# Introduction to Term Rewriting

## lecture 9

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**Sunday**

introduction, examples, abstract rewriting, equational reasoning, term rewriting

**Monday**

termination, completion

**Tuesday**

completion, termination

**Wednesday**

confluence, modularity, strategies

**Thursday**

exam, advanced topics

# Outline







## Definition

- rewrite strategy $\mathcal{S}$ is mapping that assigns to every reducible term $t$ nonempty set of finite nonempty rewrite sequences starting from $t$
- rewrite strategy $\mathcal{S}$ is deterministic if $\mathcal{S}(t)$ contains exactly one rewrite sequence for every reducible term $t$
- rewrite strategy $\mathcal{S}$ normalizes term $t$ if there are no infinite $\mathcal{S}$ rewrite sequences starting from $t$
- rewrite strategy $\mathcal{S}$ is normalizing if it normalizes every term that has normal form
- rewrite strategy $\mathcal{S}$ is perpetual if every maximal $\mathcal{S}$ rewrite sequence starting from any non-terminating term is infinite

## Lemma

*for terminating TRSs every strategy is normalizing and perpetual*

## Example

- rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 \\
\end{array}
$$

$$x+(y:z) \to y:(x+z) \qquad 0:x \to x$$
$$(x:y)+z \to x:(y+z) \qquad x:(y:z) \to (x+y):z$$

- term

$$((0:(1+2))+(3+4))+(5+6)$$

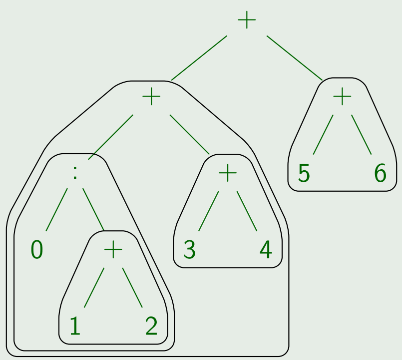




## Example (cont'd)

term $\quad 0:1+2 + 3+4 + 5+6$

tree representation




parallel/leftmost outermost/innermost strategies

## Example (cont'd)

- leftmost outermost strategy 12 redexes

$$((0:(1+2))+(3+4))+(5+6)$$
$$\to (0:((1+2)+(3+4)))+(5+6)$$
$$\to 0:(((1+2)+(3+4))+(5+6))$$
$$\to ((1+2)+(3+4))+(5+6) \to (3+(3+4))+(5+6)$$
$$\to (3+7)+(5+6) \to (1:0)+(5+6) \to 1:(0+(5+6))$$
$$\to 1:(0+(1:1)) \to 1:(1:(0+1)) \to (1+1):(0+1)$$
$$\to 2:(0+1) \to 2:1$$

- leftmost innermost strategy 10 redexes

$$(0:(1+2))+(3+4)+(5+6)$$
$$\to ((0:3)+(3+4))+(5+6) \to (3+(3+4))+(5+6)$$
$$\to (3+7)+(5+6) \to (1:0)+(5+6) \to (1:0)+(1:1)$$
$$\to 1:(0+(1:1)) \to 1:(1:(0+1)) \to 1:(1:1)$$
$$\to (1+1):1 \to 2:1$$





## Example (cont'd)

- parallel outermost strategy 12 redexes in 9 steps

$$((0:(1+2))+(3+4))+(5+6)$$
$$\twoheadrightarrow\!\!\!\!\!\!\!\!\;\;\; (0:((1+2)+(3+4)))+(1:1)$$
$$\to 0:(((1+2)+(3+4))+(1:1))$$
$$\to ((1+2)+(3+4))+(1:1)$$
$$\to 1:(((1+2)+(3+4))+1)$$
$$\twoheadrightarrow\!\!\!\!\!\!\!\!\;\;\; 1:((3+7)+1) \to 1:((1:0)+1) \to 1:(1:(0+1))$$
$$\to (1+1):(0+1) \twoheadrightarrow\!\!\!\!\!\!\!\!\;\;\; 2:1$$

- parallel innermost strategy 10 redexes in 8 steps

$$((0:(1+2))+(3+4))+(5+6)$$
$$\twoheadrightarrow\!\!\!\!\!\!\!\!\;\;\; ((0:3)+7)+(1:1) \to (3+7)+(1:1)$$
$$\to (1:0)+(1:1) \to 1:(0+(1:1)) \to 1:(1:(0+1))$$
$$\to 1:(1:1) \to (1+1):1 \to 2:1$$

## Definition

development of set of redex positions $Q$ in term $t$ is rewrite sequence starting from $t$ in which all contracted redex positions descend from position in $Q$

## Example

- rewrite rules

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- rewrite sequences

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0)$$ ☺

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to \underline{(0 \times (0 \times 0))} + (0 \times 0) \to (0 \times 0) + (0 \times 0)$$ ☹

$$\mathsf{s}(0) \times \underline{(0 \times 0)} \to \underline{\mathsf{s}(0) \times 0} \to (0 \times 0) + 0$$ ☺





## Theorem

*developments are finite*

## Definition

development $A\colon s \to^* t$ of set of redex positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$ is complete if $Q \backslash A = \varnothing$

## Lemma

*for orthogonal TRSs:* $s \to^* t$ *is complete development* $\iff$ $s \mathrel{\circ\!\!\!\!\rightarrow} t$

## Theorem

*all complete developments of* $Q$ *are permutation equivalent*

## Definition

for orthogonal TRSs full substitution strategy performs complete development of all redexes

## Example

- rewrite rules

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- full substitution strategy

$$\begin{aligned} &\mathsf{s}(\mathsf{s}(0)) \times (\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))) \\ &\quad \twoheadrightarrow (\mathsf{s}(0) \times \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \\ &\quad \twoheadrightarrow ((0 \times \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\ &\quad \to (0 + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\ &\quad \to \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \to^{+} \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))))) \end{aligned}$$





## Outline

### Theorem

*for orthogonal TRSs*

- *full substitution and parallel outermost strategies are normalizing*
- *innermost strategies are perpetual*
- *leftmost outermost strategy is not normalizing*

### Example

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

- leftmost outermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- leftmost innermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- parallel outermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{-\!\!\!\|\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \mathsf{b}$
- parallel innermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{-\!\!\!\|\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \cdots$
- full substitution $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{-\!\!\circ\!\!\twoheadrightarrow} \mathsf{f}(\mathsf{c}, \mathsf{b}) \mathrel{-\!\!\circ\!\!\twoheadrightarrow} \mathsf{b}$





### Definition

TRS is left-normal if variables do not precede function symbols in left-hand sides

### Example

- $x + (y : z) \to y : (x + z)$ ☹
- $(x : y) + z \to x : (y + z)$ ☺

### Theorem

*leftmost outermost strategy is normalizing for orthogonal left-normal TRSs*

### Remark

easy but important result: Combinatory Logic is left-normal

$$\mathsf{I}\,x \to x \qquad \mathsf{K}\,x\,y \to x \qquad \mathsf{S}\,x\,y\,z \to x\,z\,(y\,z)$$

Innsbruck





## PhD Positions in Innsbruck

1. 3-year FWF project position (advisor: Dr. René Thiemann)
   - certification of termination methods in Isabelle/HOL
2. 4-year university position from October 1, 2010
3. 4-year university position from April 1, 2011

## Requirements

- master degree in computer science or mathematics
- interest in computational logic – term rewriting – theorem proving

## Further Information

- Aart Middeldorp

# Outline

- Strategies
  - Definitions
  - Results
  - Optimality
- Strategy Annotations
- Further Reading





## Observation

parallel outermost and full-substitution are not optimal because they perform useless steps

## Example

- rewrite rules

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- rewrite sequence

$$(0 \times \mathsf{s}(0)) \times (0 + \mathsf{s}(0)) \not\rightarrow\!\!\!\!\not\rightarrow 0 \times \mathsf{s}(0) \to 0$$

- redex $0 + \mathsf{s}(0)$ is not needed

## Definition

redex $\Delta$ in term $t$ is needed if descendant of $\Delta$ is contracted in every rewrite sequence from $t$ to normal form

### Theorem

*for orthogonal TRSs*

- *every reducible term has needed redex*
- *needed reduction is normalizing*

### Unfortunately

for orthogonal TRSs it is undecidable whether redex is needed

### Remark

decidable approximations based on powerful tree automata techniques exist

### Lemma

*for left-normal orthogonal TRSs leftmost outermost redex is needed*





## Outline

## Example

rewrite rules

$$\alpha:\ \mathsf{and}(x,\mathsf{T}) \to x \qquad \gamma:\ \mathsf{or}(\mathsf{T},x) \to \mathsf{T} \qquad \epsilon:\ \infty \to \infty$$
$$\beta:\ \mathsf{and}(x,\mathsf{F}) \to \mathsf{F} \qquad \delta:\ \mathsf{or}(\mathsf{F},x) \to x$$

strategy annotation $A$

$$A(\mathsf{and}) = [2,\alpha,\beta,1] \qquad A(\mathsf{or}) = [1,\gamma,\delta,2] \qquad A(\infty) = [\epsilon]$$

evaluation

$$\begin{aligned} \mathsf{or}(\mathsf{and}(\infty,\mathsf{F}),\mathsf{or}(\mathsf{T},\infty)) &\to \mathsf{or}(\mathsf{F},\mathsf{or}(\mathsf{T},\infty)) \\ &\to \mathsf{or}(\mathsf{T},\infty) \\ &\to \mathsf{T} \end{aligned}$$





## Definitions

- strategy annotation for function symbol $f$ is finite list $A(f)$ containing
  - argument positions of $f$
  - (labels of) rewrite rules for $f$
- strategy annotation $A(f)$ for function symbol $f$ is full if $A(f)$ contains all argument positions of $f$ and all rewrite rules for $f$
- strategy annotation $A(f)$ for function symbol $f$ is in-time if argument positions are listed in $A(f)$ before rewrite rules that need them
- rewrite rule $f(s_1,\ldots,s_n) \to t$ needs argument position $i$ if
  - $s_i$ is non-variable
  - $s_i$ is variable that appears in $s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n$

## Example

rewrite rules

$$\alpha\colon\ \mathsf{if}\,(\mathsf{T},x,y) \to x \qquad \beta\colon\ \mathsf{if}\,(\mathsf{F},x,y) \to y \qquad \gamma\colon\ \mathsf{if}\,(z,x,x) \to x$$

strategy annotations

| | full | in-time |
|---|---|---|
| $A_1(\mathsf{if}) = [\,1,\alpha,\beta,2,3,\gamma\,]$ | ☺ | ☺ |
| $A_2(\mathsf{if}) = [\,\alpha,\beta,1,2,3,\gamma\,]$ | ☺ | ☹ |
| $A_3(\mathsf{if}) = [\,1,\alpha,\beta,\gamma,2,3\,]$ | ☺ | ☹ |
| $A_4(\mathsf{if}) = [\,1,\alpha,\beta\,]$ | ☹ | ☺ |





## Definition

$$\text{find-redex}_A(t) = \text{find-redex}_A(t, A(\text{root}(t)))$$

$$\text{find-redex}_A(t, [\,]) = \bot$$

$$\text{find-redex}_A(t, [\,\ell \to r \mid L\,]) = \begin{cases} (\epsilon, r\sigma) & \text{if } t = \ell\sigma \\ \text{find-redex}_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{find-redex}_A(t, [\,i \mid L\,]) = \begin{cases} (ip, u) & \text{if find-redex}_A(t|_i) = (p, u) \\ \text{find-redex}_A(t, L) & \text{otherwise} \end{cases}$$

## Theorem

$\forall$ *full strategy annotation* $A$ $\quad\forall$ *term* $t$

$$\textit{find-redex}_A(t) = \bot \iff t \textit{ is normal form}$$

## Example

rewrite rules

$$\alpha:\ \mathsf{and}(x, \mathsf{T}) \to x \qquad \gamma:\ \mathsf{or}(\mathsf{T}, x) \to \mathsf{T} \qquad \epsilon:\ \infty \to \infty$$

$$\beta:\ \mathsf{and}(x, \mathsf{F}) \to \mathsf{F} \qquad \delta:\ \mathsf{or}(\mathsf{F}, x) \to x$$

strategy annotation $A$

$$A(\mathsf{and}) = [\,2, \alpha, \beta, 1\,] \qquad A(\mathsf{or}) = [\,1, \gamma, \delta, 2\,] \qquad A(\infty) = [\,\epsilon\,]$$

evaluation

$$\text{find-redex}_A(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty)))$$

$$= \text{find-redex}_A(\mathsf{or}(\mathsf{and}(\infty, \mathsf{F}), \mathsf{or}(\mathsf{T}, \infty)), [\,1, \gamma, \delta, 2\,]) = (1, \mathsf{F})$$

$$\text{find-redex}_A(\mathsf{and}(\infty, \mathsf{F}))$$

$$= \text{find-redex}_A(\mathsf{and}(\infty, \mathsf{F}), [\,2, \alpha, \beta, 1\,])$$

$$= \text{find-redex}_A(\mathsf{and}(\infty, \mathsf{F}), [\,\alpha, \beta, 1\,])$$

$$= \text{find-redex}_A(\mathsf{and}(\infty, \mathsf{F}), [\,\beta, 1\,]) = (\epsilon, \mathsf{F})$$

$$\text{find-redex}_A(\mathsf{F}) = \bot$$





## Example

rewrite rules

$$\alpha:\ \mathsf{if}\,(\mathsf{T}, x, y) \to x \qquad \beta:\ \mathsf{if}\,(\mathsf{F}, x, y) \to y \qquad \gamma:\ \mathsf{if}\,(z, x, x) \to x$$

strategy annotation

$$A(\mathsf{if}) = [\,1, \alpha, 2, 3, \gamma\,]$$

$$\text{find-redex}_A(\mathsf{if}\,(\mathsf{F}, \mathsf{T}, \mathsf{F})) = \bot \qquad \mathsf{if}\,(\mathsf{F}, \mathsf{T}, \mathsf{F}) \to \mathsf{T}$$

## Definition

$$\text{strategy } \mathcal{S}_A(t) = \begin{cases} \varnothing & \text{if find-redex}_A(t) = \bot \\ \{t \to t[u]_p\} & \text{if find-redex}_A(t) = (p, u) \end{cases}$$

## Corollary

$\forall$ *full strategy annotation* $A$

$$\mathcal{S}_A \text{ is rewrite strategy}$$



Strategy Annotations

## Definition

$$\text{normalize}_A(t) = \text{normalize}_A(t, A(\text{root}(t)))$$

$$\text{normalize}_A(t, [\,]) = t$$

$$\text{normalize}_A(t, [\ell \to r \mid L]) = \begin{cases} \text{normalize}_A(r\sigma) & \text{if } t = \ell\sigma \\ \text{normalize}_A(t, L) & \text{otherwise} \end{cases}$$

$$\text{normalize}_A(t, [i \mid L]) = \text{normalize}_A(t[\text{normalize}_A(t|_i)]_i, L)$$

## Theorem

$\forall$ *in-time strategy annotation* $A$ $\forall$ *term* $t$

$$\mathcal{S}_A \text{ normalizes } t \iff \text{normalize}_A(t) \text{ is normal form of } t$$

## JITty

`http://www.cwi.nl/~vdpol/jitty/`

## Outline

- Strategies
- Strategy Annotations
- Further Reading






- Just-in-Time: On Strategy Annotations
  Jaco van de Pol
  Proc. 1st WRS, ENTCS 57, pp. 41–63, 2001
- A Sequential Reduction Strategy
  Sergio Antoy and Aart Middeldorp
  TCS 165(1), pp. 75 – 95, 1996