Relative Undecidability in Term Rewriting
Part 1: The Termination Hierarchy

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Abstract

For a hierarchy of properties of term rewriting systems related to termination we prove *relative undecidability*: For implications $X \Rightarrow Y$ in the hierarchy the property X is undecidable for term rewriting systems satisfying Y. For most implications we obtain this result for term rewriting systems consisting of a single rewrite rule.

1 Introduction

Termination and confluence are fundamental properties of term rewriting systems (TRSs) which are often very hard to prove. Classical results ([11, 12]) state that they are undecidable. Besides termination and confluence, a number of related properties are of interest. For termination they are ordered by implication:

$$\begin{array}{ccccccccccccccc} \mathrm{PT} & \Rightarrow & \omega\mathrm{T} & \Rightarrow & \mathrm{TT} & \Rightarrow & \mathrm{ST} & \Rightarrow & \mathrm{NSE} & \Rightarrow & \mathrm{SN} & \Rightarrow & \mathrm{NL} & \Rightarrow & \mathrm{AC} \\ &&&&&&&&&& \Downarrow &&&& \\ &&&&&&&&&& \mathrm{WN} &&&& \end{array}$$

The acronyms stand for polynomial termination (PT), ω-termination (ωT), total termination (TT), simple termination (ST), non-self-embeddingness (NSE), termination (strong normalization, SN), weak normalization (WN), non-loopingness (NL), and acyclicity (AC). We call this the *termination hierarchy*.

Apart from polynomial termination, all properties in the termination hierarchy are known to be undecidable ([11, 22, 19, 26, 7]), sometimes even for single rules ([3, 19, 15]). In this paper we show the stronger result of *relative undecidability*: For all implications $X \Rightarrow Y$ in the hierarchy except one—PT $\Rightarrow$ ωT—we prove that the property X is undecidable for TRSs satisfying Y.

We also address the question of relative undecidability for TRSs consisting of a single rewrite rule. We show that for all implications $X \Rightarrow Y$ in the termination hierarchy except two—PT $\Rightarrow$ ωT and SN $\Rightarrow$ WN—the property X is undecidable for one-rule TRSs satisfying property Y. Dauchet [3] was the first to prove undecidability of termination for one-rule TRSs, by means of a reduction of the uniform halting problem for Turing machines. Middeldorp and Gramlich [19] reduced the undecidability of simple termination, non-self-embeddingness, and non-loopingness for one-rule TRSs to the uniform halting problem for linear bounded automata. Lescanne [15] showed that Dauchet's result can also be obtained by a reduction of Post's Correspondence Problem (PCP). The results presented in this paper are stronger because (1) we obtain the same undecidability results for (much) smaller classes of one-rule TRSs, (2) we show the undecidability of total termination for one-rule (simply terminating) TRSs—solving problem 87 in [5] and rectifying a conjecture in [26]—and (3) we show the undecidability of ω-termination for one-rule totally terminating TRSs. The latter strengthens Geser's [7] result that ω-termination is an undecidable property of totally terminating TRSs, to the one-rule case.

We obtain our relative undecidability results by using PCP in the following uniform way: First we construct a TRS $\mathcal{U}(P, \mathcal{Q})$ parameterized by a PCP instance P and a TRS $\mathcal{Q}$. The TRS $\mathcal{U}(P, \mathcal{Q})$ has the following properties: (1) the left-hand sides of its rewrite rules are the same, (2) if P admits no solution then $\mathcal{U}(P, \mathcal{Q})$ is ω-terminating, and (3) if P admits a solution then $\mathcal{U}(P, \mathcal{Q})$ simulates $\mathcal{Q}$. Because of property (1) every $\mathcal{U}(P, \mathcal{Q})$ can be compressed into a one-rule TRS $\mathcal{S}(P, \mathcal{Q})$ without affecting (2) and (3). That is, if P admits no solution, then $\mathcal{S}(P, \mathcal{Q})$ is ω-terminating and if P admits a solution, then $\mathcal{S}(P, \mathcal{Q})$ simulates $\mathcal{Q}$. Finally, for all implications $X \Rightarrow Y$ in the termination hierarchy except PT $\Rightarrow$ ωT and SN $\Rightarrow$ WN we define a suitable TRS $\mathcal{Q}$ such that $\mathcal{S}(P, \mathcal{Q})$ always satisfies Y and satisfies X if and only if P admits no solution. The advantage of this approach is that the complicated part—the construction and properties of the TRS $\mathcal{U}(P, \mathcal{Q})$—is independent of the involved level in the termination hierarchy.

The paper is organized as follows. In the next section we give the definition of rewriting and PCP. The properties in the termination hierarchy are defined in Section 3. In Section 4 we define the TRS $\mathcal{U}(P, \mathcal{Q})$ and show that it simulates $\mathcal{Q}$ whenever P admits a solution. In Section 5 we present the difficult proof of ω-termination of $\mathcal{U}(P, \mathcal{Q})$ for PCP instances P that admit no solution. In the final few sections we instantiate $\mathcal{U}(P, \mathcal{Q})$ and $\mathcal{S}(P, \mathcal{Q})$ by suitable TRSs $\mathcal{Q}$ in order to conclude the desired relative undecidability results.

Some of the results in this paper were first reported in our earlier papers [9] and [10].

2 Preliminaries

For preliminaries on rewriting the reader is referred to [1, 13, 4]. We recall here the following definitions. A rewrite rule $l \to r$ is called non-erasing (variable-preserving) if the sets (multisets) of variables (variable occurrences) in l and r are the same. We call $l \to r$ collapsing if r is a variable and duplicating if some variable occurs more often in r than in l. A TRS is non-erasing (variable-preserving, non-collapsing, non-duplicating) if all its rewrite rules are so.

For the proofs we use Post's Correspondence Problem (PCP), which can be stated as follows:

> given a finite alphabet Γ and a finite set $P \subset \Gamma^+ \times \Gamma^+$, is there some natural number $n > 0$ and $(\alpha_i, \beta_i) \in P$ for $i = 1, \dots, n$ such that $\alpha_1\alpha_2 \cdots \alpha_n = \beta_1\beta_2 \cdots \beta_n$?

This problem is known to be undecidable even in the case of a two-letter alphabet (Post [23]). The set P is called an *instance* of PCP, the string $\alpha_1\alpha_2 \cdots \alpha_n = \beta_1\beta_2 \cdots \beta_n$ a *solution* for P. Without loss of generality we require P to be non-empty. Matiyasevich and Senizergues [18] recently showed that PCP is undecidable even when restricted to instances consisting of seven pairs.

3 The Termination Hierarchy

Before we can define the properties in the termination hierarchy, we need a few preliminary definitions. Throughout the following we assume that $\mathcal{F}$ is a finite signature containing at least one constant. A (strict partial) order $>$ on the set $\mathcal{T}(\mathcal{F})$ of ground terms is called *monotonic* if for all $f \in \mathcal{F}$ and $t, u \in \mathcal{T}(\mathcal{F})$ with $t > u$ we have $f(\dots, t, \dots) > f(\dots, u, \dots)$. A TRS $\mathcal{R}$ over $\mathcal{F}$ and an order $>$ on $\mathcal{T}(\mathcal{F})$ are called *compatible* if $t > u$ for all rewrite steps $t \to_{\mathcal{R}} u$. For compatibility with a monotonic order it suffices to check that $l\sigma > r\sigma$ for all rules $l \to r$ in $\mathcal{R}$ and all ground substitutions σ. It is well-known that a TRS is terminating if and only if it is compatible with a monotonic well-founded order. An $\mathcal{F}$-algebra consists of a set A and for every $f \in \mathcal{F}$ a function

$f_A\colon A^n \to A$, where n is the arity of f. A *monotone* $\mathcal{F}$-algebra $(A, >)$ is an $\mathcal{F}$-algebra A for which the underlying set is provided with an order $>$ such that every algebra operation is monotonic in all of its arguments. More precisely, for all $f \in \mathcal{F}$ and $a, b \in A$ with $a > b$ we have $f_A(\dots, a, \dots) > f_A(\dots, b, \dots)$. A monotone $\mathcal{F}$-algebra $(A, >)$ is called *well-founded* if $>$ is a well-founded order. Every monotone $\mathcal{F}$-algebra $(A, >)$ induces an order $>_A$ on the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows: $t >_A u$ if and only if $[\alpha](t) > [\alpha](u)$ for all assignments $\alpha\colon \mathcal{X} \to A$. Here $[\alpha]$ denotes the homomorphic extension of α, i.e., $[\alpha](x) = \alpha(x)$ for $x \in \mathcal{X}$ and $[\alpha](f(t_1, \dots, t_n)) = f_A([\alpha](t_1), \dots, [\alpha](t_n))$ for all n-ary $f \in \mathcal{F}$ and $t_1, \dots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. A TRS $\mathcal{R}$ and a monotone algebra $(A, >)$ are called *compatible* if $\mathcal{R}$ and $>_A$ are compatible. It is well-known that a TRS is terminating if and only if it is compatible with a well-founded monotone algebra. The set of rewrite rules $f(x_1, \dots, x_n) \to x_i$ for all $f \in \mathcal{F}$ and all $i = 1, \dots, n$, where $n \geqslant 1$ is the arity of f, is denoted by $\mathcal{E}\text{mb}(\mathcal{F})$, or simply by $\mathcal{E}\text{mb}$ when the signature $\mathcal{F}$ can be inferred from the context.

The properties in the termination hierarchy are defined as follows. A TRS is called *terminating* if it does not allow an infinite rewrite sequence. A TRS $\mathcal{R}$ over a signature $\mathcal{F}$ is called *simply terminating* if $\mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F})$ is terminating, or, equivalently (by Kruskal's Tree Theorem [14]), $\mathcal{R} \cup \mathcal{E}\text{mb}(\mathcal{F})$ has no cycle. A well-known sufficient condition for simple termination of terminating TRSs is *length-preservingness*, which means that $|l\sigma| = |r\sigma|$ for all rules $l \to r$ and all ground substitutions σ. Here $|t|$ denotes the number of (occurrences of) function symbols in t. Note that length-preservingness is equivalent to the combination of variable-preservingness and the requirement that $|l| = |r|$ for all rules $l \to r$. A TRS over a signature $\mathcal{F}$ is called *totally terminating* if it is compatible with a monotonic well-founded total order on $\mathcal{T}(\mathcal{F})$, or, equivalently, it is compatible with $>_A$ for some well-founded monotone $\mathcal{F}$-algebra $(A, >)$ in which the order $>$ is total. A TRS over a signature $\mathcal{F}$ is called *ω-terminating* if it is compatible with some well-founded monotone $\mathcal{F}$-algebra $(A, >)$ such that A is a subset of the set $\mathbb{N}$ of natural numbers and $>$ is the restriction of the usual order on $\mathbb{N}$ to A. If, in addition, every interpretation function f_A is a polynomial, we say that the TRS is *polynomially terminating*. Our definitions of ω-termination and polynomial termination differ from the ones in [25] in that we allow an arbitrary subset of $\mathbb{N}$ as carrier of the compatible algebra. For ω-termination this makes no difference: If A is an infinite subset of $\mathbb{N}$ then there exists exactly one monotonic bijection $\phi\colon A \to \mathbb{N}$. Let ψ be its (monotonic) inverse and define $f_{\mathbb{N}}(x_1, \dots, x_n) = \phi(f_A(\psi(x_1), \dots, \psi(x_n)))$ for every function symbol f. In this way we obtain a monotone algebra with carrier $\mathbb{N}$. This construction preserves all compatibility requirements, hence both definitions of ω-termination yield the same class of TRSs. (For polynomial termination this is unclear.) A TRS $\mathcal{R}$ is called *looping* if it admits a rewrite sequence $t \to_{\mathcal{R}}^{+} C[t\sigma]$ for some term t, some context C and some substitution σ. A TRS $\mathcal{R}$ is called *cyclic* if it admits a rewrite sequence $t \to_{\mathcal{R}}^{+} t$ for some term t. A TRS $\mathcal{R}$ over a signature $\mathcal{F}$ is called *self-embedding* if it admits a rewrite sequence $t \to_{\mathcal{R}}^{+} u \to_{\mathcal{E}\text{mb}(\mathcal{F})}^{*} t$ for some terms t and u. Recent investigations of these notions include [2, 6, 7, 16, 17, 21, 24, 27].

Validity of most of the implications in the termination hierarchy is direct from the definitions; only TT $\Rightarrow$ ST requires some well-known argument, see e.g. [25], and NSE

$\Rightarrow$ SN requires Kruskal's theorem. None of the implications are equivalences: for all implications $X \Rightarrow Y$ in the termination hierarchy a TRS exists satisfying Y but not X. For infinite TRSs over infinite signatures the termination hierarchy is more complicated: if the notion of embedding is not changed then NSE $\Rightarrow$ SN does not hold any more, if the notions of embedding and simple termination are adjusted as motivated in [21], then the implication TT $\Rightarrow$ ST no longer holds ([21]). In this paper however we consider only finite TRSs over finite signatures.

4 The TRS $\mathcal{U}(P, \mathcal{Q})$

We encode PCP instances P and, for each layer $X \Rightarrow Y$ of the hierarchy, a characteristic non-empty TRS $\mathcal{Q}$ into a TRS $\mathcal{U}(P, \mathcal{Q})$ such that $\mathcal{U}(P, \mathcal{Q})$ is in Y for all P, and in X if and only if P has no solution. In order to facilitate the transformation (in Section 6) of $\mathcal{U}(P, \mathcal{Q})$ into a one-rule TRS $\mathcal{S}(P, \mathcal{Q})$, we require that all rewrite rules of $\mathcal{U}(P, \mathcal{Q})$ have the same left-hand side. This property it will inherit from the TRS $\mathcal{Q}$.

The technical definition of $\mathcal{U}(P, \mathcal{Q})$ can be seen as an accumulation of a number of modifications of the following system from Zantema [26]:

$$\mathcal{S}_P = \begin{cases} F(w, \overline{a}(x), w, \overline{a}(x)) & \rightarrow & F(a(w), x, a(w), x) & \text{for all } a \in \Gamma \\ F(\alpha(w), x, \beta(y), z) & \rightarrow & F(w, \overline{\alpha}(x), y, \overline{\beta}(z)) & \text{for all } (\alpha, \beta) \in P \end{cases}$$

Here for every $a \in \Gamma$ two unary symbols a and $\overline{a}$ are defined, while

$$\alpha(t) = a_1(a_2(\cdots a_k(t) \cdots)) \quad \text{and} \quad \overline{\alpha}(t) = \overline{a}_k(\overline{a}_{k-1}(\cdots \overline{a}_1(t) \cdots))$$

for $\alpha = a_1 a_2 \ldots a_k$. The system $\mathcal{S}_P$ admits a cycle

$$F(\gamma(w), x, \gamma(w), x) \rightarrow^+ F(\gamma(w), x, \gamma(w), x)$$

if and only if γ is a solution of the PCP instance P. If P has no solution then $\mathcal{S}_P$ is totally terminating. The use of barred symbols in the second and fourth argument of F is essential for the proof of total termination. It is now straightforward to change the cyclic behaviour to any desired behaviour that can be expressed by some non-empty TRS $\mathcal{Q}$. To this end F is equipped with an additional argument. This extra argument is left unchanged, except for the step that completes the cycle when it is rewritten by a rule in $\mathcal{Q}$. To avoid unintended rewrite steps, we refine control: We distinguish two states, exhibited by function symbols G and H, which enable only steps of the first and second shape, respectively, in $\mathcal{S}_P$. A change from state G to state H is possible only if the second and fourth arguments are equal to ε. Vice versa, a change of state from H to G requires that the first and third arguments are equal to ε. This yields the TRS consisting of the rewrite rules

$$\begin{aligned} G(w, \varepsilon, y, \varepsilon, \mathsf{LHS}) &\rightarrow H(w, \varepsilon, y, \varepsilon, \mathsf{LHS}) && (1) \\ H(\alpha(w), x, \beta(y), z, \mathsf{LHS}) &\rightarrow H(w, \overline{\alpha}(x), y, \overline{\beta}(z), \mathsf{LHS}) && (2) \\ H(\varepsilon, \overline{a}(x), \varepsilon, \overline{a}(z), \mathsf{LHS}) &\rightarrow G(a(\varepsilon), x, a(\varepsilon), z, \mathsf{RHS}) && (3) \\ G(w, \overline{a}(x), y, \overline{a}(z), \mathsf{LHS}) &\rightarrow G(a(w), x, a(y), z, \mathsf{LHS}) && (4) \end{aligned}$$

for every $(\alpha, \beta) \in P$, $a \in \Gamma$, and right-hand side RHS of the rewrite rules in $\mathcal{Q}$. Here LHS denotes the unique left-hand side of the rules in $\mathcal{Q}$. The TRS is linear whenever $\mathcal{Q}$ is linear.

Throughout the remainder of the paper we assume that $\Gamma = \{0, 1\}$. This entails no loss of generality. Writing n for the size of the PCP instance P and m for the number of rules of $\mathcal{Q}$, there is one rule of type (1), there are n rules of type (2), there are $2m$ rules of type (3), and there are 2 rules of type (4), hence $n + 2m + 3$ rules in total.

In view of the one-rule construction it is necessary to have equal left-hand sides. Subsequently we describe how to code the difference between G and H, the transfer of strings from one argument position to another argument position, and the treatment of the empty string. The accumulation of all of these modifications will yield the technical definition of the system $\mathcal{U}(P, \mathcal{Q})$ again consisting of $n+2m+3$ rules, in which the single left-hand side and all right-hand sides are of the shape $A(\cdots)$ for a symbol A of high arity. The encoding is highly inspired by Lescanne [15]. Basically, some of the matching is delayed and extra parameters serve for the delayed matching. Let us demonstrate the technique at a simple example. The TRS $\{f(a) \to f(a'), f(b) \to f(b')\}$ is translated into the TRS $\{f'(a, b, x) \to f'(x, b, a'), f'(a, b, x) \to f'(a, x, b')\}$. Rewrite steps $f(t) \to f(t')$ in the original system correspond to rewrite steps $f'(a, b, t) \to f'(a, b, t')$ in the translated system. Rewrite steps that have no counterpart in the original system, e.g. $f'(a, b, a) \to f'(a, a, b')$, produce an irreducible term.

For treating the difference between G and H we use four arguments of A. The system is transformed according to the following scheme:

rule of shape	is coded as
$G(\dots) \to H(\dots)$	$A(0, 1, u, v, \dots) \to A(u, v, 1, 0, \dots)$
$H(\dots) \to H(\dots)$	$A(0, 1, u, v, \dots) \to A(v, u, 1, 0, \dots)$
$H(\dots) \to G(\dots)$	$A(0, 1, u, v, \dots) \to A(v, u, 0, 1, \dots)$
$G(\dots) \to G(\dots)$	$A(0, 1, u, v, \dots) \to A(u, v, 0, 1, \dots)$

By coding $G(\dots)$ as $A(0, 1, 0, 1, \dots)$ and $H(\dots)$ as $A(0, 1, 1, 0, \dots)$ every rewrite step in the old system transforms to a rewrite step in the new system. Conversely, every rewrite step in the new system not corresponding to this coding of G and H will result in a term $A(1, 0, \dots)$ not allowing further rewrite steps.

Next we describe the transfer of strings from one argument position to another argument position. In the system $\mathcal{S}_P$ string elements were coded by unary symbols. In order to allow variables as string elements we now choose another representation: Elements of Γ are represented by constants and combined into strings by a binary symbol cons. In order to distinguish between unbarred and barred strings as in $\mathcal{S}_P$ we introduce another binary symbol $\overline{\mathsf{cons}}$. For any term t and string $\alpha = t_1 t_2 \dots t_n$ of terms we write

$$\alpha(t) = \mathsf{cons}(t_1, \mathsf{cons}(t_2, \dots \mathsf{cons}(t_n, t) \dots))$$

and

$$\overline{\alpha}(t) = \overline{\mathsf{cons}}(t_n, \overline{\mathsf{cons}}(t_{n-1}, \dots \overline{\mathsf{cons}}(t_1, t) \dots))$$

We introduce an extra constant \$ to mark the end of a string. The intention of the rules of type (2) is to enable a rewrite sequence

$$H(\gamma(\varepsilon), \varepsilon, \dots) \to^+ H(\varepsilon, \overline{\gamma}(\varepsilon), \dots)$$

for $\gamma = \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}$ by means of rules of the shape

$$H(\alpha_i(x), y, \dots) \to H(x, \overline{\alpha_i}(y), \dots)$$

for $1 \leqslant i \leqslant n$. In the new notation the same can be achieved by a single left-hand side by adding $n+2$ arguments to the symbol A (the only $n+2$ arguments to be displayed for the moment) and choosing rules with left-hand side

$$A(\alpha_1(\varepsilon), \dots, \alpha_n(\varepsilon), w_1 \dots w_\mu(w), \overline{x_1}(x))$$

and right-hand sides

$$A(\alpha_1(\varepsilon), \dots, \alpha_{i-1}(\varepsilon), w_1 \dots w_{|\alpha_i|}(\varepsilon), \alpha_{i+1}(\varepsilon), \dots, \alpha_n(\varepsilon), w_{|\alpha_i|+1} \dots w_\mu(w), \overline{x_1\alpha_i}(x))$$

for $1 \leqslant i \leqslant n$. Here μ is a number satisfying $|\alpha_i| \leqslant \mu$ for all $1 \leqslant i \leqslant n$, and x, x_1, w, and $w_1, \dots, w_\mu$ are fresh variables. The objective of $w_1 \dots w_{|\alpha_i|}(\varepsilon)$ in the right-hand sides at the position of α_i is that rewriting can only be continued if the variables $w_1, \dots, w_{|\alpha_i|}$ are instantiated by the successive elements of the string α_i. In this way we obtain the rewrite sequence

$$\begin{array}{ll}
 & A(\alpha_1(\varepsilon), \dots, \alpha_n(\varepsilon), \gamma(t_1), \overline{\$}(x)) \\
\to & A(\alpha_1(\varepsilon), \dots, \alpha_n(\varepsilon), \alpha_{i_2} \dots \alpha_{i_k}(t_1), \overline{\$\alpha_{i_1}}(x)) \\
\to^* & A(\alpha_1(\varepsilon), \dots, \alpha_n(\varepsilon), \alpha_{i_k}(t_1), \overline{\$\alpha_{i_1} \dots \alpha_{i_{k-1}}}(x)) \\
\to & A(\alpha_1(\varepsilon), \dots, \alpha_n(\varepsilon), t_1, \overline{\$\gamma}(x))
\end{array}$$

for $\gamma = \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}$ and $t_1 = \$w_2 \dots w_\mu(w)$. Here the variables after \$ in t_1 are needed to perform the last few steps in the above rewrite sequence if the length of the remaining string to be transferred is less than μ. (Actually, any term t_1 of the form $s_1 \dots s_\mu(s)$ will do here.)

The next thing to do is to represent the elementwise backward transfer of strings as is done by the rules of type (3) and (4). This is simpler than the forward transfer described above. We need three new arguments of A to code this; besides these three also the arguments $w_1 \dots w_\mu(w)$ and $\overline{x_1}(x)$, as they occur in the left-hand side, are involved since the real string transfer has to take place here. For the moment we will only consider these five arguments of A. For transferring a '0' or a '1' by rule (3) we give rules

$$A(0, 1, \$, w_1 \dots w_\mu(w), \overline{x_1}(x)) \to A(x_1, 1, w_1, 0\$w_2 \dots w_\mu(w), x)$$

and

$$A(0, 1, \$, w_1 \dots w_\mu(w), \overline{x_1}(x)) \to A(0, x_1, w_1, 1\$w_2 \dots w_\mu(w), x)$$

For transferring a '0' or a '1' by rule (4) we give rules

$$A(0,1,\$,w_1\ldots w_\mu(w),\overline{x_1}(x)) \to A(x_1,1,\$,0w_1\ldots w_\mu(w),x)$$

and

$$A(0,1,\$,w_1\ldots w_\mu(w),\overline{x_1}(x)) \to A(0,x_1,\$,1w_1\ldots w_\mu(w),x)$$

These rules allow the full backward transfer

$$A(0,1,\$,t_1,\overline{\$\gamma}(x)) \to^+ A(0,1,\$,\gamma(t_1),\overline{\$}(x))$$

for $t_1 = \$w_2\ldots w_\mu(w)$. Note that for continuation after the first step in this rewrite sequence it is essential that t_1 starts with \$.

Just like in the rules of type (2) the α's and β's are transferred simultaneously; in our system we will similarly add $n+2$ arguments again in order to simultaneously transfer β's, and another 3 arguments for elementwise backward transfer. In this way the arity of A becomes $2n+15$: 4 arguments for coding the difference between G and H, $2(n+2+3) = 2n+10$ for transferring strings, and one final argument to contain LHS or RHS from $\mathcal{Q}$. In order to obtain rewrite sequences that have a consecutive group of non-changing arguments (which is very convenient when we present statements and proofs about the construction later on), the arguments are not ordered in the way we just introduced them. Instead they are ordered as follows:

- arguments $1, 2, 2n+9, 2n+10$ are the four arguments for coding the difference between G and H;
- arguments $6,\ldots,n+5$ are the n arguments for coding the matching with the α's;
- arguments $2n+11$ and $2n+12$ are the arguments in which the string transfer of the α's takes place as in the first two arguments of G and H;
- arguments $3, 4, 5$ are the three arguments for coding the elementwise backward transfer of the string consisting of α's;
- arguments $n+9,\ldots,2n+8$ are the n arguments for coding the matching with the β's;
- arguments $2n+13$ and $2n+14$ are the arguments in which the string transfer of the β's takes place as in the third and fourth argument of G and H;
- arguments $n+6, n+7, n+8$ are the three arguments for coding the elementwise backward transfer of the string consisting of β's;
- argument $2n+15$ contains LHS or RHS from $\mathcal{Q}$.

Combining all parts of the construction as described above in this order, we arrive at the following definition where (I), (II), (III), and (IV) refer to transformations of the rules of type (1), (2), (3), and (4), respectively.

Definition 4.1 Let $P = (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ be an arbitrary PCP instance and let $\mathcal{Q} = \{\mathsf{LHS} \to \mathsf{RHS}_1, \ldots, \mathsf{LHS} \to \mathsf{RHS}_m\}$ be a finite non-empty TRS with the property that all left-hand sides equal LHS. The maximum length of strings in P is denoted by μ: $\mu = \max\{|\alpha|, |\beta| \mid (\alpha, \beta) \in P\}$. We define the TRS $\mathcal{U}(P, \mathcal{Q})$ as follows. Its signature $\mathcal{F}_\mathcal{U}$ consists of the signature $\mathcal{F}_\mathcal{Q}$ of the TRS $\mathcal{Q}$ together with constants 0, 1, \$, and ε, binary function symbols cons and $\overline{\mathsf{cons}}$, and a function symbol A of arity $2n + 15$. The TRS $\mathcal{U}(P, \mathcal{Q})$ consists of the rewrite rules $l \to r_i$, $1 \leqslant i \leqslant n + 2m + 3$, where l and r_i are defined as follows:

$$\begin{aligned} l = A(&0, 1, 0, 1, \$, \alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon), 0, 1, \$, \beta_1(\varepsilon), \ldots, \beta_n(\varepsilon),\\ &u, v, w_1 \ldots w_\mu(w), \overline{x_1}(x), y_1 \ldots y_\mu(y), \overline{z_1}(z), \mathsf{LHS})\end{aligned}$$

$$\begin{aligned} r_1 = A(&u, v, 0, 1, x_1, \alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon), 0, 1, z_1, \beta_1(\varepsilon), \ldots, \beta_n(\varepsilon),\\ &1, 0, w_1 \ldots w_\mu(w), \overline{\$}(x), y_1 \ldots y_\mu(y), \overline{\$}(z), \mathsf{LHS})\end{aligned} \tag{I}$$

$$\begin{aligned} r_{i+1} = A(&v, u, 0, 1, \$, \alpha_1(\varepsilon), \ldots, \alpha_{i-1}(\varepsilon), w_1 \ldots w_{|\alpha_i|}(\varepsilon), \alpha_{i+1}(\varepsilon), \ldots, \alpha_n(\varepsilon),\\ &0, 1, \$, \beta_1(\varepsilon), \ldots, \beta_{i-1}(\varepsilon), y_1 \ldots y_{|\beta_i|}(\varepsilon), \beta_{i+1}(\varepsilon), \ldots, \beta_n(\varepsilon),\\ &1, 0, w_{|\alpha_i|+1} \ldots w_\mu(w), \overline{x_1 \alpha_i}(x), y_{|\beta_i|+1} \ldots y_\mu(y), \overline{z_1 \beta_i}(z), \mathsf{LHS})\end{aligned} \tag{II}$$

for all $1 \leqslant i \leqslant n$,

$$\begin{aligned} r_{n+1+j} = A(&v, u, x_1, 1, w_1, \alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon), z_1, 1, y_1, \beta_1(\varepsilon), \ldots, \beta_n(\varepsilon),\\ &0, 1, 0\$w_2 \ldots w_\mu(w), x, 0\$y_2 \ldots y_\mu(y), z, \mathsf{RHS}_j)\end{aligned} \tag{III}$$

$$\begin{aligned} r_{n+1+m+j} = A(&v, u, 0, x_1, w_1, \alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon), 0, z_1, y_1, \beta_1(\varepsilon), \ldots, \beta_n(\varepsilon),\\ &0, 1, 1\$w_2 \ldots w_\mu(w), x, 1\$y_2 \ldots y_\mu(y), z, \mathsf{RHS}_j)\end{aligned} \tag{III}$$

for all $1 \leqslant j \leqslant m$, and finally

$$\begin{aligned} r_{n+2m+2} = A(&u, v, x_1, 1, \$, \alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon), z_1, 1, \$, \beta_1(\varepsilon), \ldots, \beta_n(\varepsilon),\\ &0, 1, 0w_1 \ldots w_\mu(w), x, 0y_1 \ldots y_\mu(y), z, \mathsf{LHS})\end{aligned} \tag{IV}$$

$$\begin{aligned} r_{n+2m+3} = A(&u, v, 0, x_1, \$, \alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon), 0, z_1, \$, \beta_1(\varepsilon), \ldots, \beta_n(\varepsilon),\\ &0, 1, 1w_1 \ldots w_\mu(w), x, 1y_1 \ldots y_\mu(y), z, \mathsf{LHS})\end{aligned} \tag{IV}$$

Let $V = 0, 1, 0, 1, \$, \alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon), 0, 1, \$, \beta_1(\varepsilon), \ldots, \beta_n(\varepsilon)$ be the sequence of the first $2n + 8$ arguments of the left-hand side l and let $V_1, \ldots, V_{2n+8}$ denote its components.

The next lemma states that $\mathcal{U}(P, \mathcal{Q})$ can simulate root reductions in $\mathcal{Q}$ provided P admits a solution.

Lemma 4.2 *If the PCP instance P admits a solution then there exist terms $W_1, \ldots, W_6$ such that for every rewrite rule $\mathsf{LHS} \to \mathsf{RHS}$ in $\mathcal{Q}$ there is a rewrite sequence*

$$A(V, W_1, \ldots, W_6, \mathsf{LHS}) \to^+_{\mathcal{U}(P,\mathcal{Q})} A(V, W_1, \ldots, W_6, \mathsf{RHS}).$$

Proof Let $\gamma = \alpha_{i_1} \ldots \alpha_{i_k} = \beta_{i_1} \ldots \beta_{i_k} = \gamma' a$ be a solution of P. Define $t_1 = \$w_2 \ldots w_\mu(w)$, $t_2 = \$y_2 \ldots y_\mu(y)$, $W_1 = 0$, $W_2 = 1$, $W_3 = a(t_1)$, $W_4 = \overline{\$\gamma'}(x)$, $W_5 = a(t_2)$, and $W_6 = \overline{\$\gamma'}(z)$. It is easy to see that for every $\mathsf{LHS} \to \mathsf{RHS}$ in $\mathcal{Q}$ we have the following rewrite sequence in $\mathcal{U}(P, \mathcal{Q})$:

$$
\begin{array}{ll}
\multicolumn{2}{l}{A(V, 0, 1, a(t_1), \overline{\$\gamma'}(x), a(t_2), \overline{\$\gamma'}(z), \mathsf{LHS})} \\
\quad \to^*_{(\mathrm{IV})} & A(V, 0, 1, \gamma(t_1), \overline{\$}(x), \gamma(t_2), \overline{\$}(z), \mathsf{LHS}) \\
\quad \to_{(\mathrm{I})} & A(V, 1, 0, \gamma(t_1), \overline{\$}(x), \gamma(t_2), \overline{\$}(z), \mathsf{LHS}) \\
\quad \to_{(\mathrm{II})} & A(V, 1, 0, \alpha_{i_2} \ldots \alpha_{i_k}(t_1), \overline{\$\alpha_{i_1}}(x), \beta_{i_2} \ldots \beta_{i_k}(t_2), \overline{\$\beta_{i_1}}(z), \mathsf{LHS}) \\
\quad \to^*_{(\mathrm{II})} & A(V, 1, 0, t_1, \overline{\$\gamma}(x), t_2, \overline{\$\gamma}(z), \mathsf{LHS}) \\
\quad \to_{(\mathrm{III})} & A(V, 0, 1, a(t_1), \overline{\$\gamma'}(x), a(t_2), \overline{\$\gamma'}(z), \mathsf{RHS})
\end{array}
$$

□

Conversely, a rewrite sequence in $\mathcal{U}(P, \mathcal{Q})$ gives rise to either a rewrite sequence in $\mathcal{Q}$ or a rewrite sequence in $\mathcal{U}(P, \mathcal{Q})$ without the type (III) rules. We will denote the latter system by $\mathcal{U}_-(P, \mathcal{Q})$. From now on W and W' denote sequences of 6 arbitrary terms, and V' denotes a sequence of $2n + 8$ arbitrary terms.

Lemma 4.3 *If W and t do not contain A symbols then $A(V, W, t) \to_{\mathcal{U}(P,\mathcal{Q})} A(V', W', t')$ implies either $t \to_Q t'$ or both $t = t'$ and $A(V, W, t) \to_{\mathcal{U}_-(P,\mathcal{Q})} A(V', W', t)$.*

Proof Since there is only one A symbol in $A(V, W, t)$, the rewrite step must take place at the root position. If a rewrite rule of type (III) has been applied then $t = \mathsf{LHS}\sigma \to_Q \mathsf{RHS}\sigma = t'$ for some rewrite rule $\mathsf{LHS} \to \mathsf{RHS}$ in $\mathcal{Q}$ and substitution σ. Otherwise, $A(V, W, t) \to_{\mathcal{U}_-(P,\mathcal{Q})} A(V', W', t')$ which obviously implies $t = t'$ by the form of the rules in $\mathcal{U}_-(P, \mathcal{Q})$. □

5 ω-Termination of $\mathcal{U}(P, \mathcal{Q})$

In this somewhat lengthy section we will show the ω-termination of $\mathcal{U}(P, \mathcal{Q})$ for PCP instances P that do not have a solution and of $\mathcal{U}_-(P, \mathcal{Q})$ for arbitrary PCP instances P. Since we prefer not to treat the two cases separately, we write $\mathcal{U}'(P, \mathcal{Q})$ to denote either $\mathcal{U}(P, \mathcal{Q})$ under the assumption that P admits no solution or $\mathcal{U}_-(P, \mathcal{Q})$ without any assumptions on P.

The proof is quite complicated, so readers may want to skip it upon first reading. The basic idea of the proof is similar to the one in Geser [7], but the details are more intricate here.

First we will show that the length of rewrite sequences in $\mathcal{U}'(P, \mathcal{Q})$ is bounded. For a term t, let $\|t\|$ denote the maximal length of the "mixed" string $\zeta \in \{0, 1, \overline{0}, \overline{1}\}^*$ such that $t = \zeta(t')$ for some term t'.

Lemma 5.1 *No rewrite sequence in $\mathcal{U}'(P,\mathcal{Q})$ starting from a term $t = A(V, W, t')$ contains more than $1 + 4\|W_3\| + 3\|W_4\|$ steps at the root position.*

Proof First we consider the case that $\mathcal{U}'(P,\mathcal{Q}) = \mathcal{U}(P,\mathcal{Q})$ and thus P lacks a solution. Consider a maximal rewrite sequence in $\mathcal{U}(P,\mathcal{Q})$ starting from t. Since rewrite steps that take place inside the first $2n + 14$ arguments of t cannot create a redex at the root position, we may assume without loss of generality that there are no rewrite steps inside the first $2n + 14$ arguments. This reasoning does not apply to the last argument of t because the left-hand side LHS of the rewrite rules in $\mathcal{Q}$ need not be linear. However, by taking $t' = \mathsf{LHS}\sigma = \mathsf{RHS}\sigma$ for some substitution σ, we are assured that there are no rewrite sequences starting from $A(V, W, t'')$ that have more steps at the root position than $A(V, W, t')$. (In other words, for the purpose of proving this lemma we may assume without loss of generality that $\mathcal{Q} = \{d \to d\}$.) So all steps in the maximal rewrite sequence starting from $t = A(V, W, t')$ take place at the root position.

Below we write $\mathrm{root}'(s) = s'$ ($\mathrm{root}'(s) = \overline{s'}$) to indicate that $s = \mathsf{cons}(s', s'')$ ($s = \overline{\mathsf{cons}}(s', s'')$) for some term s''. For a proof by contradiction, consider a rewrite sequence starting from t that contains more than $1 + 4\|W_3\| + 3\|W_4\|$ steps at the root position. We are going to show that P has a solution. We must have $(W_1, W_2) = (0, 1)$ or $(W_1, W_2) = (1, 0)$. First we consider the former.

All terms of the rewrite sequence, except possibly the last, are of the form $A(V, \ldots)$. Due to the fact that there must be changes in the state (W_1, W_2), the given rewrite sequence without its last step is a prefix of a rewrite sequence of the form

$$t \to^*_{(\mathrm{IV})} t^1 \to_{(\mathrm{I})} t^2 \to^*_{(\mathrm{II})} t^3 \to_{(\mathrm{III})} t^4 \to^*_{(\mathrm{IV})} t^5 \to_{(\mathrm{I})} t^6 \to \cdots \tag{5}$$

By the forms of the rules we can reason as follows.

In (5) we must have $t^1 = A(V, 0, 1, W_3^1, W_4^1, W_5^1, W_6^1, t')$ with

$$\begin{aligned} W_3^1 &= \gamma(W_3) & \qquad W_5^1 &= \gamma(W_5) \\ \overline{\gamma}(W_4^1) &= W_4 & \qquad \overline{\gamma}(W_6^1) &= W_6 \end{aligned}$$

for some $\gamma \in \{0, 1\}^*$. Likewise, $t^2 = A(V, 1, 0, W_3^2, W_4^2, W_5^2, W_6^2, t')$ with $\mathrm{root}'(W_4^1) = \mathrm{root}'(W_6^1) = \overline{\$}$ and

$$\begin{aligned} W_3^2 &= W_3^1 & \qquad W_5^2 &= W_5^1 \\ W_4^2 &= W_4^1 & \qquad W_6^2 &= W_6^1 \end{aligned}$$

Furthermore, $t^3 = A(V, 1, 0, W_3^3, W_4^3, W_5^3, W_6^3, t')$ with

$$\begin{aligned} \alpha(W_3^3) &= W_3^2 & \qquad \beta(W_5^3) &= W_5^2 \\ W_4^3 &= \overline{\alpha}(W_4^2) & \qquad W_6^3 &= \overline{\beta}(W_6^2) \end{aligned}$$

and $\alpha = \alpha_{i_1} \dots \alpha_{i_k}$ and $\beta = \beta_{i_1} \dots \beta_{i_k}$ with $k \geqslant 0$ and $1 \leqslant i_j \leqslant n$ for all $1 \leqslant j \leqslant k$. Here k denotes the number of steps of type (II). Next, $t^4 = A(V, 0, 1, W_3^4, W_4^4, W_5^4, W_6^4, t')$ with $\text{root}'(W_3^3) = \text{root}'(W_5^3) = \$$ and

$$\begin{aligned} W_3^4 &= i(W_3^3) & \qquad W_5^4 &= i(W_5^3) \\ \bar{i}(W_4^4) &= W_4^3 & \qquad \bar{i}(W_6^4) &= W_6^3 \end{aligned}$$

for some $i \in \{0,1\}$. Next, $t^5 = A(V, 0, 1, W_3^5, W_4^5, W_5^5, W_6^5, t')$ with

$$\begin{aligned} W_3^5 &= \delta(W_3^4) & \qquad W_5^5 &= \delta(W_5^4) \\ \overline{\delta}(W_4^5) &= W_4^4 & \qquad \overline{\delta}(W_6^5) &= W_6^4 \end{aligned}$$

for some $\delta \in \{0,1\}^*$. Finally, $t^6 = A(V, 1, 0, \dots)$ with $\text{root}'(W_4^5) = \text{root}'(W_6^5) = \overline{\$}$. We have $\bar{i}(W_4^4) = \overline{\alpha}(W_4^2)$ and $\bar{i}(W_6^4) = \overline{\beta}(W_6^2)$. So there exist $\alpha', \beta' \in \{0,1\}^*$ such that $\alpha = \alpha' i$ and $\beta = \beta' i$. In particular, since $\alpha = \alpha_{i_1} \dots \alpha_{i_k}$ (and $\beta = \beta_{i_1} \dots \beta_{i_k}$), $k > 0$. We have $W_4^4 = \overline{\alpha'}(W_4^2) = \overline{\delta}(W_4^5)$ with $\text{root}'(W_4^2) = \text{root}'(W_4^5) = \overline{\$}$. This implies that $\alpha' = \delta$. In the same way we obtain $\beta' = \delta$. Hence $\alpha = \beta$. In other words, P has a solution. Since this contradicts our assumption we conclude that rewrite sequence (5) cannot go beyond t^5. It still must be shown that there are at most $1 + 4\|W_3\| + 3\|W_4\|$ steps until t_5 is reached.

The sequence from t to t^1 contains $|\gamma|$ steps and clearly $|\gamma| \leqslant \|W_4\|$. Recall that $\alpha, \beta \in \{0,1\}^+$ for all $(\alpha, \beta) \in P$. So in every step in the sequence from t^2 to t^3 at least one symbol of W_3^2 is consumed. It follows that $k \leqslant ||W_3^2||$. Because $W_3^2 = \gamma(W_3)$, we have $\|W_3^2\| = |\gamma| + \|W_3\| \leqslant \|W_3\| + \|W_4\|$. Next consider the sequence from t^4 to t^5. This part contains $|\delta|$ steps. Since $\delta i = \alpha$ and $|\alpha| \leqslant \|W_3^2\|$, we obtain $|\delta| \leqslant \|W_3\| + \|W_4\| - 1$. By putting everything together we obtain

$$\|W_4\| + 1 + (\|W_3\| + \|W_4\|) + 1 + (\|W_3\| + \|W_4\| - 1) = 1 + 2\|W_3\| + 3\|W_4\|$$

as an upper bound for the maximum length of the sequence from t to t_5 in (5) and clearly $1 + 2\|W_3\| + 3\|W_4\| \leqslant 1 + 4\|W_3\| + 3\|W_4\|$.

In the other case we have $(W_1, W_2) = (1, 0)$. By using very similar arguments as above we obtain $4\|W_3\| + 2\|W_4\|$ as an upper bound on the maximum number of rewrite steps at the root position in a rewrite sequence starting from t. Since $4\|W_3\| + 2\|W_4\| < 1 + 4\|W_3\| + 3\|W_4\|$ this proves the lemma in the case that $\mathcal{U}'(P, \mathcal{Q}) = \mathcal{U}(P, \mathcal{Q})$ and P admits no solution.

Next we consider the case that $\mathcal{U}'(P, \mathcal{Q}) = \mathcal{U}_{-}(P, \mathcal{Q})$. If $(W_1, W_2) = (0, 1)$ then we obtain $1 + \|W_3\| + 2\|W_4\|$ as an upper bound on the maximum number of rewrite steps at the root position in a rewrite sequence starting from t; note that rewrite sequence (5) above cannot go beyond t^3 since there are no rules of type (III) in $\mathcal{U}_{-}(P, \mathcal{Q})$. Similarly, if $(W_1, W_2) = (1, 0)$ then we obtain the upper bound $\|W_3\|$. Since both bounds do not exceed $1 + \|W_3\| + 2\|W_4\|$, the lemma also holds for $\mathcal{U}'(P, \mathcal{Q}) = \mathcal{U}_{-}(P, \mathcal{Q})$. □

Definition 5.2 Let $\mathsf{len}(W)$ denote the maximum number of root rewrite steps in any rewrite sequence in $\mathcal{U}'(P, \mathcal{Q})$ starting from a term of the form $A(V, W, t)$.

The following result is an immediate consequence of Lemma 5.1.

Corollary 5.3 *The function* len *satisfies* $\mathsf{len}(W) \leqslant 1 + 4\|W_3\| + 3\|W_4\|$. $\square$

Below we define an interpretation [ ] into the positive integers which is capable of orienting all ground instances of the rewrite rules in $\mathcal{U}'(P, \mathcal{Q})$ from left to right.

We start by defining a few useful auxiliary functions on the positive integers $\mathbb{N}_+$. Let $\ell(x)$ denote the number of digits in the decimal representation of a positive integer x and let $\ell(0) = 0$. Define two binary operators $\circ$ and $\uparrow$ on non-negative integers by

$$\begin{aligned} x \circ y &= 10^{\ell(y)} \cdot x + y \\ x \uparrow 0 &= 0 \\ x \uparrow (y+1) &= x \uparrow y \circ x \end{aligned}$$

We will assume that $\circ$ binds weaker than $+$ and $\uparrow$. Informally, $x \circ y$ yields the concatenation of the decimal representations of x and y without leading zeros and the expression $x \uparrow y$ denotes the y-fold repetition of x. Both functions are strictly monotonic in all arguments, $\circ$ is associative, and the identities $\ell(x \circ y) = \ell(x) + \ell(y)$ and $\ell(x \uparrow y) = y \cdot \ell(x)$ hold. A function $\mathsf{code}\colon \mathbb{N}_+ \to \mathbb{N}_+$ is defined to take the octal representation of its argument and adds 2 to every digit greater than 4. The resulting digit sequence is the decimal representation of the result. For instance, $\mathsf{code}(11209) = 27911$ as $(11209)_{10} = (25711)_8$. Note that code is strictly monotonic. Note furthermore that $\mathsf{code}(x)$ does not contain the digits 5 and 6. It is not difficult to see that code is the smallest function with these two properties.

Below we will define the interpretation of all function symbols except A.

Definition 5.4 We interpret function symbols in $\{\varepsilon, \$, 0, 1, \mathsf{cons}, \overline{\mathsf{cons}}\}$ as follows:

$$\begin{aligned} [\varepsilon] &= 1 \\ [\$] &= 2 \\ [0] &= 3 \\ [1] &= 4 \\ [\mathsf{cons}](x, y) &= y \circ 5 \circ x \circ 6 \\ [\overline{\mathsf{cons}}](x, y) &= 5 \circ x \circ 6 \circ y \circ 7 \uparrow (2 + \ell(x)) \end{aligned}$$

Every k-ary function symbol f in the signature of $\mathcal{Q}$ is interpreted as follows:

$$[f](x_1, \ldots, x_k) = \begin{cases} 10 \cdot \mathsf{code}(x_1 + \cdots + x_k) & \text{if } k > 0 \\ 10 & \text{if } k = 0 \end{cases}$$

Before we can extend the interpretation to A, we need a few further auxiliary functions, some of which depend on the interpretation of terms over $\mathcal{F}_{\mathcal{U}} \setminus \{A\}$.

For the treatment of the last argument of A we first define two unary functions $\phi_{\mathcal{Q}}$ and $\psi_{\mathcal{Q}}$ which depend on the TRS $\mathcal{Q}$. Function $\phi_{\mathcal{Q}}$ estimates the growth of the last argument of A caused by an application of a type (III) rewrite rule of $\mathcal{U}'(P,\mathcal{Q})$: For every upper bound x of the interpretations of LHS, $\phi_{\mathcal{Q}}(x)$ is an upper bound of the corresponding interpretations of RHS, for all $\mathsf{LHS} \to \mathsf{RHS} \in \mathcal{Q}$. Function $\phi_{\mathcal{Q}} \colon \mathbb{N}_+ \to \mathbb{N}_+$ is defined as follows:

$$\phi_{\mathcal{Q}}(x) = \max\ \{x\} \cup \{[\theta](\mathsf{RHS}) \mid \mathsf{LHS} \to \mathsf{RHS} \in \mathcal{Q} \text{ and } [\theta](\mathsf{LHS}) \leqslant x\}$$

Since LHS and RHS only contain function symbols in the signature of $\mathcal{Q}$, we can compute $[\theta](\mathsf{LHS})$ and $[\theta](\mathsf{RHS})$ for every assignment θ of positive integers for the variables in $\mathsf{LHS} \to \mathsf{RHS}$. Here $[\theta](t)$ for $t \in \mathcal{T}(\mathcal{F}_{\mathcal{Q}},\mathcal{X})$ is inductively defined as follows:

$$[\theta](t) = \begin{cases} [f]([\theta](t_1),\ldots,[\theta](t_k)) & \text{if } t = f(t_1,\ldots,t_k) \text{ with } k \geqslant 0 \\ \theta(t) & \text{if } t \in \mathcal{X} \end{cases}$$

Because the interpretation of every function symbol in $\mathcal{Q}$ is strictly monotonic in all its arguments—(an immediate consequence of the strict monotonicity of $\circ$ and code)—there can only be a finite number of assignments θ such that $[\theta](\mathsf{LHS}) \leqslant x$ for a given $x \in \mathbb{N}_+$. Hence the maximum is formed over a finite computable set and thus the function $\phi_{\mathcal{Q}}$ is well-defined. Note that for every ground instance $t \to u$ of a rewrite rule in $\mathcal{Q}$, we have $\phi_{\mathcal{Q}}([t]) \geqslant [u]$. It is easy to check that $\phi_{\mathcal{Q}}$ is monotonic and that $\phi_{\mathcal{Q}}(x) \geqslant x$ holds. Function $\psi_{\mathcal{Q}} \colon \mathbb{N} \times \mathbb{N}_+ \to \mathbb{N}_+$ is used to compensate a potential increase of the last argument of A along an application of a type (III) rewrite rule of $\mathcal{U}'(P,\mathcal{Q})$. It is defined inductively as follows:

$$\begin{aligned} \psi_{\mathcal{Q}}(0,y) &= y+1 \\ \psi_{\mathcal{Q}}(x+1,y) &= y + \psi_{\mathcal{Q}}(x,\phi_{\mathcal{Q}}(y)) \end{aligned}$$

One easily verifies that $\psi_{\mathcal{Q}}$ is strictly monotonic in both arguments.

One more auxiliary function is needed before we can define the interpretation of function symbol A. The interpretation defined above has the property that for a ground term t not containing any A symbol, the top part of t that consists of symbols in $\{\mathsf{cons},\overline{\mathsf{cons}},\varepsilon,\$,0,1\}$ can be extracted from $[t]$. A suitable extraction function π is defined next.

Definition 5.5 The function $\pi \colon \mathbb{N}_+ \to \mathcal{T}(\mathcal{F})$ from positive integers to ground terms is inductively defined as follows:

$$\pi(x) = \begin{cases} \varepsilon & \text{if } x = 1 \\ \$ & \text{if } x = 2 \\ 0 & \text{if } x = 3 \\ 1 & \text{if } x = 4 \\ \mathsf{cons}(\pi(y),\pi(z)) & \text{if } x = z \circ 5 \circ y \circ 6 \text{ with } y > 0 \text{ well-balanced} \\ \overline{\mathsf{cons}}(\pi(y),\pi(z)) & \text{if } x = 5 \circ y \circ 6 \circ z \circ 7 \uparrow (2+\ell(y)) \text{ with } y > 0 \text{ well-balanced} \\ A(\varepsilon,\ldots,\varepsilon) & \text{otherwise} \end{cases}$$

Here for well-definedness we require that the digits 5 ("left parenthesis") and 6 ("right parenthesis") form a well-balanced sequence in the decimal representation of y in the fifth and sixth clause of the definition. Formally, the decimal representation of a natural number is well-balanced if it is generated by the context-free grammar

$$\begin{array}{rcl} S & \rightarrow & 5S6 \mid SS \mid T \mid 56 \\ T & \rightarrow & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 7 \mid 8 \mid 9 \end{array}$$

with start symbol S. For example, 123 and 1556576236 are well-balanced numbers but 65 is not.

The premise on y in the fifth and sixth clause of the definition ensures well-definedness of the definition of π.

Lemma 5.6 *The function π is well-defined.*

Proof First we show that there is no ambiguity in the fifth clause of the definition of π. Suppose to the contrary that there exists $x \in \mathbb{N}_+$ such that $x = z \circ 5 \circ y \circ 6 = z' \circ 5 \circ y' \circ 6$ with different well-balanced $y, y' > 0$. Without loss of generality, assume that $y > y'$. Then $y = z'' \circ 5 \circ y'$ for some $z'' \in \mathbb{N}$. However, since y' is well-balanced, y cannot be well-balanced (as there is no 6 in y' that corresponds to the displayed 5 in y). A similar argument shows that there is no ambiguity in the sixth clause of the definition of π. $\square$

For instance,

$$\begin{aligned} \pi(156655635626) &= \pi(1566 \circ 5 \circ 563562 \circ 6) = \mathsf{cons}(\pi(563562), \pi(1566)) \\ &= \mathsf{cons}(A(\varepsilon, \ldots, \varepsilon), A(\varepsilon, \ldots, \varepsilon)) \end{aligned}$$

and $\pi(5462777536) = \mathsf{cons}(\pi(3), \pi(5462777)) = \mathsf{cons}(0, \overline{\mathsf{cons}}(1, \$))$.

We need two more definitions:

$$\begin{aligned} \mathsf{revc}(x, y) &= x \circ y \circ 7 \uparrow \ell(x) \\ \mathsf{bound}(x, y) &= 6\ell(x) + 3\ell(y) \end{aligned}$$

The function revc (for "reverse concatenation") is strictly monotonic and bound is monotonic in both arguments.

Definition 5.7 The interpretation $[A]$ is defined as follows:

$$[A](x_1, \ldots, x_{2n+15}) = \mathsf{code}(\psi_{\mathcal{Q}}(D(x_1, \ldots, x_{2n+14}), x_{2n+15})) \circ 8$$

Here $D(x_1, \ldots, x_{2n+14})$ denotes the expression

$$\left(\prod_{i=1}^{2n+8} \chi(x_i \geq [V_i])\right) \cdot E(x_1, \ldots, x_{2n+14}) + \left(1 - \prod_{i=1}^{2n+8} \chi(x_i \geq [V_i])\right) \cdot \left(\sum_{i=1}^{2n+14} x_i\right)$$

$\chi\colon \{\mathsf{false}, \mathsf{true}\} \to \{0,1\}$ is defined by $\chi(\mathsf{false}) = 0$ and $\chi(\mathsf{true}) = 1$, and $E(x_1, \dots, x_{2n+14})$ denotes the expression

$$\mathsf{len}(\pi(x_{2n+9}), \dots, \pi(x_{2n+14})) + \mathsf{bound}(x_{2n+11}, x_{2n+12}) \cdot \mathsf{factor}(x_1, \dots, x_{2n+14})$$

where $\mathsf{factor}(x_1, \dots, x_{2n+14})$ is an abbreviation for

$$2(x_{2n+9} + x_{2n+10}) + \mathsf{revc}(x_{2n+11}, x_{2n+12}) + \mathsf{revc}(x_{2n+13}, x_{2n+14}) + \sum_{i=1}^{2n+8} x_i$$

Note that $D(x_1, \dots, x_{2n+14}) = E(x_1, \dots, x_{2n+14})$ if $x_i \geqslant [V_i]$ for all $1 \leqslant i \leqslant 2n+8$. If there is at least one $1 \leqslant i \leqslant 2n+8$ such that $x_i < [V_i]$, then $D(x_1, \dots, x_{2n+14}) = \sum_{i=1}^{2n+14} x_i$.

Lemma 5.8 *For every ground term t the decimal representation of its interpretation $[t]$ has a well-balanced sequence of 5 and 6 digits.*

Proof Easy induction on the structure of t. If $t \in \{\varepsilon, \$, 0, 1\}$ then the decimal representation of $[t]$ does not contain any 5 or 6. If $t = \mathsf{cons}(t_1, t_2)$ or $t = \overline{\mathsf{cons}}(t_1, t_2)$ then the result follows from the induction hypothesis. If $t = A(t_1, \dots, t_{2n+15})$ or $t = f(t_1, \dots, t_k)$ with $f \in \mathcal{F}_{\mathcal{Q}}$ then the decimal representation of $[t]$ does not contain any 5 or 6 by the definition of the function code. □

The next lemma states that $\pi([t])$ is sufficiently close to t.

Definition 5.9 Let $\sim$ be the smallest congruence on ground terms such that $t \sim t'$ holds if $\mathrm{root}(t), \mathrm{root}(t') \in \{A\} \cup \mathcal{F}_{\mathcal{Q}}$. In other words, $t \sim t'$ if the top parts consisting of symbols in $\{\varepsilon, \$, 0, 1, \mathsf{cons}, \overline{\mathsf{cons}}\}$ in t and t' coincide. Let us call a term over the restricted signature $\{\varepsilon, \$, 0, 1, \mathsf{cons}, \overline{\mathsf{cons}}\}$ *pure*. We extend $\sim$ to sequences of terms componentwise.

For instance, if $f \in \mathcal{F}_{\mathcal{Q}}$ then $\mathsf{cons}(0, \mathsf{cons}(1, A(\dots))) \sim \mathsf{cons}(0, \mathsf{cons}(1, f(\dots)))$ for all sequences of arguments of A and f. On the other hand, $\overline{\mathsf{cons}}(0, \varepsilon) \not\sim \overline{\mathsf{cons}}(1, \varepsilon)$.

Lemma 5.10 *If t is a ground term then $\pi([t]) \sim t$. In addition, if t is pure then $\pi([t]) = t$.*

Proof Let t be a ground term. We prove that $\pi([t]) \sim t$ by induction on the structure of t. The base case is an immediate consequence of the definitions of $[\,]$ and π. Suppose $t = \mathsf{cons}(t_1, t_2)$. We have $[t] = [t_2] \circ 5 \circ [t_1] \circ 6$. According to Lemma 5.8 the subsequence of the digits 5 and 6 in $[t_1]$ is well-balanced. Hence $\pi([t]) = \mathsf{cons}(\pi([t_1]), \pi([t_2])) \sim \mathsf{cons}(t_1, t_2)$ by the definition of π and the induction hypothesis. The case $t = \overline{\mathsf{cons}}(t_1, t_2)$ is just as easy. If $\mathrm{root}(t) = A$ or $\mathrm{root}(t) \in \mathcal{F}_{\mathcal{Q}}$ then the decimal representation of $[t]$ ends with the digit 8 or 0. Hence $\pi([t]) = A(\varepsilon, \dots, \varepsilon)$ and thus $\pi([t]) \sim t$ by the definition of $\sim$.

To conclude the latter statement, according to the former statement and the definition of $\sim$ it is sufficient to show that $\pi([t])$ is pure whenever t is pure. This is easily proved by induction on the structure of t, similar to the above proof. □

Lemma 5.11 *If $W \sim W'$ then* $\mathsf{len}(W) = \mathsf{len}(W')$.

Proof Let $A(V, W, t) \to A(V_1, W_1, t_1)$ and $W \sim W'$. Due to the fact that the arguments of the left-hand side of the rewrite rules in $\mathcal{U}'(P, \mathcal{Q})$ are terms over the signature $\{\varepsilon, \$, 0, 1, \mathsf{cons}, \overline{\mathsf{cons}}\}$, it follows that $A(V, W', t)$ also matches the left-hand side. If we apply the same rewrite rule, we obtain $A(V, W', t) \to A(V_1', W_1', t_1)$ with $V_1 \sim V_1'$ and $W_1 \sim W_1'$. If one of V_1, V_1' equals V then both are equal to V. From this observation we easily obtain $\mathsf{len}(W) = \mathsf{len}(W')$. □

Next we are going to show that the interpretation functions $[f]$ for $f \in \mathcal{F}_{\mathcal{U}}$ are strictly monotonic in all arguments. The proof of this statement for function symbol A relies on the following lemma.

Lemma 5.12 *For all $x_1, \ldots, x_6 \in \mathbb{N}_+$,* $\mathsf{len}(\pi(x_1), \ldots, \pi(x_6)) < \mathsf{bound}(x_3, x_4)$.

Proof First we show that $\|\pi(x)\| \leqslant \ell(x)$ by induction on $x \in \mathbb{N}_+$ according to the definition of π. If $\pi(x) \in \{\varepsilon, \$, 0, 1, A(\varepsilon, \ldots, \varepsilon)\}$ then $\|\pi(x)\| = 0$. Suppose $x = z \circ 5 \circ y \circ 6$ with $y > 0$ being well-balanced, so $\pi(x) = \mathsf{cons}(\pi(y), \pi(z))$. We have $\|\pi(x)\| = 1 + \|\pi(z)\|$ if $\pi(y) \in \{0, 1\}$ and $\|\pi(x)\| = 0$ otherwise. (Here we exploit the fact that ζ in the definition of $\|t\|$ is a mixed string.) In the former case we obtain the desired $\|\pi(x)\| \leqslant \ell(x)$ from the induction hypothesis (applied to z). In the latter case the inequality $\|\pi(x)\| \leqslant \ell(x)$ is trivial. If $x = 5 \circ y \circ 6 \circ z \circ 7 \uparrow (2 + \ell(y))$ with $y > 0$ well-balanced, and thus $\pi(x) = \overline{\mathsf{cons}}(\pi(y), \pi(z))$, we obtain $\|\pi(x)\| \leqslant \ell(x)$ in exactly the same way. Using Corollary 5.3 we now obtain

$$\begin{aligned}
\mathsf{len}(\pi(x_1), \ldots, \pi(x_6)) &\leqslant 1 + 4\|\pi(x_3)\| + 3\|\pi(x_4)\| \\
&\leqslant 1 + 4\ell(x_3) + 3\ell(x_4) \\
&< 6\ell(x_3) + 3\ell(x_4) \qquad (6) \\
&= \mathsf{bound}(x_3, x_4)
\end{aligned}$$

Here (6) follows from the fact that $x_3 > 0$ and thus $\ell(x_3) > 0$. □

Lemma 5.13 *For every $f \in \mathcal{F}_{\mathcal{U}}$, the interpretation function $[f]$ is strictly monotonic in all its arguments.*

Proof For constants in $\mathcal{F}_{\mathcal{U}}$ there is nothing to show. For $[\mathsf{cons}]$ and $[\overline{\mathsf{cons}}]$ the result follows directly from the strict monotonicity of $\circ$. We already observed that every $[f]$ with $f \in \mathcal{F}_{\mathcal{Q}}$ is strictly monotonic. For function symbol A more effort is required. The strict monotonicity of $[A]$ in its last argument follows from the strict monotonicity of $\psi_{\mathcal{Q}}$, code, and $\circ$. For the other arguments we reason as follows. Let $x_i \geqslant y_i$ for $1 \leqslant i \leqslant 2n + 14$ where at least one of these inequalities is strict. We distinguish three cases. If

$$\prod_{i=1}^{2n+8} \chi(y_i \geqslant [V_i]) = 1$$

then also

$$\prod_{i=1}^{2n+8} \chi(x_i \geqslant [V_i]) = 1$$

because $x_i \geqslant y_i$. We have

$$\mathsf{factor}(x_1, \ldots x_{2n+14}) > \mathsf{factor}(y_1, \ldots, y_{2n+14})$$

by the strict monotonicity of revc and $+$, and

$$\mathsf{bound}(x_{2n+11}, x_{2n+12}) \geqslant \mathsf{bound}(y_{2n+11}, y_{2n+12})$$

by the monotonicity of bound. Using Lemma 5.12, it follows that

$$\begin{aligned}
&E(x_1, \ldots, x_{2n+14}) \\
&= \mathsf{len}(\pi(x_{2n+9}), \ldots, \pi(x_{2n+14})) + \mathsf{bound}(x_{2n+11}, x_{2n+12}) \cdot \mathsf{factor}(x_1, \ldots, x_{2n+14}) \\
&\geqslant \mathsf{bound}(x_{2n+11}, x_{2n+12}) \cdot \mathsf{factor}(x_1, \ldots, x_{2n+14}) \\
&\geqslant \mathsf{bound}(y_{2n+11}, y_{2n+12}) \cdot \mathsf{factor}(x_1, \ldots, x_{2n+14}) \\
&\geqslant \mathsf{bound}(y_{2n+11}, y_{2n+12}) \cdot (1 + \mathsf{factor}(y_1, \ldots, y_{2n+14})) \\
&= \mathsf{bound}(y_{2n+11}, y_{2n+12}) + \mathsf{bound}(y_{2n+11}, y_{2n+12}) \cdot \mathsf{factor}(y_1, \ldots, y_{2n+14}) \\
&> \mathsf{len}(\pi(y_{2n+9}), \ldots, \pi(y_{2n+14})) + \mathsf{bound}(y_{2n+11}, y_{2n+12}) \cdot \mathsf{factor}(y_1, \ldots, y_{2n+14}) \\
&= E(y_1, \ldots, y_{2n+14})
\end{aligned}$$

and so, by the strict monotonicity of $\psi_{\mathcal{Q}}$, code, and $\circ$,

$$\begin{aligned}
[A](x_1, \ldots, x_{2n+14}, x_{2n+15}) &= \mathsf{code}(\psi_{\mathcal{Q}}(E(x_1, \ldots, x_{2n+14}), x_{2n+15})) \circ 8 \\
&> \mathsf{code}(\psi_{\mathcal{Q}}(E(y_1, \ldots, y_{2n+14}), x_{2n+15})) \circ 8 \\
&= [A](y_1, \ldots, y_{2n+14}, x_{2n+15})
\end{aligned}$$

Suppose

$$\prod_{i=1}^{2n+8} \chi(y_i \geqslant [V_i]) = \prod_{i=1}^{2n+8} \chi(x_i \geqslant [V_i]) = 0$$

We have

$$D(x_1, \ldots, x_{2n+14}) = \sum_{i=1}^{2n+14} x_i > \sum_{i=1}^{2n+14} y_i = D(y_1, \ldots, y_{2n+14})$$

and hence the assertion follows from the strict monotonicity of $\psi_{\mathcal{Q}}$, code, and $\circ$. Finally, suppose

$$\prod_{i=1}^{2n+8} \chi(y_i \geqslant [V_i]) = 0$$

and

$$\prod_{i=1}^{2n+8} \chi(x_i \geqslant [V_i]) = 1$$

In this case $D(x_1, \ldots, x_{2n+14})$ equals

$$\mathsf{len}(\pi(x_{2n+9}), \ldots, \pi(x_{2n+14})) + \mathsf{bound}(x_{2n+11}, x_{2n+12}) \cdot \mathsf{factor}(x_1, \ldots, x_{2n+14})$$

and

$$D(y_1, \ldots, y_{2n+14}) = \sum_{i=1}^{2n+14} y_i$$

Since $\mathsf{revc}(x_{2n+11}, x_{2n+12}) \geqslant x_{2n+11} + x_{2n+12}$ and $\mathsf{revc}(x_{2n+13}, x_{2n+14}) \geqslant x_{2n+13} + x_{2n+14}$, it follows that

$$\mathsf{factor}(x_1, \ldots, x_{2n+14}) \geqslant \sum_{i=1}^{2n+14} x_i$$

and thus $D(x_1, \ldots, x_{2n+14}) > D(y_1, \ldots, y_{2n+14})$. The desired result now follows from the strict monotonicity of $\psi_{\mathcal{Q}}$, code, and $\circ$. $\square$

In the final part of this section we will make good on our claim that the interpretation [ ] is capable of orienting all ground instances of the rewrite rules in $\mathcal{U}'(P, \mathcal{Q})$ from left to right. We need a few preliminary results, concerning the interplay of revc and [ ].

Lemma 5.14 *Let α be a sequence of ground terms. For all ground terms s and t we have* $\mathsf{revc}([\alpha(s)], [t]) = \mathsf{revc}([s], [\overline{\alpha}(t)])$.

Proof By induction on the length of α. If α is the empty sequence, then the lemma is trivially true. Let $\alpha = u\beta$. Then

$$\begin{aligned}
\mathsf{revc}([\alpha(s)], [t]) &= \mathsf{revc}([\mathsf{cons}(u, \beta(s))], [t]) \\
&= \mathsf{revc}([\mathsf{cons}]([u], [\beta(s)]), [t]) \\
&= \mathsf{revc}([\beta(s)] \circ 5 \circ [u] \circ 6, [t]) \\
&= [\beta(s)] \circ 5 \circ [u] \circ 6 \circ [t] \circ 7 \uparrow \ell([\beta(s)] \circ 5 \circ [u] \circ 6) \\
&= \mathsf{revc}([\beta(s)], 5 \circ [u] \circ 6 \circ [t] \circ 7 \uparrow \ell(5 \circ [u] \circ 6)) \\
&= \mathsf{revc}([\beta(s)], [\overline{\mathsf{cons}}]([u], [t])) \\
&= \mathsf{revc}([\beta(s)], [\overline{\mathsf{cons}}(u, t)]) \\
&= \mathsf{revc}([s], [\overline{\beta}(\overline{\mathsf{cons}}(u, t))]) && (7) \\
&= \mathsf{revc}([s], [\overline{\beta}(\overline{u}(t))]) \\
&= \mathsf{revc}([s], [\overline{u\beta}(t)]) \\
&= \mathsf{revc}([s], [\overline{\alpha}(t)])
\end{aligned}$$

Here (7) follows from the induction hypothesis. $\square$

Lemma 5.15 *Let s, t, and u be ground terms and x a positive integer. If $[s] \geqslant [t]$ then*

$$\mathsf{revc}([\mathsf{cons}(s,u)],x) + [t] \geqslant \mathsf{revc}([\mathsf{cons}(t,u)],x) + [s]$$
$$\mathsf{revc}(x,[\overline{\mathsf{cons}}(s,u)]) + [t] \geqslant \mathsf{revc}(x,[\overline{\mathsf{cons}}(t,u)]) + [s]$$

Moreover, if $[s] > [t]$ then both inequalities are strict.

Proof The first statement is obtained as follows:

$$\begin{aligned}
\mathsf{revc}([\mathsf{cons}(s,u)],x) + [t] &= [\mathsf{cons}(s,u)] \circ x \circ 7 \uparrow \ell([\mathsf{cons}(s,u)]) + [t] \\
&= [u] \circ 5 \circ [s] \circ 6 \circ x \circ 7 \uparrow \ell([\mathsf{cons}(s,u)]) + [t] \\
&\geqslant [u] \circ 5 \circ [t] \circ 6 \circ x \circ 7 \uparrow \ell([\mathsf{cons}(s,u)]) + [s] && (8) \\
&\geqslant [u] \circ 5 \circ [t] \circ 6 \circ x \circ 7 \uparrow \ell([\mathsf{cons}(t,u)]) + [s] && (9) \\
&= [\mathsf{cons}(t,u)] \circ x \circ 7 \uparrow \ell([\mathsf{cons}(t,u)]) + [s] \\
&= \mathsf{revc}([\mathsf{cons}(t,u)],x) + [s]
\end{aligned}$$

Here (8) follows from the fact that, for all $x, y, z \in \mathbb{N}_+$, $x \circ y \circ z + y' \geqslant x \circ y' \circ z + y$ whenever $y \geqslant y'$ and (9) follows from the monotonicity of ℓ, $\circ$, and $7^{\circ(\cdot)}$. If $[s] > [t]$ then (9) becomes strict, and so

$$\mathsf{revc}([\mathsf{cons}(s,u)],x) + [t] > \mathsf{revc}([\mathsf{cons}(t,u)],x) + [s]$$

The other two statements are obtained in a similar fashion. $\square$

The last preliminary result is a variant of the previous lemma.

Lemma 5.16 *Let α and β be sequences of ground terms, t a ground term, and x a positive integer. If $[\alpha(\varepsilon)] \geqslant [\beta(\varepsilon)]$ then*

$$\mathsf{revc}([\alpha(t)],x) + [\beta(\varepsilon)] \geqslant \mathsf{revc}([\beta(t)],x) + [\alpha(\varepsilon)]$$

Moreover, if $[\alpha(\varepsilon)] > [\beta(\varepsilon)]$ then the inequality is strict.

Proof Write $\alpha = s_1 \ldots s_k$ and $\beta = t_1 \ldots t_l$. We have

$$\begin{aligned}
[\alpha(\varepsilon)] &= 1 \circ 5 \circ [s_k] \circ 6 \cdots \circ 5 \circ [s_1] \circ 6 \\
&\geqslant 1 \circ 5 \circ [t_l] \circ 6 \cdots \circ 5 \circ [t_1] \circ 6 = [\beta(\varepsilon)]
\end{aligned}$$

and thus

$$a = 5 \circ [s_k] \circ 6 \cdots \circ 5 \circ [s_1] \circ 6 \geqslant 5 \circ [t_l] \circ 6 \cdots \circ 5 \circ [t_1] \circ 6 = b$$

which implies $[\alpha(t)] = [t] \circ a \geqslant [t] \circ b = [\beta(t)]$. Now the desired inequality is obtained as in the proof of Lemma 5.15:

$$\begin{aligned}
\mathsf{revc}([\alpha(t)],x) + [\beta(\varepsilon)] &= [\alpha(t)] \circ x \circ 7 \uparrow \ell([\alpha(t)]) + [\beta(\varepsilon)] \\
&= [t] \circ a \circ x \circ 7 \uparrow \ell([\alpha(t)]) + 1 \circ b \\
&\geqslant [t] \circ b \circ x \circ 7 \uparrow \ell([\alpha(t)]) + 1 \circ a && (10) \\
&\geqslant [t] \circ b \circ x \circ 7 \uparrow \ell([\beta(t)]) + 1 \circ a \\
&= [\beta(t)] \circ x \circ 7 \uparrow \ell([\beta(t)]) + [\alpha(\varepsilon)] \\
&= \mathsf{revc}([\beta(t)],x) + [\alpha(\varepsilon)]
\end{aligned}$$

Here (10) follows from the fact that, for all $x, y, z \in \mathbb{N}_+$, $x \circ y \circ z + 1 \circ y' \geqslant x \circ y' \circ z + 1 \circ y$ whenever $y \geqslant y'$. Note that the second inequality in the above derivation becomes strict if $[\alpha(\varepsilon)] > [\beta(\varepsilon)]$. □

Theorem 5.17 *For every ground instance*

$$l\sigma = A(t_1, \ldots, t_{2n+15}) \to A(u_1, \ldots, u_{2n+15}) = r\sigma$$

of a rewrite rule $l \to r$ in $\mathcal{U}'(P, \mathcal{Q})$ we have $[l\sigma] > [r\sigma]$.

Proof First we consider the case that $[u_i] \geqslant [t_i]$ for all $1 \leqslant i \leqslant 2n+8$. By definition of $\mathcal{U}'(P, \mathcal{Q})$ we have $t_1, \ldots, t_{2n+8} = V$, $t_{2n+15} = \mathsf{LHS}\sigma$, and either $u_{2n+15} = \mathsf{LHS}\sigma$ or $u_{2n+15} = \mathsf{RHS}_j\sigma$ for some $1 \leqslant j \leqslant m$. Positive integers E_1 and F_1 are defined as follows:

$$E_1 = [t_1] + [t_2] + 2([t_{2n+9}] + [t_{2n+10}])$$
$$F_1 = [u_1] + [u_2] + 2([u_{2n+9}] + [u_{2n+10}])$$

We claim that

$$E_1 \geqslant F_1 \tag{11}$$

We claim moreover that, if $[u_i] > [t_i]$ for some $i = 1, 2$ then $E_1 > F_1$. Inspection of the rewrite rules shows that $[t_1]+[t_2] = [u_{2n+9}]+[u_{2n+10}]$, and $[t_{2n+9}]+[t_{2n+10}] = [u_1]+[u_2]$. Hence $E_1 = [t_1]+[t_2]+2([u_1]+[u_2])$ and $F_1 = [u_1]+[u_2]+2([t_1]+[t_2])$. By assumption $[u_1] + [u_2] \geqslant [t_1] + [t_2]$ and thus $E_1 \geqslant F_1$. Clearly, either $[u_1] > [t_1]$ or $[u_2] > [t_2]$ is sufficient to conclude that $E_1 > F_1$.

Positive integers E_2 and F_2 are defined as follows:

$$E_2 = \mathsf{revc}([t_{2n+11}], [t_{2n+12}]) + \sum_{i=3}^{n+5} [t_i]$$
$$F_2 = \mathsf{revc}([u_{2n+11}], [u_{2n+12}]) + \sum_{i=3}^{n+5} [u_i]$$

We claim that

$$E_2 \geqslant F_2 \tag{12}$$

Moreover we claim that, if $[u_i] > [t_i]$ for some $3 \leqslant i \leqslant n+5$ then $E_2 > F_2$. To prove this claim, we distinguish between the four types of rewrite rules.

Suppose a rule of type (I) is used. In this case the sequences of terms $t_3, \ldots, t_{n+5}$ and $u_3, \ldots, u_{n+5}$ differ only in their third terms: $t_5 = \$$ and $u_5 = x_1\sigma$. Hence $E_2 - F_2$ equals

$$\mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [\$] - (\mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{\$}(x)\sigma]) + [x_1\sigma])$$

which is non-negative according to Lemma 5.15. Recall here that $[x_1\sigma] = [u_5] \geqslant [t_5] = [\$]$ as we are in the case that $[u_i] \geqslant [t_i]$ for all $1 \leqslant i \leqslant 2n+8$. In addition to that, if $[x_1\sigma] = [u_5] > [t_5] = [\$]$ then $E_2 - F_2 > 0$.

Next suppose that a rule of type (II) is used. More precisely suppose rule $l \to r_{i+1}$ $(1 \leqslant i \leqslant n)$ is used. In this case the sequences of terms $t_3, \ldots, t_{n+5}$ and $u_3, \ldots, u_{n+5}$ differ only in their $i+3$-th terms: $t_{i+5} = \alpha_i(\varepsilon)$ and $u_{i+5} = w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma$. Hence

$$
\begin{aligned}
E_2 - F_2 = {} & \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [\alpha_i(\varepsilon)] - \\
& (\mathsf{revc}([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma], [\overline{x_1 \alpha_i}(x)\sigma]) + [w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma])
\end{aligned}
$$

From Lemmata 5.16 (with $\alpha = w_1 \ldots w_{|\alpha_i|}$, $\beta = \alpha_i$, and $t = w_{|\alpha_i|+1} \ldots w_\mu(w)$) and 5.14 it follows that

$$
\begin{aligned}
& \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [\alpha_i(\varepsilon)\sigma] \\
& \geqslant \mathsf{revc}([\alpha_i w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma] \\
& = \mathsf{revc}([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma], [\overline{\alpha_i}(\overline{x_1}(x))\sigma]) + [w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma] \\
& = \mathsf{revc}([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma], [\overline{x_1 \alpha_i}(x)\sigma]) + [w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma]
\end{aligned}
$$

and thus $E_2 - F_2 \geqslant 0$. We obtain $E_2 - F_2 > 0$ if $[u_{i+5}] > [t_{i+5}]$.

Suppose that a rule of type (III) is used. In this case the difference between the sequences of terms $t_3, \ldots, t_{n+5}$ and $u_3, \ldots, u_{n+5}$ is the third term and either the first or second term. Here we will consider the former (so a rule r_{n+1+j} with $1 \leqslant j \leqslant m$ is used); the latter is proved in exactly the same way. So $t_3 = 0$, $t_5 = \$$, $u_3 = x_1\sigma$, and $u_5 = w_1\sigma$. Hence

$$
\begin{aligned}
E_2 - F_2 = {} & \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [0] + [\$] - \\
& (\mathsf{revc}([0\$w_2 \ldots w_\mu(w)\sigma], [x\sigma]) + [x_1\sigma] + [w_1\sigma])
\end{aligned}
$$

Two applications of Lemma 5.15 and a single application of Lemma 5.14 yield

$$
\begin{aligned}
& \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [0] + [\$] \\
& \geqslant \mathsf{revc}([\$w_2 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [0] + [w_1\sigma] \\
& \geqslant \mathsf{revc}([\$w_2 \ldots w_\mu(w)\sigma], [\overline{0}(x)\sigma]) + [x_1\sigma] + [w_1\sigma] \\
& = \mathsf{revc}([0\$w_2 \ldots w_\mu(w)\sigma], [x\sigma]) + [x_1\sigma] + [w_1\sigma]
\end{aligned}
$$

and thus $E_2 - F_2 \geqslant 0$. Moreover, if $[u_3] > [t_3]$ or $[u_5] > [t_5]$ then $E_2 - F_2 > 0$.

Finally, suppose that a rule of type (IV) is used. In this case the difference between the sequences of terms $t_3, \ldots, t_{n+5}$ and $u_3, \ldots, u_{n+5}$ is either the first or the second term. We consider here the latter (so the rule r_{n+2m+3} is used); the former is proved in exactly the same way. So $t_4 = 1$ and $u_4 = x_1\sigma$. Hence

$$
\begin{aligned}
E_2 - F_2 = {} & \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [1] - \\
& (\mathsf{revc}([1w_1 \ldots w_\mu(w)\sigma], [x\sigma]) + [x_1\sigma])
\end{aligned}
$$

From Lemmata 5.15 (recall that $[x_1\sigma] = [u_4] \geqslant [t_4] = [1]$) and 5.14 we obtain

$$
\begin{aligned}
& \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [1] \\
& \geqslant \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{1}(x)\sigma]) + [x_1\sigma] \\
& = \mathsf{revc}([1w_1 \ldots w_\mu(w)\sigma], [x\sigma]) + [x_1\sigma]
\end{aligned}
$$

and thus $E_2 - F_2 \geqslant 0$. Moreover, if $[u_4] > [t_4]$ then $E_2 - F_2 > 0$.

This concludes the proof of claim (12). Positive integers E_3 and F_3 are defined as follows:

$$E_3 = \mathsf{revc}([t_{2n+13}], [t_{2n+14}]) + \sum_{i=n+6}^{2n+8} [t_i]$$

$$F_3 = \mathsf{revc}([u_{2n+13}], [u_{2n+14}]) + \sum_{i=n+6}^{2n+8} [u_i]$$

We claim that

$$E_3 \geqslant F_3 \tag{13}$$

Moreover, if $[u_i] > [t_i]$ for some $n + 6 \leqslant i \leqslant 2n + 8$ then $E_3 > F_3$. The proof of this claim is very similar to the proof of (12) and hence omitted.

From (11), (12), and (13) we immediately obtain

$$\begin{aligned} \mathsf{factor}([t_1], \ldots, [t_{2n+14}]) &= E_1 + E_2 + E_3 \\ &\geqslant F_1 + F_2 + F_3 = \mathsf{factor}([u_1], \ldots, [u_{2n+14}]) \end{aligned} \tag{14}$$

and if additionally $[u_i] > [t_i]$ for some $1 \leqslant i \leqslant 2n + 8$ then

$$\mathsf{factor}([t_1], \ldots, [t_{2n+14}]) > \mathsf{factor}([u_1], \ldots, [u_{2n+14}]) \tag{15}$$

From (12) we easily obtain

$$\mathsf{revc}([t_{2n+11}], [t_{2n+12}]) \geqslant \mathsf{revc}([u_{2n+11}], [u_{2n+12}])$$

Hence

$$2\ell([t_{2n+11}]) + \ell([t_{2n+12}]) \geqslant 2\ell([u_{2n+11}]) + \ell([u_{2n+12}])$$

by the monotonicity of ℓ and the fact that $\ell(\mathsf{revc}(x, y)) = 2\ell(x) + \ell(y)$ for all $x, y \in \mathbb{N}_+$. Therefore

$$\mathsf{bound}([t_{2n+11}], [t_{2n+12}]) \geqslant \mathsf{bound}([u_{2n+11}], [u_{2n+12}]) \tag{16}$$

Now if $[u_i] > [t_i]$ for some $1 \leqslant i \leqslant 2n + 8$ then the statement of the theorem follows from (15) and (16) as in the proof of Lemma 5.13. Otherwise, we have $[u_i] = [t_i]$ for all $1 \leqslant i \leqslant 2n + 8$. From the first part of Lemma 5.10 we obtain $u_i \sim \pi([u_i]) = \pi([t_i]) \sim t_i = V_i$. Since V_i is a pure ground term, the second part yields $\pi([u_i]) = V_i$ and hence $u_i = V_i$ by the definition of $\sim$. Hence $r\sigma = A(V, u_{2n+9}, \ldots, u_{2n+15})$ and therefore

$$\mathsf{len}(t_{2n+9}, \ldots, t_{2n+14}) > \mathsf{len}(u_{2n+9}, \ldots, u_{2n+14}) \tag{17}$$

by the definition of len. From (14), (16), and (17) we obtain

$$\begin{aligned} D([t_1], \ldots, [t_{2n+14}]) &= E([t_1], \ldots, [t_{2n+14}]) \\ &> E([u_1], \ldots, [u_{2n+14}]) = D([u_1], \ldots, [u_{2n+14}]) \end{aligned}$$

With help of the strict monotonicity of the various functions, we now obtain

$$\begin{aligned}
[l\sigma] &= \mathsf{code}(\psi_{\mathcal{Q}}(D([t_1],\ldots,[t_{2n+14}]),[t_{2n+15}])) \circ 8 \\
&\geqslant \mathsf{code}(\psi_{\mathcal{Q}}(D([u_1],\ldots,[u_{2n+14}])+1,[t_{2n+15}])) \circ 8 \\
&= \mathsf{code}([t_{2n+15}] + \psi_{\mathcal{Q}}(D([u_1],\ldots,[u_{2n+14}]),\phi_{\mathcal{Q}}([t_{2n+15}]))) \circ 8 \\
&> \mathsf{code}(\psi_{\mathcal{Q}}(D([u_1],\ldots,[u_{2n+14}]),\phi_{\mathcal{Q}}([t_{2n+15}]))) \circ 8 \\
&\geqslant \mathsf{code}(\psi_{\mathcal{Q}}(D([u_1],\ldots,[u_{2n+14}]),[u_{2n+15}])) \circ 8 \\
&= [r\sigma]
\end{aligned}$$

Note that $\phi_{\mathcal{Q}}([t_{2n+15}]) \geqslant [u_{2n+15}]$ because either $t_{2n+15} \to u_{2n+15}$ is a ground instance of a rewrite rule in $\mathcal{Q}$ in which case the inequality follows from the definition of $\phi_{\mathcal{Q}}$ or $t_{2n+15} = u_{2n+15}$ in which case the inequality follows from $\phi_{\mathcal{Q}}(x) \geqslant x$.

In the second half of the proof we consider the case that $[u_i] < [t_i]$ for at least one $1 \leqslant i \leqslant 2n+8$. In this case $D([t_1],\ldots,[t_{2n+14}])$ equals

$$\mathsf{len}(\pi([t_{2n+9}]),\ldots,\pi([t_{2n+14}])) + \mathsf{bound}([t_{2n+11}],[t_{2n+12}]) \cdot \mathsf{factor}([t_1],\ldots,[t_{2n+14}])$$

and $D([u_1],\ldots,[u_{2n+14}])$ equals $\sum_{i=1}^{2n+14}[u_i]$. If we show that

$$\mathsf{factor}([t_1],\ldots,[t_{2n+14}]) \geqslant \sum_{i=1}^{2n+14} [u_i] \tag{18}$$

then $D([t_1],\ldots,[t_{2n+14}]) > D([u_1],\ldots,[u_{2n+14}])$ because $\mathsf{bound}([t_{2n+11}],[t_{2n+12}]) > 0$ and thus we obtain the desired $[l\sigma] > [r\sigma]$ as in the preceding case. The proof of (18) has the same structure as the proof of (14) above. Positive integers are defined as follows:

$$\begin{aligned}
E_1 &= [t_1] + [t_2] + 2([t_{2n+9}] + [t_{2n+10}]) \\
F_1' &= [u_1] + [u_2] + [u_{2n+9}] + [u_{2n+10}] \\
E_2 &= \mathsf{revc}([t_{2n+11}],[t_{2n+12}]) + \sum_{i=3}^{n+5}[t_i] \\
F_2' &= [u_{2n+11}] + [u_{2n+12}] + \sum_{i=3}^{n+5}[u_i] \\
E_3 &= \mathsf{revc}([t_{2n+13}],[t_{2n+14}]) + \sum_{i=n+6}^{2n+8} [t_i] \\
F_3' &= [u_{2n+13}] + [u_{2n+14}] + \sum_{i=n+6}^{2n+8} [u_i]
\end{aligned}$$

Note that $\mathsf{factor}([t_1],\ldots,[t_{2n+14}]) = E_1 + E_2 + E_3$ and

$$\sum_{i=1}^{2n+14} [u_i] = F_1' + F_2' + F_3'$$

In order to show (18) it is sufficient to show that $E_1 > F_1'$, $E_2 > F_2'$, and $E_3 > F_3'$. For every rewrite rule in $\mathcal{U}'(P, \mathcal{Q})$ we have

$$E_1 = [0] + [1] + 2([u\sigma] + [v\sigma]) > [u\sigma] + [v\sigma] + [0] + [1] = F_1'$$

We show $E_2 > F_2'$ and $E_3 > F_3'$ by distinguishing between the four types of rewrite rules. Actually, we only show $E_2 > F_2'$ for rules of type (II) and $E_3 > F_3'$ for rules of type (III). The other cases are very similar.

We start with $E_2 > F_2'$. Suppose that rule $l \to r_{i+1}$ ($1 \leqslant i \leqslant n$) is used. In this case the only difference between the sequences of terms $t_3, \ldots, t_{n+5}$ and $u_3, \ldots, u_{n+5}$ is the $i+3$-th term: $t_{i+5} = \alpha_i(\varepsilon)$ and $u_{i+5} = w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma$. Hence

$$\begin{aligned} E_2 - F_2' = {} & \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) + [\alpha_i(\varepsilon)] - \\ & ([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma] + [\overline{x_1\alpha_i}(x)\sigma] + [w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma]) \end{aligned}$$

If $[w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma] \geqslant [\alpha_i(\varepsilon)]$, then we obtain $E_2 - F_2 \geqslant 0$ from the first half of this proof (claim (12)) and therefore $E_2 - F_2' > 0$ as $\mathsf{revc}(a, b) > a + b$ for all positive integers a and b. So suppose that $[w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma] < [\alpha_i(\varepsilon)]$. Since the decimal representation of the interpretation of a term that is not a constant has at least two digits, this is possible only if $w_j\sigma$ is a constant for every $1 \leqslant j \leqslant |\alpha_i|$. This implies that the number of digits in $[w_1 \ldots w_{|\alpha_i|}(\varepsilon)\sigma]$ and $[\alpha_i(\varepsilon)]$ coincide. We have

$$E_2 - F_2' > \mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) - ([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma] + [\overline{x_1\alpha_i}(x)\sigma])$$

From Lemma 5.14 we obtain

$$\mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma]) = \mathsf{revc}([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma], [\overline{x_1 w_1 \ldots w_{|\alpha_i|}}(x)\sigma])$$

Hence $\mathsf{revc}([w_1 \ldots w_\mu(w)\sigma], [\overline{x_1}(x)\sigma])$ has

$$\ell_1 = 2 \cdot \ell([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma]) + \ell([\overline{x_1 w_1 \ldots w_{|\alpha_i|}}(x)\sigma])$$

digits. On the other hand, $[w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma] + [\overline{x_1\alpha_i}(x)\sigma]$ has at most

$$\ell_2 = 1 + \max\{\ell([w_{|\alpha_i|+1} \ldots w_\mu(w)\sigma]), \ell([\overline{x_1\alpha_i}(x)\sigma])\}$$

digits. Because $\ell([\overline{x_1 w_1 \ldots w_{|\alpha_i|}}(x)\sigma]) = \ell([\overline{x_1\alpha_i}(x)\sigma])$ it follows that $\ell_1 > \ell_2$ and thus $E_2 > F_2'$.

Next we show that $E_3 > F_3'$ for rules of type (III). In this case the difference between the sequences of terms $t_{n+6}, \ldots, t_{2n+8}$ and $u_{n+6}, \ldots, u_{2n+8}$ is the third term and either the first or second term. We consider here the latter (so a rule $r_{n+1+m+j}$ with $1 \leqslant j \leqslant m$ is used); the former is proved in exactly the same way. So $t_{n+7} = 1$, $t_{n+8} = \$$, $u_{n+7} = z_1\sigma$, and $u_{n+8} = y_1\sigma$. Hence

$$\begin{aligned} E_3 - F_3' = {} & \mathsf{revc}([y_1 \ldots y_\mu(y)\sigma], [\overline{z_1}(z)\sigma]) + [1] + [\$] - \\ & ([1\$y_2 \ldots y_\mu(y)\sigma] + [z\sigma] + [z_1\sigma] + [y_1\sigma]) \end{aligned}$$

If both $[y_1\sigma] \geqslant [\$]$ and $[z_1\sigma] \geqslant [1]$ then we obtain

$$\begin{aligned}
&\mathsf{revc}([y_1 \dots y_\mu(y)\sigma], [\overline{z_1}(z)\sigma]) + [1] + [\$] \\
&\geqslant \mathsf{revc}([\$y_2 \dots y_\mu(y)\sigma], [\overline{z_1}(z)\sigma]) + [1] + [y_1\sigma] \\
&\geqslant \mathsf{revc}([\$y_2 \dots y_\mu(y)\sigma], [\overline{1}(z)\sigma]) + [z_1\sigma] + [y_1\sigma] \\
&= \mathsf{revc}([1\$y_2 \dots y_\mu(y)\sigma], [z\sigma]) + [z_1\sigma] + [y_1\sigma] \\
&> [1\$y_2 \dots y_\mu(y)\sigma] + [z\sigma] + [z_1\sigma] + [y_1\sigma]
\end{aligned}$$

by two applications of Lemma 5.15, a single application of Lemma 5.14, and the fact that $\mathsf{revc}(a, b) > a + b$ for all positive integers a and b. Consequently $E_3 - F'_3 \geqslant 0$. If neither $[y_1\sigma] \geqslant [\$]$ nor $[z_1\sigma] \geqslant [1]$ then $y_1\sigma$ and $z_1\sigma$ are constants. From Lemma 5.14 we obtain

$$\mathsf{revc}([y_1 \dots y_\mu(y)\sigma], [\overline{z_1}(z)\sigma]) = \mathsf{revc}([z_1 y_1 \dots y_\mu(y)\sigma], [z\sigma])$$

Hence $\mathsf{revc}([y_1 \dots y_\mu(y)\sigma], [\overline{z_1}(z)\sigma]) + [1] + [\$]$ has at least

$$\ell_1 = 2 \cdot \ell([z_1 y_1 \dots y_\mu(y)\sigma]) + \ell([z\sigma])$$

digits. On the other hand, since $[z_1\sigma] + [y_1\sigma] \leqslant [1] + [\$] = 6$, $[1\$y_2 \dots y_\mu(y)\sigma] + [z\sigma] + [z_1\sigma] + [y_1\sigma]$ has at most

$$\ell_2 = 1 + \max\{\ell([1\$y_2 \dots y_\mu(y)\sigma]), \ell([z\sigma])\}$$

digits. Because $\ell([z_1 y_1 \dots y_\mu(y)\sigma]) = \ell([1\$y_2 \dots y_\mu(y)\sigma])$ it follows that $\ell_1 > \ell_2$ and thus $E_3 > F'_3$. If either $[y_1\sigma] \geqslant [\$]$ or $[z_1\sigma] \geqslant [1]$ then we obtain the desired $E_3 > F'_3$ by combining the argumentation for the preceding two cases. □

Theorem 5.18 *The TRS $\mathcal{U}'(P, \mathcal{Q})$ is ω-terminating.*

Proof Let $A = \{[t] \mid t \in \mathcal{T}(\mathcal{F}_\mathcal{U})\}$ be the set of all natural numbers that are the interpretation of some ground term. Note that the interpretation functions $[f]$ for $f \in \mathcal{F}_\mathcal{U}$ are well-defined on A. According to the preceding theorem, $\mathcal{U}'(P, \mathcal{Q})$ is compatible with $(A, >)$ and thus ω-terminating by definition. □

Corollary 5.19 *The TRS $\mathcal{U}(P, \mathcal{Q})$ is ω-terminating if P has no solution.* □

Corollary 5.20 *The TRS $\mathcal{U}_-(P, \mathcal{Q})$ is ω-terminating for every PCP instance P.* □

6 One-Rule Term Rewriting Systems

Transforming $\mathcal{U}(P, \mathcal{Q})$ into a single-rule TRS $\mathcal{S}(P, \mathcal{Q})$ is easy: We define $\mathcal{S}(P, \mathcal{Q})$ as the rule

$$l \to B(r_1, \dots, r_{n+2m+3})$$

where B is a fresh function symbol of arity $n + 2m + 3$.

The transformation from $\mathcal{S}(P,\mathcal{Q})$ to $\mathcal{U}(P,\mathcal{Q})$ is an instance of the *distribution elimination* technique of Zantema [25]. Below we make use of the following results. (Actually, the right-linearity requirement can be dropped from the first [20] and third [25] statements.)

Lemma 6.1 *Let $\mathcal{Q}$ be a right-linear TRS.*

1. *If $\mathcal{U}(P,\mathcal{Q})$ is terminating then $\mathcal{S}(P,\mathcal{Q})$ is terminating.*
2. *$\mathcal{U}(P,\mathcal{Q})$ is simply terminating if and only if $\mathcal{S}(P,\mathcal{Q})$ is simply terminating.*
3. *$\mathcal{U}(P,\mathcal{Q})$ is totally terminating if and only if $\mathcal{S}(P,\mathcal{Q})$ is totally terminating.*

Proof Since $\mathcal{U}(P,\mathcal{Q})$ inherits right-linearity from $\mathcal{Q}$, this is an immediate consequence of [25, Theorem 12] (by noting that $\mathcal{U}(P,\mathcal{Q}) = E_B(\mathcal{S}(P,\mathcal{Q}))$). $\square$

We would like to strengthen the last statement of the preceding lemma to ω-termination. One direction is easy.

Lemma 6.2 *If $\mathcal{S}(P,\mathcal{Q})$ is ω-terminating then $\mathcal{U}(P,\mathcal{Q})$ is ω-terminating.*
Proof By definition $l >_{\mathcal{A}} B(r_1,\ldots,r_{n+2m+3})$ for some monotone algebra $\mathcal{A} = (A,>)$ with $A \subseteq \mathbb{N}$. In [25, Proposition 7] it is shown that every well-founded monotone algebra $\mathcal{A} = (A,>)$ with the property that the order $>$ is total on A is simple. Hence $B(r_1,\ldots,r_{n+2m+3}) \geqslant_{\mathcal{A}} r_i$ and thus $l >_{\mathcal{A}} r_i$ for every $1 \leqslant i \leqslant n+2m+3$. We conclude that $\mathcal{A}$ is compatible with $\mathcal{U}(P,\mathcal{Q})$. In other words, $\mathcal{U}(P,\mathcal{Q})$ is ω-terminating. $\square$

We do not know whether the reverse direction holds for ω-termination. The following partial result however suffices for our purposes.

Lemma 6.3 *The TRS $\mathcal{S}(P,\mathcal{Q})$ is ω-terminating if P does not have a solution.*
Proof We refine the interpretation $[\ ]$ that was used in the previous section to show ω-termination of $\mathcal{U}(P,\mathcal{Q})$ into an interpretation $[\![\ ]\!]$ in the positive integers that orients $\mathcal{S}(P,\mathcal{Q})$. The interpretation of every k-ary function symbol $f \in \mathcal{F}_{\mathcal{U}} \setminus \{A\}$ is unchanged:

$$[\![f]\!](x_1,\ldots,x_k) = [f](x_1,\ldots,x_k)$$

The interpretation $[\![A]\!]$ of A is given by

$$[\![A]\!](x_1,\ldots,x_{2n+15}) = \lambda([A](x_1,\ldots,x_{2n+15}))$$

where $\lambda\colon \mathbb{N}_+ \to \mathbb{N}_+$ is the strictly monotonic function inductively defined by

$$\lambda(x) = \begin{cases} 1 & \text{if } x = 1 \\ 10 \cdot (n+2m+3)^2 \cdot \lambda(x-1)^2 & \text{if } x > 1 \end{cases}$$

and the interpretation $[\![B]\!]$ of B is defined as

$$[\![B]\!](x_1,\ldots,x_{n+2m+3}) = 10 \cdot \mathsf{code}\left(\sum_{i=1}^{n+2m+3} x_i\right)$$

The interpretation of A is chosen in such a way that the inequality $[\![l\sigma]\!] > [\![r\sigma]\!]$ is easily proved for every ground substitution σ. The definition of $[\![B]\!]$ ensures that the interpretation $[\![t]\!]$ of every ground term t with root symbol B ends with a 0 and does not contain the digits 5 and 6. This is essential for the extension of Theorem 5.17 mentioned below. Using Lemma 5.13 we easily obtain the strict monotonicity of every interpretation function in all arguments. Let

$$l\sigma = A(t_1, \dots, t_{2n+15}) \to B(r_1\sigma, \dots, r_{n+2m+3}\sigma) = r\sigma$$

be a ground instance of the only rewrite rule of $\mathcal{S}(P, \mathcal{Q})$. We have to show that $[\![l\sigma]\!] >$ $[\![r\sigma]\!]$. Write $r_i = A(u_1^i, \dots, u_{2n+15}^i)$. By definition

$$[\![l\sigma]\!] = \lambda([A]([\![t_1]\!], \dots, [\![t_{2n+15}]\!]))$$

and

$$[\![r\sigma]\!] = 10 \cdot \mathsf{code}\left(\sum_{i=1}^{n+2m+3} \lambda([A]([\![u_1^i]\!], \dots, [\![u_{2n+15}^i]\!]))\right)$$

After extending the congruence relation $\sim$ of Definition 5.9 by defining $t \sim t'$ if $\mathrm{root}(t), \mathrm{root}(t') \in \{A, B\} \cup \mathcal{F}_{\mathcal{Q}}$, the proof of Theorem 5.17 can be reused to obtain

$$p = [A]([\![t_1]\!], \dots, [\![t_{2n+15}]\!]) > [A]([\![u_1^i]\!], \dots, [\![u_{2n+15}^i]\!]) = q_i$$

Hence

$$\sum_{i=1}^{n+2m+3} \lambda(q_i) \leqslant \sum_{i=1}^{n+2m+3} \lambda(p-1) = (n+2m+3) \cdot \lambda(p-1)$$

and thus

$$\begin{aligned}
[\![r\sigma]\!] &\leqslant 10 \cdot \mathsf{code}((n+2m+3) \cdot \lambda(p-1)) \\
&< 10 \cdot (n+2m+3)^2 \cdot \lambda(p-1)^2 \\
&= \lambda(p) \\
&= [\![l\sigma]\!]
\end{aligned}$$

because $\mathsf{code}(x) < x^2$ for every integer $x > 1$, which we show by induction on x as follows. For $x < 8$ one directly verifies that $\mathsf{code}(x) < x^2$. Suppose $x \geqslant 8$. Let y be the natural number uniquely determined by $8(y-1) \leqslant x < 8y$. We have $\mathsf{code}(x) < \mathsf{code}(8y)$ by the strict monotonicity of code. Since $(8y)_8 = 10(y)_8$ it follows that $\mathsf{code}(8y) = 10 \cdot \mathsf{code}(y)$ and hence $\mathsf{code}(x) < 10 \cdot y^2$ by the induction hypothesis. From $y \geqslant 2$ we infer that $y/(y-1) \leqslant 2$ and thus $y^2/(y-1)^2 \leqslant 4 < 64/10$. Therefore $\mathsf{code}(x) < 64 \cdot (y-1)^2 = (8 \cdot (y-1))^2 \leqslant x^2$ as desired. □

In the following we also need results about the relation between $\mathcal{U}(P, \mathcal{Q})$ and $\mathcal{S}(P, \mathcal{Q})$ for the other properties in the termination hierarchy. These results will be stated and proved in the respective sections.

The final result of this section will be used to transform rewrite sequences in $\mathcal{U}(P, \mathcal{Q})$ to rewrite sequences in $\mathcal{S}(P, \mathcal{Q})$ for PCP instances P that admit a solution.

Lemma 6.4 *If W and t do not contain A symbols and $A(V,W,t) \to^+_{\mathcal{U}(P,\mathcal{Q})} A(V',W',t')$ then $A(V,W,t) \to^+_{\mathcal{S}(P,\mathcal{Q})} C[A(V',W',t)]$ for some context C.*

Proof Straightforward induction on the length of $A(V,W,t) \to^+_{\mathcal{U}(P,\mathcal{Q})} A(V',W',t')$. $\square$

7 NL $\Rightarrow$ AC

Let $\mathcal{Q}_1 = \{d \to d\}$.

Lemma 7.1 *The TRS $\mathcal{S}(P,\mathcal{Q}_1)$ is acyclic for every PCP instance P.*

Proof Since every rewrite step in $\mathcal{S}(P,\mathcal{Q}_1)$ increases the size of terms, acyclicity is obvious. $\square$

Lemma 7.2 *The TRS $\mathcal{S}(P,\mathcal{Q}_1)$ is non-looping if and only if P admits no solution.*

Proof If P admits a solution then there exists a sextuple W such that $A(V,W,d) \to^+_{\mathcal{U}(P,\mathcal{Q}_1)} A(V,W,d)$ according to Lemma 4.2 and thus $A(V,W,d) \to^+_{\mathcal{S}(P,\mathcal{Q}_1)} C[A(V,W,d)]$ for some context C by Lemma 6.4, hence $\mathcal{S}(P,\mathcal{Q}_1)$ is looping. On the other hand, if P has no solution then $\mathcal{S}(P,\mathcal{Q}_1)$ is ω-terminating by Lemma 6.3 and thus also non-looping. $\square$

8 SN $\Rightarrow$ NL

Before defining the TRS $\mathcal{Q}_2$ used for the relative undecidability of SN $\Rightarrow$ NL, we will present two simple but useful facts about non-loopingness.

Lemma 8.1 *Every term in a loop is looping.*

Proof Let $t \to^* u \to^* C[t\sigma]$ be a non-empty rewrite sequence. Since rewriting is closed under substitution, we obtain $u \to^* C[t\sigma] \to^* C[u\sigma]$, which shows that u is looping. $\square$

Lemma 8.2 *Every looping TRS admits a loop that starts with a root rewrite step.*

Proof Let $t \to^+ C[t\sigma]$ be any loop. We show by induction on the structure of t that there exists a loop (not necessarily starting at t) which contains a root rewrite step. In the base case (t is a constant) the first step of the given loop must take place at the root position. For the induction step, suppose $t = f(t_1,\dots,t_k)$ with $k \geqslant 1$ and no step in $t \to^+ C[t\sigma]$ takes place at the root position. So $C[t\sigma] = f(u_1,\dots,u_k)$ with $t_i \to^* u_i$ for all $1 \leqslant i \leqslant k$ and $t_j \to^+ u_j$ for at least one $1 \leqslant j \leqslant k$. If the context C is empty then $u_j = t_j\sigma$ and we obtain the desired loop from the induction hypothesis (applied to t_j). Otherwise there exist a context C' and an index $1 \leqslant l \leqslant k$ such that $C = f(v_1,\dots,v_{l-1},C',v_{l+1},\dots,v_k)$ and $t_l \to^* C'[t\sigma] = C'[f(\dots,t_l\sigma,\dots)]$. Since $t_l \neq C'[f(\dots,t_l\sigma,\dots)]$ we can apply the induction hypothesis to t_l, yielding a loop which contains a root rewrite step. Now the result follows from the preceding lemma. $\square$

Let

$$\mathcal{Q}_2 = \left\{ \begin{array}{lcl} f(d,b(x),y) & \to & f(d,x,b(y)) \\ f(d,b(x),y) & \to & f(x,y,b(b(d))) \end{array} \right.$$

Lemma 8.3 ([27]) *The TRS $\mathcal{Q}_2$ is non-looping and not terminating.*

Proof First we show that $\mathcal{Q}_2$ is not terminating. Define terms $t_i = f(d, b(d), b^i(d))$ for all $i \geqslant 1$. We have $t_i \to^+ t_{i+1}$ by one application of the second rewrite rule followed by $i-1$ applications of the first rewrite rule. Hence $\mathcal{Q}_2$ admits the infinite rewrite sequence $t_1 \to^+ t_2 \to^+ t_3 \to^+ \cdots$. Next we show that $\mathcal{Q}_2$ is non-looping. For a proof by contradiction suppose that $\mathcal{Q}_2$ is looping. According to Lemma 8.2 there must be a loop that starts with a root rewrite step, which is only possible if the loop is of the form

$$f(d, b(t_1), t_2) \to^+ C[f(d, b(t_1\sigma), t_2\sigma)] \tag{19}$$

Since rewrite steps do not change the number of f symbols, the context C must be empty. Moreover, the substitution σ does not assign terms that contain any f symbols to the variables in t_1 and t_2. We may assume that t_1 and t_2 do not contain f symbols. (If they do, we replace their outermost f symbols by a fresh variable, resulting in a loop that has the desired property.) Consequently, all rewrite steps in (19) take place at the root position. We claim that t_1 and t_2 are ground terms. First note that the second rule of $\mathcal{Q}_2$ must be used in (19) as the first rule constitutes a terminating (and hence non-looping) TRS. Hence we can render (19) as follows:

$$f(d, b(t_1), t_2) \to^* f(d, b(u_1), u_2) \to f(u_1, u_2, b(b(d))) \to^* f(d, b(t_1\sigma), t_2\sigma) \tag{20}$$

such that $t_1 = b^k(u_1)$ and $b^k(t_2) = u_2$ for some $k \geqslant 0$. Because of the form of the left-hand sides of the rules of $\mathcal{Q}_2$ we must have $u_1 = d$ and hence $t_1 = b^k(d)$ is a ground term. Repeating the same reasoning for the loop

$$f(d, u_2, b(b(d))) \to^+ f(d, u_2\sigma, b(b(d))) \tag{21}$$

whose existence is guaranteed by (the proof of) Lemma 8.1 shows that u_2, and thus also t_2, is a ground term. Hence $t_1\sigma = t_1$, $t_2\sigma = t_2$, and thus (19) is actually a cycle. Since applications of the second rewrite rule increase the size of terms whereas applications of the first rewrite rule do not change the size of terms, only the first rewrite rule can be used. However, we have already observed that the first rule constitutes a terminating (and thus acyclic) TRS. Therefore $\mathcal{Q}_2$ is non-looping. $\square$

We want to show that $\mathcal{S}(P, \mathcal{Q}_2)$ is non-looping for every PCP instance P. Since it is easier to reason about $\mathcal{U}(P, \mathcal{Q}_2)$, we will show how to transform a loop in $\mathcal{S}(P, \mathcal{Q}_2)$ into a loop in $\mathcal{U}(P, \mathcal{Q}_2)$. Actually, we present a more general statement which will also be used in Section 10.

Definition 8.4 We define two (partial) mappings ϕ and ψ as follows:

$$\begin{aligned}
\phi(A(t_1, \ldots, t_{2n+15})) &= A(\psi(t_1), \ldots, \psi(t_{2n+15})), \\
\phi(B(t_1, \ldots, t_{n+2m+3})) &= B(\phi(t_1), \ldots, \phi(t_{n+2m+3})), \\
\psi(A(t_1, \ldots, t_{2n+15})) &= \psi(B(t_1, \ldots, t_{n+2m+3})) = z,
\end{aligned}$$

$\psi(g(t_1, \ldots, t_k)) = g(\psi(t_1), \ldots, \psi(t_k))$ for all function symbols g different from A and B, and $\psi(x) = x$ for all variables x. Here z is a designated fresh variable.

The purpose of these mappings is to simplify the structure of $\mathcal{S}(P, \mathcal{Q})$ rewrite sequences by replacing all descendants of non-outermost A symbols by the variable z.

Lemma 8.5 *If $t \to^+_{\mathcal{S}(P,\mathcal{Q})} u$ with $\mathrm{root}(t) = A$ contains a root rewrite step then $\phi(t) \to^+_{\mathcal{S}(P,\mathcal{Q})} \phi(u)$. Moreover, if v is a maximal subterm of u with root symbol A then $\phi(v)$ is a subterm of $\phi(u)$ and $\phi(t) \to^+_{\mathcal{U}(P,\mathcal{Q})} \phi(v)$.*

Proof It is easy to see that for every step $t' \to_{\mathcal{S}(P,\mathcal{Q}_2)} u'$ in the given rewrite sequence we get $\phi(t') \to_{\mathcal{S}(P,\mathcal{Q}_2)} \phi(u')$ if the contracted redex is outermost and $\phi(t') = \phi(u')$ otherwise. Hence $\phi(t) \to^+_{\mathcal{S}(P,\mathcal{Q})} \phi(u)$. The term u can be (uniquely) written as $C[v_1, \ldots, v_k]$ such that C is a context consisting of B symbols and every v_i starts with an A symbol. So $v_1, \ldots, v_k$ are the maximal subterms of u with root symbol A. By definition $\phi(u) = C[\phi(v_1), \ldots, \phi(v_k)]$. A straightforward induction on the length of $t \to^*_{\mathcal{S}(P,\mathcal{Q})} u$ yields $\phi(t) \to^*_{\mathcal{U}(P,\mathcal{Q})} \phi(v_i)$ for every $1 \leqslant i \leqslant k$. $\square$

Lemma 8.6 *If $\mathcal{U}(P, \mathcal{Q}_2)$ is non-looping then $\mathcal{S}(P, \mathcal{Q}_2)$ is non-looping.*

Proof Suppose on the contrary that $\mathcal{S}(P, \mathcal{Q}_2)$ is looping. According to Lemma 8.2 there exists a loop $t \to^+_{\mathcal{S}(P,\mathcal{Q}_2)} C[t\sigma]$ that starts with a root rewrite step. This implies that $\mathrm{root}(t) = A$. Since the maximum nesting of A symbols does not change by $\mathcal{S}(P, \mathcal{Q}_2)$ rewrite steps, there cannot be A symbols above the position of the hole in the context C. In other words, $t\sigma$ is a maximal subterm of $C[t\sigma]$ with root symbol A. Lemma 8.5 yields

$$\phi(t) \to^+_{\mathcal{U}(P,\mathcal{Q}_2)} \phi(t\sigma) \tag{22}$$

Note that $\phi(t\sigma) = \phi(t)\sigma'$ where the substitution σ' is defined as the composition of σ and ψ. Hence (22) is a loop, which contradicts the assumption. $\square$

Lemma 8.7 *The TRS $\mathcal{S}(P, \mathcal{Q}_2)$ is non-looping for every PCP instance P.*

Proof Assume $\mathcal{S}(P, \mathcal{Q}_2)$ admits a loop. From Lemma 8.6 we obtain a loop, $t \to^+ C[t\sigma]$, in $\mathcal{U}(P, \mathcal{Q}_2)$. According to Lemma 8.2 we may assume that this loop starts with a root-rewrite step. This is only possible if t is a redex and hence we may write $t = A(V, W, t')$. The linear interpretation ϕ is defined by $\phi(b(t)) = \phi(t)$ and $\phi(g(t_1, \ldots, t_k)) = \phi(t_1) + \cdots + \phi(t_k) + 1$ for every other function symbol g of arity k. Clearly, $s \to_{\mathcal{U}(P,\mathcal{Q}_2)} s'$ implies $\phi(s) = \phi(s')$ for all terms s and s', hence C consists of b symbols only. Another linear interpretation ψ is defined by $\psi(b(t)) = \psi(t) + 1$ and $\psi(g(t_1, \ldots, t_k)) = 0$ for every other function symbol g of arity k. For all terms s and s', if $s \to_{\mathcal{U}(P,\mathcal{Q}_2)} s'$ then $\psi(s) = \psi(s')$, hence C is empty. We conclude that the loop must be of the form $A(V, W, t') \to^+ A(V, W\sigma, t'\sigma)$. Since $A(V, W, t') \to^+_{\mathcal{U}_-(P,\mathcal{Q})} A(V, W\sigma, t'\sigma)$ contradicts the (ω-)termination of $\mathcal{U}_-(P, \mathcal{Q})$ (Corollary 5.20), we obtain $t' \to^+_{\mathcal{Q}_2} t'\sigma$ from Lemma 4.3. This is impossible as $\mathcal{Q}_2$ is non-looping (Lemma 8.3). $\square$

Lemma 8.8 *The TRS $\mathcal{S}(P, \mathcal{Q}_2)$ is terminating if and only if P admits no solution.*

Proof Suppose P has a solution. From (the proof of) Lemma 8.3 we know there exists an infinite rewrite sequence $t_1 \to_{\mathcal{Q}_2} t_2 \to_{\mathcal{Q}_2} t_3 \to_{\mathcal{Q}_2} \cdots$ in which all steps take place at the root position. According to Lemmata 4.2 and 6.4 this sequence can be transformed into an infinite rewrite sequence in $\mathcal{S}(P, \mathcal{Q}_2)$:

$$A(V, W, t_1) \to^+ C_1[A(V, W, t_2)] \to^+ C_1[C_2[A(V, W, t_3)]] \to \cdots$$

Note that for the applicability of Lemma 4.2 it is essential that all steps in the infinite $\mathcal{Q}_2$-rewrite sequence take place at the root position. Conversely, if P has no solution then $\mathcal{S}(P, \mathcal{Q}_2)$ is ω-terminating by Lemma 6.3 and therefore also terminating. $\square$

9 NSE $\Rightarrow$ SN

Let $\mathcal{Q}_3 = \{f(d) \to f(g(d))\}$. This TRS is terminating and self-embedding.

Lemma 9.1 *The TRS $\mathcal{S}(P, \mathcal{Q}_3)$ is terminating for every PCP instance P.*

Proof According to Lemma 6.1(1) it suffices to show that $\mathcal{U}(P, \mathcal{Q}_3)$ is terminating. There are several ways to achieve this. We use type introduction ([25]), which is possible since $\mathcal{U}(P, \mathcal{Q}_3)$ lacks collapsing (and duplicating) rules. Hence we may assume that the function symbols come from a many-sorted signature such that the left and right-hand side of any rewrite rule are well-typed and of the same type. We use two sorts 1 and 2 with A of type $1 \times \cdots \times 1 \to 2$ and all other function symbols of type $1 \times \cdots \times 1 \to 1$. Terms of type 1 are in normal form. So if $\mathcal{U}(P, \mathcal{Q}_3)$ is not terminating then there exists an infinite rewrite sequence consisting of terms of type 2. Hence all steps take place at the root position. Since terms of type 2 do not contain occurrences of A below the root position, Lemma 4.3 applies. Because $\mathcal{U}_{-}(P, \mathcal{Q})$ is (simply) terminating (Lemma 5.20), we obtain an infinite rewrite sequence in $\mathcal{Q}_3$, contradicting its termination. $\square$

Lemma 9.2 *The TRS $\mathcal{S}(P, \mathcal{Q}_3)$ is non-self-embedding if and only if P admits no solution.*

Proof If P admits a solution then we obtain

$$A(V, W, f(d)) \to^+_{\mathcal{S}(P, \mathcal{Q}_3)} C[A(V, W, f(g(d)))]$$

from Lemmata 4.2 and 6.4. Since $A(V, W, f(d))$ is embedded in $C[A(V, W, f(g(d)))]$, this shows that $\mathcal{S}(P, \mathcal{Q}_3)$ is self-embedding. Conversely, if P has no solution then $\mathcal{S}(P, \mathcal{Q}_3)$ is ω-terminating and thus non-self-embedding by Lemma 6.3. $\square$

10 ST $\Rightarrow$ NSE

Before defining the TRS $\mathcal{Q}_4$ used for the relative undecidability of ST $\Rightarrow$ NSE, we present a simple fact about non-self-embeddingness.

Lemma 10.1 *If a TRS $\mathcal{R}$ is self-embedding then it admits a rewrite sequence $t \to^+_{\mathcal{R}} u \to^*_{\mathcal{E}\mathrm{mb}} t$ such that the subsequence from t to u contains a root rewrite step.*

Proof Similar to the proof of Lemma 8.2, but note that we cannot prove that there exists a rewrite sequence $t \to^+_{\mathcal{R}} u \to^*_{\mathcal{E}\mathrm{mb}} t$ that starts with root rewrite step: consider the TRS $\{f(a) \to f(g(h(b))), g(b) \to a\}$. $\square$

We define

$$\mathcal{Q}_4 = \left\{ \begin{array}{lcl} f(d,e,x) & \to & f(x,g(e),e) \\ f(d,e,x) & \to & f(g(d),x,d) \end{array} \right.$$

Lemma 10.2 *The TRS $\mathcal{Q}_4$ is non-self-embedding.*

Proof If $\mathcal{Q}_4$ is self-embedding then by Lemma 10.1 there exists a rewrite sequence $t \to^+_{\mathcal{Q}_4} u \to^*_{\mathcal{E}\mathrm{mb}} t$ such that the subsequence from t to u contains a root rewrite step. By the form of the rules in $\mathcal{Q}_4$, the latter condition requires that the first step in the subsequence from t to u is a root rewrite step. Moreover, later steps must take place below the root. It follows that $t = f(d,e,s)$, $s \to^*_{\mathcal{Q}_4} s'$ and either $u = f(s',g(e),e)$ or $u = f(g(d),s',d)$. But then t can only be embedded in u if $s = e$ and $s' \to^*_{\mathcal{E}\mathrm{mb}} d$ or $s = d$ and $s' \to^*_{\mathcal{E}\mathrm{mb}} e$. Note that t cannot be embedded in s' as t contains one more f symbol than s'. However, both cases contradict $s \to^*_{\mathcal{Q}_4} s'$. Hence $\mathcal{Q}_4$ is non-self-embedding. $\square$

Lemma 10.3 *The TRS $\mathcal{S}(P,\mathcal{Q}_4)$ is non-self-embedding for every PCP instance P.*

Proof Suppose on the contrary that $\mathcal{S}(P,\mathcal{Q}_4)$ is self-embedding. According to Lemma 10.1 there exists a rewrite sequence

$$t \to^+_{\mathcal{S}(P,\mathcal{Q}_4)} u \to^*_{\mathcal{E}\mathrm{mb}} t \tag{23}$$

such that its first part contains a step at the root position. By the form of the rules in $\mathcal{S}(P,\mathcal{Q}_4)$ this implies that t is a redex, so we may write $t = A(V,W,f(d,e,t'))$. The term u can be written (cf. the proof of Lemma 8.5) as $C[v_1,\ldots,v_k]$ such that C is a non-empty context consisting of B symbols and every v_i starts with an A symbol. We can rearrange the second part of (23) into $u \to^+_{\mathcal{E}\mathrm{mb}} v_i \to^*_{\mathcal{E}\mathrm{mb}} t$ for suitable $1 \leqslant i \leqslant k$. Lemma 8.5 yields $\phi(t) \to^+_{\mathcal{U}(P,\mathcal{Q}_4)} \phi(v_i)$. Since $\mathcal{Q}_4$ is non-duplicating and variable-preserving, v_i has the same number of A symbols as t and hence no A symbol is erased in $v_i \to^*_{\mathcal{E}\mathrm{mb}} t$. This implies that $\phi(v_i) \to^*_{\mathcal{E}\mathrm{mb}} \phi(t)$. We have $\phi(t) = A(V,\phi(W),f(d,e,\phi(t')))$ and $\phi(v_i) = A(\phi(V'),\phi(W'),\phi(u'))$ for a certain $2n+8$-tuple V', sextuple W', and term u'. We must have $\phi(V') \to^*_{\mathcal{E}\mathrm{mb}} V$, $\phi(W') \to^*_{\mathcal{E}\mathrm{mb}} \phi(W)$, and $\phi(u') \to^*_{\mathcal{E}\mathrm{mb}} f(d,e,\phi(t'))$. From Lemma 4.3 we infer that either $f(d,e,\phi(t')) \to^+_{\mathcal{Q}_4} \phi(u')$ or $\phi(t) \to^+_{\mathcal{U}_-(P,\mathcal{Q})} \phi(v_i)$ (and $f(d,e,\phi(t')) = \phi(u')$). The former contradicts the fact that $\mathcal{Q}_4$ is non-self-embedding (Lemma 10.2), the latter the simple termination of $\mathcal{U}_-(P,\mathcal{Q})$ (which follows from Corollary 5.20). $\square$

Lemma 10.4 *The TRS $\mathcal{S}(P,\mathcal{Q}_4)$ is simply terminating if and only if P admits no solution.*

Proof If P admits a solution then with the help of Lemmata 4.2 and 6.4 we obtain the following cycle in $\mathcal{S}(P,\mathcal{Q}_4) \cup \mathcal{E}\mathrm{mb}$:

$$\begin{aligned} A(V,W,f(d,e,d)) &\to^+ C_1[A(V,W,f(d,g(e),e))] \to^+ A(V,W,f(d,e,e)) \\ &\to^+ C_2[A(V,W,f(g(d),e,d))] \to^+ A(V,W,f(d,e,d)) \end{aligned}$$

So $\mathcal{S}(P,\mathcal{Q}_4)$ is not simply terminating. Conversely, if P has no solution, then $\mathcal{S}(P,\mathcal{Q}_4)$ is ω-terminating and thus simply terminating by Lemma 6.3. $\square$

11 TT $\Rightarrow$ ST

Let

$$\mathcal{Q}_5 = \begin{cases} f(d,e) & \to & f(e,e) \\ f(d,e) & \to & f(d,d) \end{cases}$$

This TRS is simply terminating (because it is terminating and length-preserving) but not totally terminating (as d and e are incomparable).

Lemma 11.1 *The TRS $\mathcal{S}(P,\mathcal{Q}_5)$ is simply terminating for every PCP instance P.*

Proof According to Lemma 6.1(2) it is sufficient to show that $\mathcal{U}(P,\mathcal{Q}_5)$ is simply terminating. Since $\mathcal{U}(P,\mathcal{Q}_5)$ is length-preserving, simple termination follows from termination. By using typing, the termination of $\mathcal{U}(P,\mathcal{Q}_5)$ follows from the termination of $\mathcal{Q}_5$, just as in the proof of Lemma 9.1. $\square$

Lemma 11.2 *The TRS $\mathcal{S}(P,\mathcal{Q}_5)$ is totally terminating if and only if P admits no solution.*

Proof If P has no solution then ω-termination, and thus also total termination, of $\mathcal{S}(P,\mathcal{Q}_5)$ follows from Lemma 6.3. Let P have a solution. We show that $\mathcal{S}(P,\mathcal{Q}_5)$ is not totally terminating. According to Lemma 6.1(3) this is equivalent to showing that $\mathcal{U}(P,\mathcal{Q}_5)$ is not totally terminating. Suppose on the contrary that $\mathcal{U}(P,\mathcal{Q}_5)$ is totally terminating. So there exists a compatible total reduction order $>$. Because by Lemma 4.2 both $A(V,W,f(d,e)) \to^+ A(V,W,f(e,e))$ and $A(V,W,f(d,e)) \to^+ A(V,W,f(d,d))$, we have $A(V,W,f(d,e)) > A(V,W,f(e,e))$ and $A(V,W,f(d,e)) > A(V,W,f(d,d))$ by compatibility. By the truncation rule for total reduction orders ([25, Proposition 9]) one may remove a context C from an inequation $C[t] > C[t']$. By removing $A(V,W,f(\ ,e))$ and $A(V,W,f(d,\ ))$, respectively, we obtain the impossible $d > e$ and $e > d$. $\square$

12 ωT $\Rightarrow$ TT

Geser [7] showed the undecidability of ω-termination for totally terminating TRSs. In this section we will show that ω-termination is an undecidable property of one-rule totally terminating TRSs.

Let $\mathcal{Q}_6 = \{f(g(x)) \to g(f(f(x)))\}$. This TRS is totally terminating but not ω-terminating (Zantema [25, Proposition 11]).

Lemma 12.1 *The TRS $\mathcal{S}(P,\mathcal{Q}_6)$ is totally terminating for every PCP instance P.*

Proof First we show that $\mathcal{U}(P,\mathcal{Q}_6)$ is totally terminating. Let the interpretation $[\ ]'$ in $\mathbb{N}_+^2$ be defined by $[f]'(x,y) = (x, x+y)$, $[g]'(x,y) = (2x+1, y)$, and

$$[h]'(x_1,y_1),\ldots,(x_n,y_n)) = (1 + \sum_{i=1}^{n} x_i, 1 + \sum_{i=1}^{n} y_i)$$

for every function symbol $h \in \mathcal{F}_{\mathcal{U}} \setminus \{f, g\}$. We claim that the interpretation $[\![\ ]\!]$ in $\mathbb{N}_+^3$ (ordered lexicographically) defined by $[\![\ ]\!] = ([\ ]', [\ ])$ proves total termination of $\mathcal{U}(P, \mathcal{Q}_6)$. For rules $l \to r$ of type (III) and ground substitutions σ we have $[l\sigma]' > [r\sigma]'$. For the rules in $\mathcal{U}_-(P, \mathcal{Q}_6)$ we have $[l\sigma]' = [r\sigma]'$ and $[l\sigma] > [r\sigma]$ by Theorem 5.17. From Lemma 6.1(3) it follows that $\mathcal{S}(P, \mathcal{Q}_6)$ is totally terminating. □

Lemma 12.2 *The TRS $\mathcal{S}(P, \mathcal{Q}_6)$ is ω-terminating if and only if P admits no solution.*

Proof If P has no solution then ω-termination of $\mathcal{S}(P, \mathcal{Q}_6)$ follows from Lemma 6.3. Let P have a solution. We show that $\mathcal{S}(P, \mathcal{Q}_6)$ is not ω-terminating. According to Lemma 6.2 it is sufficient to show that $\mathcal{U}(P, \mathcal{Q}_6)$ is not ω-terminating. Suppose on the contrary that $\mathcal{U}(P, \mathcal{Q}_6)$ is ω-terminating. So there exists a compatible well-founded monotone algebra $\mathcal{A} = (\mathbb{N}, >)$. According to Lemma 4.2 we have

$$A(V, W, f(g(t))) \to^+ A(V, W, g(f(f(t))))$$

and thus $A(V, W, f(g(t))) >_{\mathcal{A}} A(V, W, g(f(f(t))))$ for every ground term t. Because the interpretation of A is strictly monotone in its final argument, this is only possible if $f(g(t)) >_{\mathcal{A}} g(f(f(t)))$ for every ground term t, which contradicts the fact that $\mathcal{Q}_6$ is not ω-terminating. □

13 PT ⇒ ωT

At present it is unknown as to whether polynomial termination is an undecidable property of (ω-terminating) TRSs. The following TRS is ω-terminating but not polynomially terminating (Zantema [25, Proposition 10]):

$$f(g(h(x))) \quad \to \quad g(f(h(g(x))))$$

So the polynomially terminating TRSs form a proper subclass of the ω-terminating TRSs. We conjecture that the implication PT ⇒ ωT is relative undecidable, even for one-rule TRSs.

14 SN ⇒ WN

Lemma 14.1 *The TRS $\mathcal{U}(P, \mathcal{Q})$ is weakly normalizing for every PCP instance P and TRS $\mathcal{Q}$.*

Proof We show that every term t has a normal form by induction on the structure of t. The only interesting case is $t = A(t_1, \dots, t_{2n+15})$. According to the induction hypothesis every t_i has a normal form u_i. So t rewrites to $t' = A(u_1, \dots, u_{2n+15})$. If t' is a not redex then we are done. If t' is a redex then we consider u_{2n+9} and u_{2n+10}. If $u_{2n+9} = 0$ and $u_{2n+10} = 1$ then we apply any rewrite rule of type (III). If $u_{2n+9} = 1$ and $u_{2n+10} = 0$ then we apply the only rewrite rule of type (I). Otherwise, we apply an arbitrary rewrite rule. In all cases we obtain a term of the form $A(1, 0, \dots)$ which does not match the unique left-hand side l of $\mathcal{U}(P, \mathcal{Q})$, in other words, a normal form. □

Since $\mathcal{U}(P, \mathcal{Q}_2)$ is terminating if and only if P admits no solution—this follows from the proof of Lemma 8.8—we obtain the relative undecidability of the implication SN $\Rightarrow$ WN. (Instead of $\mathcal{Q}_2$ the simpler $\mathcal{Q}_7 = \{a \to a, a \to b\}$ also works.) However, with the constructions given in this paper we cannot strengthen this result to one-rule systems. The reason is that weak normalization of $\mathcal{U}(P, \mathcal{Q})$ does not imply weak normalization of $\mathcal{S}(P, \mathcal{Q})$. As a matter of fact, for all TRSs $\mathcal{Q}$ presented so far the TRS $\mathcal{S}(P, \mathcal{Q})$ is both orthogonal and non-erasing. According to the following well-known result (see [13]) such systems can never be used for obtaining the relative undecidability of the implication SN $\Rightarrow$ WN. (Note that unlike $\mathcal{S}(P, \mathcal{Q})$ the TRS $\mathcal{U}(P, \mathcal{Q})$ is not orthogonal.)

Theorem 14.2 *An orthogonal and non-erasing TRS is weakly normalizing if and only if it is strongly normalizing.* □

The implication SN $\Rightarrow$ WN is known to be strict even for one-rule TRSs. An example of a weakly but not strongly normalizing one-rule TRS is the system

$$\mathcal{Q}_8 = \{f(a, f(x, y)) \to f(x, f(x, f(b, b)))\}$$

of Akkerman (see [13]). Note that this TRS is neither orthogonal nor non-erasing. However, $\mathcal{Q}_8$ cannot be used either: Since $\mathcal{U}(P, \mathcal{Q})$ and $\mathcal{S}(P, \mathcal{Q})$ only simulate *root* rewrite sequences in $\mathcal{Q}$ (for PCP instances P that admit a solution) and root rewriting in $\mathcal{Q}_8$ is not weakly normalizing, viz. $f(a, f(a, f(b, b))) \to f(a, f(a, f(b, b)))$, it follows that $\mathcal{S}(P, \mathcal{Q}_8)$ is not weakly normalizing for solvable P.

Conclusion

The results proved in this paper are summarized below.

Theorem 14.3 *For the following implications $X \Rightarrow Y$, X is an undecidable property of one-rule TRSs that satisfy property Y:*

$$\omega T \;\Rightarrow\; TT \;\Rightarrow\; ST \;\Rightarrow\; NSE \;\Rightarrow\; SN \;\Rightarrow\; NL \;\Rightarrow\; AC$$

In addition, SN is an undecidable property of TRSs that satisfy WN. □

We already identified the following open problems: Is the implication SN $\Rightarrow$ WN relative undecidable for *one-rule* TRSs? Is the implication PT $\Rightarrow$ ωT relative undecidable? Another problem is whether the results obtained in this paper can be strengthened to string rewriting systems.

In part 2 of this paper [8] we present relative undecidability results for properties related to confluence.

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