 0 \times y \rightarrow 0 \\ s(x) + y \rightarrow s(x + y) & s(x) \times y \rightarrow (x \times y) + y \end{array}$$

- rewrite sequences

$$\underline{s(0) \times (0 \times 0)} \rightarrow (0 \times \underline{(0 \times 0)}) + (0 \times 0) \rightarrow (0 \times 0) + (0 \times 0) \quad \text{☺}$$

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Theorem

developments are finite



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Definition

development $A: s \rightarrow^* t$ of set of redex positions $Q \subseteq \mathcal{P}\text{os}(s)$ is **complete** if $Q \setminus A = \emptyset$



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Lemma

for orthogonal TRSs: $s \rightarrow^* t$ is complete development $\iff s \twoheadrightarrow t$



Theorem

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Lemma

for orthogonal TRSs: $s \rightarrow^* t$ is complete development $\iff s \twoheadrightarrow t$

Theorem

*all complete developments of Q are **permutation equivalent***

Definition

for orthogonal TRSs **full substitution** strategy performs **complete development** of all redexes



Definition

for orthogonal TRSs full substitution strategy performs complete development of all redexes

Example

- rewrite rules

$$0 + y \rightarrow y$$

$$s(x) + y \rightarrow s(x + y)$$

$$0 \times y \rightarrow 0$$

$$s(x) \times y \rightarrow (x \times y) + y$$

- full substitution strategy

$$s(s(0)) \times (s(0) + s(s(0)))$$

Definition

for orthogonal TRSs full substitution strategy performs complete development of all redexes

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$$\begin{array}{l}
 s(s(0)) \times (s(0) + s(s(0))) \\
 \rightarrow (s(0) \times s(0 + s(s(0)))) + s(0 + s(s(0)))
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 & s(s(0)) \times (s(0) + s(s(0))) \\
 & \quad \Leftrightarrow (s(0) \times s(0 + s(s(0)))) + s(0 + s(s(0))) \\
 & \quad \Leftrightarrow ((0 \times s(s(s(0)))) + s(s(s(0)))) + s(s(s(0)))
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 & \quad \Leftrightarrow ((0 \times s(s(s(0)))) + s(s(s(0)))) + s(s(s(0))) \\
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 & \quad \rightarrow s(s(s(0))) + s(s(s(0))) \rightarrow^+ s(s(s(s(s(0))))))
 \end{aligned}$$

Outline

- **Strategies**
 - Definitions
 - **Results**
 - Optimality
- Strategy Annotations
- Further Reading



Theorem

for orthogonal TRSs

- *full substitution and parallel outermost strategies are **normalizing***



Theorem

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- *full substitution* and *parallel outermost* strategies are normalizing
- *innermost* strategies are *perpetual*



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Example

$a \rightarrow b$

$c \rightarrow c$

$f(x, b) \rightarrow b$

- | | |
|----------------------|-----------|
| • leftmost outermost | $f(c, a)$ |
| • leftmost innermost | $f(c, a)$ |
| • parallel outermost | $f(c, a)$ |
| • parallel innermost | $f(c, a)$ |
| • full substitution | $f(c, a)$ |

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| • parallel outermost | $f(c, a) \Downarrow f(c, b) \rightarrow b$ |
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Definition

TRS is **left-normal** if variables do not precede function symbols in left-hand sides



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Example

- $x + (y : z) \rightarrow y : (x + z)$



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Example

- $x + (y : z) \rightarrow y : (x + z)$ ☹️



Definition

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- $x + (y : z) \rightarrow y : (x + z)$ ☹️
- $(x : y) + z \rightarrow x : (y + z)$



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Theorem

leftmost outermost strategy is normalizing for orthogonal left-normal TRSs



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- $(x : y) + z \rightarrow x : (y + z)$ 😊

Theorem

leftmost outermost strategy is normalizing for orthogonal left-normal TRSs

Remark

easy but **important** result: **Combinatory Logic** is left-normal

$$I x \rightarrow x$$

$$K x y \rightarrow x$$

$$S x y z \rightarrow x z (y z)$$





Innsbruck

PhD Positions in Innsbruck

- 1** 3-year FWF project position (advisor: Dr. René Thiemann)
 - certification of termination methods in Isabelle/HOL
- 2** 4-year university position from October 1, 2010
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Further Information

- Aart Middeldorp

Outline

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 - **Optimality**
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Observation

parallel outermost and full-substitution are not **optimal** because they perform **useless** steps



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Example

- rewrite rules

$$0 + y \rightarrow y$$

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$$0 \times y \rightarrow 0$$

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- rewrite sequence

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parallel outermost and full-substitution are not **optimal** because they perform **useless** steps

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- redex $0 + s(0)$ is not **needed**

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Definition

redex Δ in term t is **needed** if descendant of Δ is contracted in every rewrite sequence from t to normal form

Theorem

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- *every reducible term has needed redex*
- *needed reduction is normalizing*



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for orthogonal TRSs it is **undecidable** whether redex is needed



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decidable approximations based on powerful tree automata techniques exist

Lemma

for left-normal orthogonal TRSs leftmost outermost redex is needed

Outline

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Example

rewrite rules

$$\begin{array}{lll} \alpha: \text{and}(x, T) \rightarrow x & \gamma: \text{or}(T, x) \rightarrow T & \epsilon: \infty \rightarrow \infty \\ \beta: \text{and}(x, F) \rightarrow F & \delta: \text{or}(F, x) \rightarrow x & \end{array}$$

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strategy annotation A

$$A(\text{and}) = [2, \alpha, \beta, 1]$$

$$A(\text{or}) = [1, \gamma, \delta, 2]$$

$$A(\infty) = [\epsilon]$$

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rewrite rules

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$$\text{or}(\text{and}(\infty, F), \text{or}(T, \infty))$$

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evaluation

$$\text{or}(\text{and}(\infty, F), \text{or}(T, \infty)) \rightarrow \text{or}(F, \text{or}(T, \infty))$$

Example

rewrite rules

$$\begin{array}{lll} \alpha: \text{and}(x, T) \rightarrow x & \gamma: \text{or}(T, x) \rightarrow T & \epsilon: \infty \rightarrow \infty \\ \beta: \text{and}(x, F) \rightarrow F & \delta: \text{or}(F, x) \rightarrow x & \end{array}$$

strategy annotation A

$$A(\text{and}) = [2, \alpha, \beta, 1] \quad A(\text{or}) = [1, \gamma, \delta, 2] \quad A(\infty) = [\epsilon]$$

evaluation

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Definitions

- **strategy annotation** for function symbol f is finite list $A(f)$ containing
 - argument positions of f
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Example

rewrite rules

$$\alpha: \text{if}(T, x, y) \rightarrow x \quad \beta: \text{if}(F, x, y) \rightarrow y \quad \gamma: \text{if}(z, x, x) \rightarrow x$$

strategy annotations

$$A_1(\text{if}) = [1, \alpha, \beta, 2, 3, \gamma]$$

$$A_2(\text{if}) = [\alpha, \beta, 1, 2, 3, \gamma]$$

$$A_3(\text{if}) = [1, \alpha, \beta, \gamma, 2, 3]$$

$$A_4(\text{if}) = [1, \alpha, \beta]$$

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strategy annotations

full

$$A_1(\text{if}) = [1, \alpha, \beta, 2, 3, \gamma] \quad \text{☺}$$

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Definitions

- strategy annotation for function symbol f is finite list $A(f)$ containing
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- strategy annotation $A(f)$ for function symbol f is **in-time** if argument positions are listed in $A(f)$ before rewrite rules that need them

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- rewrite rule $f(s_1, \dots, s_n) \rightarrow t$ **needs** argument position i if
 - s_i is non-variable
 - s_i is variable that appears in $s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n$

Example

rewrite rules

$$\alpha: \text{if}(T, x, y) \rightarrow x \quad \beta: \text{if}(F, x, y) \rightarrow y \quad \gamma: \text{if}(z, x, x) \rightarrow x$$

strategy annotations

	full	in-time
$A_1(\text{if}) = [1, \alpha, \beta, 2, 3, \gamma]$	☺	
$A_2(\text{if}) = [\alpha, \beta, 1, 2, 3, \gamma]$	☺	
$A_3(\text{if}) = [1, \alpha, \beta, \gamma, 2, 3]$	☺	
$A_4(\text{if}) = [1, \alpha, \beta]$	☹	

Example

rewrite rules

$$\alpha: \text{if}(T, x, y) \rightarrow x \quad \beta: \text{if}(F, x, y) \rightarrow y \quad \gamma: \text{if}(z, x, x) \rightarrow x$$

strategy annotations

	full	in-time
$A_1(\text{if}) = [1, \alpha, \beta, 2, 3, \gamma]$	☺	☺
$A_2(\text{if}) = [\alpha, \beta, 1, 2, 3, \gamma]$	☺	☹
$A_3(\text{if}) = [1, \alpha, \beta, \gamma, 2, 3]$	☺	☹
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Example

rewrite rules

$$\alpha: \text{and}(x, T) \rightarrow x \quad \gamma: \text{or}(T, x) \rightarrow T \quad \epsilon: \infty \rightarrow \infty$$

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strategy annotation A

$$A(\text{and}) = [2, \alpha, \beta, 1] \quad A(\text{or}) = [1, \gamma, \delta, 2] \quad A(\infty) = [\epsilon]$$

evaluation

$$\text{find-redex}_A(\text{or}(\text{and}(\infty, F), \text{or}(T, \infty)))$$

Example

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$$\text{find-redex}_A(F)$$

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$$\begin{aligned} & \text{find-redex}_A(\text{and}(\infty, F)) \\ &= \text{find-redex}_A(\text{and}(\infty, F), [2, \alpha, \beta, 1]) \\ &= \text{find-redex}_A(\text{and}(\infty, F), [\alpha, \beta, 1]) \\ &= \text{find-redex}_A(\text{and}(\infty, F), [\beta, 1]) = (\epsilon, F) \end{aligned}$$

$$\text{find-redex}_A(F) = \perp$$

Example

rewrite rules

$$\begin{array}{lll} \alpha: \text{and}(x, T) \rightarrow x & \gamma: \text{or}(T, x) \rightarrow T & \epsilon: \infty \rightarrow \infty \\ \beta: \text{and}(x, F) \rightarrow F & \delta: \text{or}(F, x) \rightarrow x & \end{array}$$

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$$\text{find-redex}_A(F) = \perp$$

Definition

$$\text{find-redex}_A(t) = \text{find-redex}_A(t, A(\text{root}(t)))$$

$$\text{find-redex}_A(t, []) = \perp$$

$$\text{find-redex}_A(t, [\ell \rightarrow r \mid L]) = \begin{cases} (\epsilon, r\sigma) & \text{if } t = \ell\sigma \\ \text{find-redex}_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{find-redex}_A(t, [i \mid L]) = \begin{cases} (ip, u) & \text{if } \text{find-redex}_A(t|i) = (p, u) \\ \text{find-redex}_A(t, L) & \text{otherwise} \end{cases}$$

Theorem

\forall *full strategy annotation* $A \quad \forall$ *term* t

$$\text{find-redex}_A(t) = \perp \iff t \text{ is normal form}$$

Example

rewrite rules

$$\alpha: \text{if}(T, x, y) \rightarrow x \quad \beta: \text{if}(F, x, y) \rightarrow y \quad \gamma: \text{if}(z, x, x) \rightarrow x$$

strategy annotation

$$A(\text{if}) = [1, \alpha, 2, 3, \gamma]$$

$$\text{find-redex}_A(\text{if}(F, T, F)) = \perp \quad \text{if}(F, T, F) \rightarrow T$$

Definition

$$\text{strategy } \mathcal{S}_A(t) = \begin{cases} \emptyset & \text{if } \text{find-redex}_A(t) = \perp \\ \{t \rightarrow t[u]_p\} & \text{if } \text{find-redex}_A(t) = (p, u) \end{cases}$$



Definition

$$\text{strategy } \mathcal{S}_A(t) = \begin{cases} \emptyset & \text{if } \text{find-redex}_A(t) = \perp \\ \{t \rightarrow t[u]_p\} & \text{if } \text{find-redex}_A(t) = (p, u) \end{cases}$$

Corollary

\forall *full* strategy annotation A

\mathcal{S}_A is rewrite strategy

Definition

$$\text{normalize}_A(t) = \text{normalize}_A(t, A(\text{root}(t)))$$

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$$\text{normalize}_A(t, [\ell \rightarrow r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \\ \text{normalize}_A(t, L) & \text{otherwise} \end{cases}$$



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$$\text{normalize}_A(t) = \text{normalize}_A(t, A(\text{root}(t)))$$

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$$\text{normalize}_A(t, [i \mid L]) = \text{normalize}_A(t[\text{normalize}_A(t|i)], i, L)$$



Definition

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$$\text{normalize}_A(t, [\ell \rightarrow r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \\ \text{normalize}_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}_A(t, [i \mid L]) = \text{normalize}_A(t[\text{normalize}_A(t|_i)], L)$$

Theorem

\forall *in-time* strategy annotation $A \quad \forall$ term t

$$S_A \text{ normalizes } t \iff \text{normalize}_A(t) \text{ is normal form of } t$$

Definition

$$\text{normalize}_A(t) = \text{normalize}_A(t, A(\text{root}(t)))$$

$$\text{normalize}_A(t, []) = t$$

$$\text{normalize}_A(t, [\ell \rightarrow r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \\ \text{normalize}_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}_A(t, [i \mid L]) = \text{normalize}_A(t[\text{normalize}_A(t|i)], L)$$

Theorem

\forall *in-time* strategy annotation $A \quad \forall$ term t

$$S_A \text{ normalizes } t \iff \text{normalize}_A(t) \text{ is normal form of } t$$

JITty

<http://www.cwi.nl/~vdpol/jitty/>

Outline

- Strategies
- Strategy Annotations
- Further Reading





Just-in-Time: On Strategy Annotations

Jaco van de Pol

Proc. 1st WRS, ENTCS 57, pp. 41–63, 2001



A Sequential Reduction Strategy

Sergio Antoy and Aart Middeldorp

TCS 165(1), pp. 75 – 95, 1996

