Relative Undecidability in Term Rewriting Part 2: The Confluence Hierarchy

Alfons Geser

ICASE, Mail Stop 132C
NASA Langley Research Center
Hampton VA 23681, USA

Aart Middeldorp

Institute of Information Sciences and Electronics
University of Tsukuba
Tsukuba 305-8573, Japan

Enno Ohlebusch

Faculty of Technology
University of Bielefeld
P.O. Box 10 01 31, 33501 Bielefeld, Germany

Hans Zantema

Department of Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

April 24, 2002

Abstract

For a hierarchy of properties of term rewriting systems related to confluence we prove *relative undecidability*, i.e., for implications $X \Rightarrow Y$ in the hierarchy the property X is undecidable for term rewriting systems satisfying Y. For some of the implications either X or $\neg X$ is semi-decidable, for others neither X nor $\neg X$ is semi-decidable. We prove most of these results for linear term rewrite systems.

1 Introduction

Termination and confluence are fundamental properties of term rewriting systems (TRSs) which are often very hard to prove. Classical results ([7, 8]) state that they are undecidable. Besides termination and confluence, a number of related properties are of interest. In our companion paper ([4]) properties related to termination have been studied extensively. In the present paper we study properties connected to confluence, mutually related in the *confluence hierarchy*:

$$
\begin{array}{ccccccccc}
 & & \mathrm{GCR} & & & & & & \\
 & & \Uparrow & & & & & & \\
\mathrm{SCR} & \Rightarrow & \mathrm{CR} & \Rightarrow & \mathrm{NF} & \Rightarrow & \mathrm{UN} & \Rightarrow & \mathrm{UN}^{\rightarrow} \\
 & & \Downarrow & & & & \Downarrow & & \Downarrow \\
 & & \mathrm{WCR} & & & & \mathrm{CON} & \Rightarrow & \mathrm{CON}^{\rightarrow}
\end{array}
$$

The acronyms stand for strong confluence (SCR), confluence (or the Church-Rosser property, CR), local confluence (or weak Church-Rosser, WCR), ground confluence (or ground Church-Rosser, GCR), the normal form property (NF), unique normal forms (UN), unique normal forms with respect to reduction (UN$^{\rightarrow}$), consistency (CON), and consistency with respect to reduction (CON$^{\rightarrow}$). Precise definitions are given in Section 2. For weakly normalizing systems the properties CR, NF, UN, and UN$^{\rightarrow}$ coincide. For terminating systems also WCR and CR coincide and are decidable.

Undecidability of confluence is well-known ([8]), for the other properties in the confluence hierarchy it is easy to see too. Ground confluence is known to be undecidable for terminating systems (Kapur *et al.* [9]). In this paper we show the stronger result of *relative undecidability*: For all implications $X \Rightarrow Y$ in the confluence hierarchy we prove that the property X is undecidable for TRSs satisfying Y.

It is well-known that a property X is undecidable if and only if X is not semi-decidable or $\neg X$ is not semi-decidable. In the latter case we also say that X is not co-semi-decidable. Hence we can distinguish the following three possibilities for an implication $X \Rightarrow Y$:

1. Both X and $\neg X$ are not semi-decidable for TRSs satisfying Y. This is the case for the implications CR $\Rightarrow$ WCR, CR $\Rightarrow$ GCR, CR $\Rightarrow$ NF, and NF $\Rightarrow$ UN.

2. Property X is semi-decidable and $\neg X$ is not semi-decidable for TRSs satisfying Y. This is the case for the implication SCR $\Rightarrow$ CR.

3. Property $\neg X$ is semi-decidable and X is not semi-decidable for TRSs satisfying Y. This is the case for the remaining implications UN $\Rightarrow$ UN$^{\rightarrow}$, UN $\Rightarrow$ CON, UN$^{\rightarrow}$ $\Rightarrow$ CON$^{\rightarrow}$, and CON $\Rightarrow$ CON$^{\rightarrow}$.

We prove all non-semi-decidability results for linear TRSs. We prove the non-semi-decidability of CON for TRSs satisfying CON$^{\rightarrow}$ for linear terminating TRSs. We prove the semi-decidability results for arbitrary (finite) TRSs, except that we need linearity for the semi-decidability of SCR. We also consider (non-)semi-decidability for the properties

WCR, GCR, and CON$^{\rightarrow}$ that do not appear in an X position of an implication $X \Rightarrow Y$ in the confluence hierarchy.

The paper is organized in such a way that it can be read independently of the companion paper [4] on the termination hierarchy. In Section 2 we give the main definitions and present some basic tools that we use in the sequel. In the next nine sections we treat the nine implications of the confluence hierarchy. In Section 12 we investigate (semi-)decidability issues for the three properties in the confluence hierarchy that do not appear in an X position of an implication $X \Rightarrow Y$. Finally, in Section 13 we summarize our results and point to further research. Some of the results in this paper were first reported in our earlier paper [5].

2 Preliminaries

For all our undecidability proofs we use Post's Correspondence Problem (PCP), which can be stated as follows:

> given a finite alphabet Γ and a finite list $P = (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ of pairs of non-empty strings over Γ, is there some natural number $m > 0$ and indices $1 \leqslant i_1, \ldots, i_m \leqslant n$ such that $\alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$?

The list P is called an *instance* of PCP, the string $\alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$ a *solution* for P. We write $(\alpha, \beta) \in P$ if $(\alpha, \beta) = (\alpha_i, \beta_i)$ for some $1 \leqslant i \leqslant n$. Without loss of generality we require P to be non-empty. PCP is known to be undecidable even in the case of a two-letter alphabet (Post [15]). Matiyasevich and Senizergues [12] showed that PCP is undecidable even when restricted to instances consisting of seven pairs. An obvious breadth-first search procedure yields the semi-decidability of PCP and hence the complement of PCP is not semi-decidable.

For preliminaries on rewriting the reader is referred to [2, 3, 10]. Here we define the properties in the confluence hierarchy and recall some definitions of other properties of TRSs. All our TRSs are assumed to be finite.

A TRS $\mathcal{R}$ is called *confluent* (or Church-Rosser, CR) if ${}_{\mathcal{R}}^{*}\!\leftarrow \cdot \rightarrow_{\mathcal{R}}^{*} \subseteq \rightarrow_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow$. Confluence is equivalent to the the property stating that every two convertible terms have a common reduct. A TRS $\mathcal{R}$ is called *locally confluent* (or weakly Church-Rosser, WCR) if ${}_{\mathcal{R}}\!\leftarrow \cdot \rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow$. A TRS $\mathcal{R}$ is called *strongly confluent* (or strongly Church-Rosser, SCR) if ${}_{\mathcal{R}}\!\leftarrow \cdot \rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}}^{=} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow$. A TRS is said to have the *normal form property* (NF) if every term convertible to a normal form rewrites to that normal form. The normal form property is equivalent to the property stating that every term that has a normal form is confluent. A TRS is said to have *unique normal forms* (UN) if different normal forms are not convertible. A TRS is said to have *unique normal forms with respect to reduction* (UN$^{\rightarrow}$) if every term has at most one normal form. A TRS is called *consistent* (CON) if distinct variables are not convertible. A TRS is consistent if and only if not all terms are convertible. A TRS is called *consistent with respect to reduction* (CON$^{\rightarrow}$) if no term rewrites to two distinct variables. All properties in the confluence hierarchy, except UN and CON, can be defined for individual terms in the obvious way. For instance, a term t in a TRS $\mathcal{R}$ is confluent if and only if s and u have a common reduct whenever $s \ {}_{\mathcal{R}}^{*}\!\!\leftarrow t \rightarrow_{\mathcal{R}}^{*} u$. A TRS $\mathcal{R}$ is called *ground confluent* (or ground Church-Rosser, GCR) if all its ground terms are confluent.

The above definition of strong confluence originates from Huet [6] and is different from the one in Dershowitz and Jouannaud [3]. They call a TRS $\mathcal{R}$ strongly confluent if ${}_{\mathcal{R}}\!\leftarrow \cdot \rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}}^{=} \cdot {}_{\mathcal{R}}^{=}\!\!\leftarrow$. Klop [10] calls the latter property subcommutativity (WCR$^{\leqslant 1}$). The definitions of CR and WCR are standard [10, 2]. For the definitions of UN, UN$^{\rightarrow}$, and NF we follow Klop [10]. The notion of CON goes back to Schmidt-Schauß [17]; for the notion CON$^{\rightarrow}$ modularity has been studied [18].

A rewrite rule $l \rightarrow r$ is called *variable preserving* if the sets of variables in l and r are the same. We call $l \rightarrow r$ *collapsing* if r is a variable and *linear* if both l and r are linear terms; a term is linear if no variable occurs more than once in it. A TRS is variable preserving (linear) if all its rewrite rules are so. A TRS is collapsing if it contains a collapsing rewrite rule.

Next we illustrate how we relate PCP to rewriting. An arbitrary PCP instance P admits a solution if and only if $\mathsf{A} \rightarrow_{\mathcal{R}_0(P)}^{*} \mathsf{B}$ for the TRS

$$\mathcal{R}_0(P) = \begin{cases} \mathsf{A} \ \rightarrow \ \mathsf{f}(\alpha(\mathsf{c}), \beta(\mathsf{c})) & \text{for all } (\alpha, \beta) \in P \\ \mathsf{f}(x, y) \ \rightarrow \ \mathsf{f}(\alpha(x), \beta(y)) & \text{for all } (\alpha, \beta) \in P \\ \mathsf{f}(x, x) \ \rightarrow \ \mathsf{B} & \end{cases}$$

This observation is quite simple: B can be reached if and only if a term of the shape $\mathsf{f}(t, t)$ can be reached, and this can be reached from A if and only if a PCP solution for P exists. To arrive at results for linear TRSs and for some technical convenience the basic system $\mathcal{R}_0(P)$ is replaced by the TRS $\mathcal{R}_1(P)$ that consists of the rules

$$\begin{array}{rcl@{\qquad\qquad}rcl} \mathsf{A} & \rightarrow & \mathsf{f}(\alpha(\mathsf{c}), \beta(\mathsf{c})) & & & \\ \mathsf{f}(x, y) & \rightarrow & \mathsf{f}(\alpha(x), \beta(y)) & \mathsf{g}(x, y) & \rightarrow & \mathsf{A} \\ \mathsf{f}(x, y) & \rightarrow & \mathsf{g}(x, y) & \mathsf{g}(a(x), a(y)) & \rightarrow & \mathsf{g}(x, y) \\ \mathsf{f}(x, y) & \rightarrow & \mathsf{A} & \mathsf{g}(\mathsf{c}, \mathsf{c}) & \rightarrow & \mathsf{B} \end{array}$$

The signature of $\mathcal{R}_1(P)$ consists of the constants A, B, and c, binary function symbols f and g, and for every symbol a from the alphabet Γ of the PCP instance P a unary function symbol a. So all terms must be constructed from the function symbols occurring in the rewrite rules of $\mathcal{R}_1(P)$. This convention is also adopted for the TRSs defined in later sections. We use $\mathsf{sans\ serif}$ font for fixed function symbols like A and c but not for function symbols that range over elements of certain sets (like $a \in \Gamma$). Moreover, we drop the quantification in rule schemata by adopting the following conventions:

- α and β range over all pairs (α, β) in P,
- a ranges over all elements of Γ.

So $\mathsf{A} \rightarrow \mathsf{f}(\alpha(\mathsf{c}), \beta(\mathsf{c}))$ stands for the n rules $\mathsf{A} \rightarrow \mathsf{f}(\alpha_1(\mathsf{c}), \beta_1(\mathsf{c}))$, ..., $\mathsf{A} \rightarrow \mathsf{f}(\alpha_n(\mathsf{c}), \beta_n(\mathsf{c}))$.

The TRS $\mathcal{R}_1(P)$ is the basis for many of our relative undecidability results. We show that it shares the desired key property with $\mathcal{R}_0(P)$.

Lemma 2.1 $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$ *if and only if* P *admits a solution.*

Proof Suppose $\gamma \in \Gamma^+$ is a solution for P. So $\gamma = \alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$ for some $m \geqslant 1$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$. We have the following rewrite sequence in $\mathcal{R}_1(P)$:

$$\begin{aligned} \mathsf{A} &\to \mathsf{f}(\alpha_{i_m}(\mathsf{c}), \beta_{i_m}(\mathsf{c})) \to^* \mathsf{f}(\alpha_{i_1} \cdots \alpha_{i_m}(\mathsf{c}), \beta_{i_1} \cdots \beta_{i_m}(\mathsf{c})) = \mathsf{f}(\gamma(\mathsf{c}), \gamma(\mathsf{c})) \\ &\to \mathsf{g}(\gamma(\mathsf{c}), \gamma(\mathsf{c})) \to^+ \mathsf{g}(\mathsf{c}, \mathsf{c}) \to \mathsf{B}. \end{aligned}$$

Conversely, suppose that $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$. Beyond the last A occurring in this rewrite sequence it is of the form

$$\begin{aligned} \mathsf{A} &\to \mathsf{f}(\alpha_{i_m}(\mathsf{c}), \beta_{i_m}(\mathsf{c})) \to^* \mathsf{f}(\alpha_{i_1} \cdots \alpha_{i_m}(\mathsf{c}), \beta_{i_1} \cdots \beta_{i_m}(\mathsf{c})) \\ &\to \underbrace{\mathsf{g}(\alpha_{i_1} \cdots \alpha_{i_m}(\mathsf{c}), \beta_{i_1} \cdots \beta_{i_m}(\mathsf{c})) \to^* \mathsf{g}(\mathsf{c}, \mathsf{c})} \to \mathsf{B} \end{aligned}$$

for some $m \geqslant 1$ with $1 \leqslant i_1, \ldots, i_m \leqslant n$. In the underbraced part only rewrite rules of the form $\mathsf{g}(a(x), a(y)) \to \mathsf{g}(x, y)$ are used. Hence $\alpha_{i_1} \cdots \alpha_{i_m}(\mathsf{c}) = \beta_{i_1} \cdots \beta_{i_m}(\mathsf{c})$, giving a solution for P. $\square$

Next we mention two theorems that will be used a number of times in the sequel. The first one is a result of Huet [6].

Theorem 2.2 *A linear TRS is strongly confluent if and only if it is strongly closed.* $\square$

Here a TRS $\mathcal{R}$ is called *strongly closed* if both $s \to^=_{\mathcal{R}} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow t$ and $t \to^=_{\mathcal{R}} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow s$ for every critical pair $\langle s, t \rangle$ of $\mathcal{R}$.

We use the preceding result to relate $\mathcal{R}_1(P)$ to some properties in the confluence hierarchy.

Lemma 2.3 *The following statements are equivalent:*

1. *The TRS* $\mathcal{R}_1(P)$ *has the normal form property.*
2. *The TRS* $\mathcal{R}_1(P)$ *is locally confluent.*
3. *The TRS* $\mathcal{R}_1(P)$ *is ground confluent.*
4. *The TRS* $\mathcal{R}_1(P)$ *is confluent.*
5. *The PCP instance* P *admits a solution.*

Proof Since confluence implies the normal form property, local confluence, and ground confluence, according to Lemma 2.1 it suffices to show that (i) $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$ whenever $\mathcal{R}_1(P)$ has the normal form property, is locally confluent, or is ground confluent, and (ii) $\mathcal{R}_1(P)$ is confluent whenever $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$. For (i) we note that $\mathsf{A} \leftarrow \mathsf{g}(\mathsf{c}, \mathsf{c}) \to \mathsf{B}$ in $\mathcal{R}_1(P)$ with B a normal form, hence $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$ by definition of the normal form property, local confluence, or ground confluence. For (ii) we consider the TRS $\mathcal{R}_1'(P) = \mathcal{R}_1(P) \cup \{\mathsf{A} \to \mathsf{B}, \mathsf{f}(x, y) \to \mathsf{B}, \mathsf{g}(x, y) \to \mathsf{B}\}$. Since $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$, the relations $\to^*_{\mathcal{R}_1(P)}$ and $\to^*_{\mathcal{R}_1'(P)}$ coincide. The TRS $\mathcal{R}_1'(P)$ is linear and strongly closed and thus (strongly) confluent by Theorem 2.2. Hence $\mathcal{R}_1(P)$ is confluent too. $\square$

The second theorem states that proving confluence is equivalent to proving confluence of the well-typed terms according to any many-sorted type discipline which is compatible with the rewrite system under consideration. This is known as the *persistence* ([20]) of confluence.

Theorem 2.4 *Confluence is a persistent property.* □

For instance, to prove confluence of TRS $\mathcal{R}_0(P)$ above it suffices to prove confluence of every well-typed term according to the many-sorted type declarations: $\mathsf{A}, \mathsf{B}\colon 1$, $\mathsf{f}\colon 2 \times 2 \to 1$, $\mathsf{c}\colon 2$, and $a\colon 2 \to 2$ for every a of the alphabet of P. Note that these declarations are compatible with $\mathcal{R}_0(P)$ since for every rewrite rule $l \to r$ both terms l and r are well-typed (by assigning type 2 to each variable) and have the same type 1. So in a confluence proof we do not have to consider ill-typed terms like $\mathsf{f}(\mathsf{f}(\mathsf{c}, \mathsf{A}), \mathsf{B})$.

Persistence is closely related to modularity. A property of TRSs is said to be modular if the union of two TRSs with the property and disjoint signatures has the property. Van de Pol [14] showed that a component closed property is persistent if and only if it is modular for many-sorted TRSs. (A property P of TRSs is component closed if a TRS $\mathcal{R}$ satisfies P if and only if every equivalence class of $\leftrightarrow^*_{\mathcal{R}}$ has the property P.) So proving persistence amounts to type-checking modularity proofs. For the special case of confluence this is done in [1] for the modularity proof in [11].

To prove some of the relative undecidability results for properties dealing with conversion (UN and CON), we need to relate solvability of P to the existence of a *conversion* between A and B. TRS $\mathcal{R}_1(P)$ is not suitable for this purpose because in $\mathcal{R}_1(P)$ the terms A and B may be convertible even if P admits no solution. For instance, in $\mathcal{R}_1(\{(100, 10), (10, 1)\})$ we have $\mathsf{A} \to \mathsf{f}(100(\mathsf{c}), 10(\mathsf{c})) \leftarrow \mathsf{f}(0(\mathsf{c}), 0(\mathsf{c})) \to^* \mathsf{B}$.

Let $\mathcal{R}_2(P) = \mathcal{R}_2^1(P) \cup \mathcal{R}_2^2(P)$ with $\mathcal{R}_2^1(P)$ consisting of the rules

$$\begin{aligned} \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, \mathsf{c}) &\to \mathsf{A} \\ \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, i(w)) &\to \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, w) \end{aligned}$$

and $\mathcal{R}_2^2(P)$ consisting of the rules

$$\begin{aligned} \mathsf{f}(\alpha_i(x), \beta_i(y), i(z), w) &\to \mathsf{f}(x, y, z, i(w)) & \qquad \mathsf{g}(a(x), a(y), z) &\to \mathsf{g}(x, y, z) \\ \mathsf{f}(x, y, i(z), \mathsf{c}) &\to \mathsf{g}(x, y, i(z)) & \mathsf{g}(\mathsf{c}, \mathsf{c}, i(z)) &\to \mathsf{g}(\mathsf{c}, \mathsf{c}, z) \\ & & \mathsf{g}(\mathsf{c}, \mathsf{c}, \mathsf{c}) &\to \mathsf{B} \end{aligned}$$

where i ranges over $1, \ldots, n$. Note that $\mathcal{R}_2(P)$ is terminating for all PCP instances P.

Lemma 2.5 $\mathsf{A} \leftrightarrow^*_{\mathcal{R}_2(P)} \mathsf{B}$ *if and only if P admits a solution.*

Proof First suppose that P admits a solution $\gamma = \alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$ for some $m \geqslant 1$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$. Then we have the following conversion between A and B:

$$\begin{aligned} &\mathsf{A} \leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, \mathsf{c}) \ {}^*\!\!\leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, i_m \cdots i_1(\mathsf{c})) \ {}^*\!\!\leftarrow \mathsf{f}(\gamma(\mathsf{c}), \gamma(\mathsf{c}), i_1 \cdots i_m(\mathsf{c}), \mathsf{c}) \\ &\quad \to \mathsf{g}(\gamma(\mathsf{c}), \gamma(\mathsf{c}), i_1 \cdots i_m(\mathsf{c})) \to^* \mathsf{g}(\mathsf{c}, \mathsf{c}, i_1 \cdots i_m(\mathsf{c})) \to^* \mathsf{g}(\mathsf{c}, \mathsf{c}, \mathsf{c}) \to \mathsf{B}. \end{aligned}$$

Next suppose that A and B are convertible. Because $\mathcal{R}_2(P)$ is variable preserving and non-collapsing, it follows that all steps in a conversion between A and B take place at the root position. It is also easy to see that any term of the form $\mathsf{f}(t_1, t_2, t_3, t_4)$ in a conversion between A and B satisfies $t_4 = i_m \cdots i_1(\mathsf{c})$ with $1 \leqslant i_1, \ldots, i_m \leqslant n$ for some $m \geqslant 0$. Furthermore, there exists a conversion in $\mathcal{R}_2^1(P)$ between A and $\mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, i_m \cdots i_1(\mathsf{c}))$ for all $m \geqslant 0$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$. We claim that there exists a conversion

$$\mathsf{A} \leftrightarrow^*_{\mathcal{R}_2^1(P)} t \leftrightarrow^*_{\mathcal{R}_2^2(P)} \mathsf{B}$$

for some term $t = \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, i_m \cdots i_1(\mathsf{c}))$. This easily follows from the preceding observations by considering the last application of a rewrite rule of $\mathcal{R}_2^1(P)$ in a shortest conversion between A and B. Let $\mathcal{R}_2^3(P)$ be the TRS obtained from $\mathcal{R}_2^2(P)$ by orienting all

$$\mathsf{f}(\alpha_i(x), \beta_i(y), i(z), w) \to \mathsf{f}(x, y, z, i(w))$$

rules from right to left. Clearly, $\mathcal{R}_2^2(P)$ and $\mathcal{R}_2^3(P)$ generate the same conversion relation. The crucial observation is that $\mathcal{R}_2^1(P)$ and $\mathcal{R}_2^3(P)$ are orthogonal and thus confluent. Because A and B are normal forms, the above conversion between A and B has the form

$$\mathsf{A} \;{}_{\mathcal{R}_2^1(P)}{\leftarrow^*}\; t \to^*_{\mathcal{R}_2^3(P)} \mathsf{B}.$$

It follows that

$$\begin{aligned} \mathsf{A} &\leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, \mathsf{c}) \;{}^*{\leftarrow}\; \mathsf{f}(\mathsf{c}, \mathsf{c}, \mathsf{c}, i_m \cdots i_1(\mathsf{c})) \to^* \mathsf{f}(\gamma_1(\mathsf{c}), \gamma_2(\mathsf{c}), i_1 \cdots i_m(\mathsf{c}), \mathsf{c}) \\ &\to \mathsf{g}(\gamma_1(\mathsf{c}), \gamma_2(\mathsf{c}), i_1 \cdots i_m(\mathsf{c})) \to^* \mathsf{g}(\mathsf{c}, \mathsf{c}, i_1 \cdots i_m(\mathsf{c})) \to^* \mathsf{g}(\mathsf{c}, \mathsf{c}, \mathsf{c}) \to \mathsf{B} \end{aligned}$$

with $\gamma_1 = \alpha_{i_1} \cdots \alpha_{i_m}$ and $\gamma_2 = \beta_{i_1} \cdots \beta_{i_m}$. The step $\mathsf{f}(\cdots) \to \mathsf{g}(\cdots)$ is possible only if $m \geqslant 1$ and the sequence from $\mathsf{g}(\gamma_1(\mathsf{c}), \gamma_2(\mathsf{c}), i_1 \cdots i_m(\mathsf{c}))$ to $\mathsf{g}(\mathsf{c}, \mathsf{c}, i_1 \cdots i_m(\mathsf{c}))$ entails that $\gamma_1 = \gamma_2$. We conclude that P admits a solution. $\square$

3 SCR ⇒ CR

Let $\mathcal{R}_3(P) = \mathcal{R}_1(P) \cup \{\mathsf{B} \to \mathsf{C}, \mathsf{C} \to \mathsf{A}\}$.

Lemma 3.1 *The TRS $\mathcal{R}_3(P)$ is confluent for every PCP instance P.*

Proof One easily checks that the linear TRS $\mathcal{R}_3'(P) = \mathcal{R}_3(P) \cup \{\mathsf{B} \to \mathsf{A}\}$ is strongly closed hence (strongly) confluent by Theorem 2.2. Since the relations $\to^*_{\mathcal{R}_3(P)}$ and $\to^*_{\mathcal{R}_3'(P)}$ coincide, $\mathcal{R}_3(P)$ is confluent. $\square$

Lemma 3.2 *The TRS $\mathcal{R}_3(P)$ is strongly confluent if and only if P admits a solution.*

Proof In a shortest rewrite sequence from A to B the rules $\mathsf{B} \to \mathsf{C}$ and $\mathsf{C} \to \mathsf{A}$ are not used. Hence $\mathsf{A} \to^*_{\mathcal{R}_3(P)} \mathsf{B}$ if and only if $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$. According to Lemma 2.1 we have to show that $\mathcal{R}_3(P)$ is strongly confluent if and only if $\mathsf{A} \to^*_{\mathcal{R}_3(P)} \mathsf{B}$. In $\mathcal{R}_3(P)$ we have

$\mathsf{B} \leftarrow \mathsf{g}(\mathsf{c},\mathsf{c}) \to \mathsf{A}$. If $\mathcal{R}_3(P)$ is strongly confluent then $\mathsf{B} \to^{=} \cdot {}^{*}\!\leftarrow \mathsf{A}$, so either $\mathsf{B} \;{}^{*}\!\leftarrow \mathsf{A}$ or $\mathsf{B} \to \mathsf{C} \;{}^{*}\!\leftarrow \mathsf{A}$. Since any reduction sequence from A to C must pass through B, in both cases we have the desired $\mathsf{A} \to^{*} \mathsf{B}$.

Conversely, if $\mathsf{A} \to^{*} \mathsf{B}$ then one easily checks that $\mathcal{R}_3(P)$ is strongly closed and therefore strongly confluent by Theorem 2.2. □

Corollary 3.3 *Strong confluence is not co-semi-decidable for linear confluent TRSs.* □

In the proofs of semi-decidability in this paper we make use of the observation that the set of conversions is recursively enumerable. This observation is an easy consequence of the fact that there are only countable infinitely many terms (because the set of variables is assumed to be countably infinite) and conversions.

Theorem 3.4 *Strong confluence is semi-decidable for linear TRSs.*

Proof Let $\mathcal{R}$ be an arbitrary finite linear TRS. According to Theorem 2.2 $\mathcal{R}$ is strongly confluent if and only if $s \to_{\mathcal{R}}^{=} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow t$ and $t \to_{\mathcal{R}}^{=} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow s$ for every critical pair $\langle s, t\rangle$ of $\mathcal{R}$. By enumerating and inspecting all conversions between s and t we easily obtain a semi-decision procedure for the problem whether both $s \to_{\mathcal{R}}^{=} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow t$ and $t \to_{\mathcal{R}}^{=} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow s$. Since $\mathcal{R}$ has finitely many critical pairs we obtain a semi-decision procedure for strong confluence by applying the previous procedure to all critical pairs in parallel. □

Linearity is essential in the above proof. We conjecture that for arbitrary TRSs strong confluence is not semi-decidable.

4 CR ⇒ WCR

Let $\mathcal{R}_4(P) = \mathcal{R}_1(P) \cup \{\mathsf{B} \to \mathsf{f}(\mathsf{c},\mathsf{c}), \mathsf{B} \to \mathsf{C}\}$.

Lemma 4.1 *The TRS $\mathcal{R}_4(P)$ is locally confluent for every PCP instance P.*

Proof One easily checks that all critical pairs of $\mathcal{R}_4(P)$ are joinable. □

Lemma 4.2 *The TRS $\mathcal{R}_4(P)$ is confluent if and only if P admits a solution.*

Proof In a shortest $\mathcal{R}_4(P)$-reduction sequence from A to B the rewrite rules $\mathsf{B} \to \mathsf{f}(\mathsf{c},\mathsf{c})$ and $\mathsf{B} \to \mathsf{C}$ are not used. Hence $\mathsf{A} \to_{\mathcal{R}_4(P)}^{*} \mathsf{B}$ if and only if $\mathsf{A} \to_{\mathcal{R}_1(P)}^{*} \mathsf{B}$. According to Lemma 2.1 we have to show that $\mathcal{R}_4(P)$ is confluent if and only if $\mathsf{A} \to_{\mathcal{R}_4(P)}^{*} \mathsf{B}$. In $\mathcal{R}_4(P)$ we have $\mathsf{A} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \leftarrow \mathsf{B} \to \mathsf{C}$. If $\mathcal{R}_4(P)$ is confluent then $\mathsf{A} \to^{*} \mathsf{C}$ which is equivalent to $\mathsf{A} \to^{*} \mathsf{B}$.

Conversely, if $\mathsf{A} \to^{*} \mathsf{B}$ then we obtain confluence by considering the TRS $\mathcal{R}_4'(P) = \mathcal{R}_4(P) \cup \{\mathsf{A} \to \mathsf{C}, \mathsf{f}(x,y) \to \mathsf{C}, \mathsf{g}(x,y) \to \mathsf{C}\}$. One easily shows that $\mathcal{R}_4'(P)$ is linear and strongly closed and thus (strongly) confluent by Theorem 2.2. Since the relations $\to_{\mathcal{R}_4(P)}^{*}$ and $\to_{\mathcal{R}_4'(P)}^{*}$ coincide, $\mathcal{R}_4(P)$ is confluent. □

Corollary 4.3 *Confluence is not co-semi-decidable for linear locally confluent TRSs.* □

To show the non-semi-decidability of confluence for the class of linear locally confluent TRSs we need a different construction. Let $\mathcal{R}_5(P)$ consist of the rules

$$
\begin{array}{lrcl@{\qquad}lrcll}
(1) & \mathsf{f}(\mathsf{c},\mathsf{c},i(z)) & \to & \mathsf{A} & (4) & \mathsf{g}(a(x),a(y)) & \to & \mathsf{g}(x,y) & \\
(2) & \mathsf{f}(x,y,i(z)) & \to & \mathsf{f}(\alpha_i(x),\beta_i(y),z) & & \mathsf{g}(a(x),b(y)) & \to & \mathsf{A} & \text{if } a \neq b \\
(2) & \mathsf{f}(\alpha_i(x),\beta_i(y),z) & \to & \mathsf{f}(x,y,i(z)) & & \mathsf{g}(a(x),\mathsf{c}) & \to & \mathsf{A} & \\
(3) & \mathsf{f}(x,y,\mathsf{c}) & \to & \mathsf{g}(x,y) & & \mathsf{g}(\mathsf{c},a(x)) & \to & \mathsf{A} &
\end{array}
$$

with b ranging over all elements of the alphabet Γ of P. The numbers in front of some of the rewrite rules are used for reference in the proof of Lemma 4.5 below.

Lemma 4.4 *The TRS $\mathcal{R}_5(P)$ is locally confluent for every PCP instance P.*

Proof One easily checks that all critical pairs of $\mathcal{R}_5(P)$ are joinable. □

Lemma 4.5 *The TRS $\mathcal{R}_5(P)$ is confluent if and only if P admits no solution.*

Proof If P admits a solution $\gamma = \alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$ for some $m \geqslant 1$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$ then we obtain the diverging reductions in $\mathcal{R}_5(P)$ consisting of

$$\mathsf{f}(\gamma(\mathsf{c}),\gamma(\mathsf{c}),\mathsf{c}) \to^+ \mathsf{f}(\mathsf{c},\mathsf{c},i_m \cdots i_1(\mathsf{c})) \to \mathsf{A}$$

and

$$\mathsf{f}(\gamma(\mathsf{c}),\gamma(\mathsf{c}),\mathsf{c}) \to \mathsf{g}(\gamma(\mathsf{c}),\gamma(\mathsf{c})) \to^+ \mathsf{g}(\mathsf{c},\mathsf{c}).$$

Since A and $\mathsf{g}(\mathsf{c},\mathsf{c})$ are different normal forms, $\mathcal{R}_5(P)$ is not confluent.

Conversely, assume that $\mathcal{R}_5(P)$ is not confluent; we have to prove that P admits a solution. Consider the many-sorted type declarations $\mathsf{A}\colon 1$, $\mathsf{c}\colon 2$, $i\colon 2 \to 2$ for all $i \in \{1,\ldots,n\}$, $a\colon 2 \to 2$ for all $a \in \Gamma$, $\mathsf{f}\colon 2 \times 2 \times 2 \to 1$, and $\mathsf{g}\colon 2 \times 2 \to 1$, which is compatible with $\mathcal{R}_5(P)$ by assigning type 2 to every variable. From Theorem 2.4 we conclude that $\mathcal{R}_5(P)$ admits diverging well-sorted reductions $u \mathrel{{}^*{\leftarrow}} t \to^* v$ for which u and v do not have a common reduct. Since well-sorted terms of type 2 are in normal form, t must have type 1. Hence the root symbol of t is f, g, or A, and t does not contain any of these symbols below the root. This implies that all rewrite steps in any reduction starting from t take place at the root position. Since A is in normal form and the TRS consisting of all g rules is orthogonal and hence confluent, we conclude that the root symbol of t is f. We have

$$u \mathrel{{}^*{\leftarrow}} \underbrace{u' \mathrel{{}^*{\leftarrow}} t \to^* v'} \to^* v$$

such that in the underbraced part only rules of type (2) are used and $u' \to^* u$ and $v' \to^* v$, if non-empty, start with a rule of type (1) or rule (3). Because rules of type (2) are reversible we easily obtain $u' \to^* v'$ by using only rules of type (2). It follows that $u' = \mathsf{f}(u_1,u_2,u_3)$, $v' = \mathsf{f}(v_1,v_2,v_3)$, and either $v_1 = \alpha_{i_m} \cdots \alpha_{i_1}(u_1)$, $v_2 = \beta_{i_m} \cdots \beta_{i_1}(u_2)$, and $u_3 = i_1 \cdots i_m(v_3)$ or $u_1 = \alpha_{i_m} \cdots \alpha_{i_1}(v_1)$, $u_2 = \beta_{i_m} \cdots \beta_{i_1}(v_2)$, and $v_3 = i_1 \cdots i_m(u_3)$, for some $m \geqslant 0$ and indices $i_1,\ldots,i_m \in \{1,\ldots,n\}$. Without loss of generality we consider the former. Now, if $u' \to^* u$ or $v' \to^* v$ is empty then u and v have a common reduct, contradicting the assumption. Hence both $u' \to^* u$ and $v' \to^* v$ are non-empty. If both sequences start with an application of a rule of type (1) then $u = v = \mathsf{A}$, contradicting the assumption. If both sequences start with rule (3) then $u_3 = v_3 = \mathsf{c}$, which is only possible if $m = 0$ and thus $u' = v'$. Consequently, one of the sequences $u' \to u$ and $v' \to^* v$ starts with a rule of type (1) and the other starts with rule (3). First suppose that $u' \to u$ starts with a rule of type (1). Then we may write

$$u = \mathsf{A} \leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}, i(u_3')) = u' \to^* v' = \mathsf{f}(v_1, v_2, \mathsf{c}) \to \mathsf{g}(v_1, v_2) \to^* v$$

for some $i \in \Gamma$ and thus $u_1 = u_2 = v_3 = \mathsf{c}$ and $u_3 = i(u_3')$. Hence $v_1 = \alpha_{i_m} \cdots \alpha_{i_1}(\mathsf{c})$ and $v_2 = \beta_{i_m} \cdots \beta_{i_1}(\mathsf{c})$. Because u and v are assumed to have no common reduct, v and thus also $\mathsf{g}(v_1, v_2)$ do not rewrite to A and this implies that only type (4) rules are applicable. It follows that $v_1 = v_2$ and thus $\alpha_{i_m} \cdots \alpha_{i_1} = \beta_{i_m} \cdots \beta_{i_1}$. From $i(u_3') = i_1 \cdots i_m(\mathsf{c})$ we infer that $m \geqslant 1$. So P has a solution. Next suppose that $u' \to u$ starts with a rule of type (3). Then we may write

$$u \;{}^*\!\leftarrow \mathsf{g}(u_1, u_2) \leftarrow \mathsf{f}(u_1, u_2, \mathsf{c}) = u' \to^* v' = \mathsf{f}(\mathsf{c}, \mathsf{c}, i(v_3')) \to \mathsf{A} = v$$

for some $i \in \Gamma$ and thus $u_3 = v_1 = v_2 = \mathsf{c}$ and $v_3 = i(v_3')$. This is only possible if $m = 0$ but then $v_3 = u_3$ contradicts $v_3 = i(v_3')$ and $u_3 = \mathsf{c}$. □

Corollary 4.6 *Confluence is not semi-decidable for linear locally confluent TRSs.* □

5 CR ⇒ GCR

Let $\mathcal{R}_6(P) = \mathcal{R}_6^1(P) \cup \mathcal{R}_6^2$ with $\mathcal{R}_6^1(P)$ consisting of the rules

$$\begin{array}{rcl@{\qquad}lcl}
\mathsf{A}(z) & \to & \mathsf{f}(\alpha(\mathsf{c}), \beta(\mathsf{c}), z) & & & \\
\mathsf{f}(x, y, z) & \to & \mathsf{f}(\alpha(x), \beta(y), z) & \mathsf{g}(x, y, z) & \to & \mathsf{A}(z) \\
\mathsf{f}(x, y, z) & \to & \mathsf{g}(x, y, z) & \mathsf{g}(a(x), a(y), z) & \to & \mathsf{g}(x, y, z) \\
\mathsf{f}(x, y, z) & \to & \mathsf{A}(z) & \mathsf{g}(\mathsf{c}, \mathsf{c}, z) & \to & \mathsf{B}(z)
\end{array}$$

and $\mathcal{R}_6^2$ of the rules

$$\begin{array}{rcl@{\qquad}rcl}
\mathsf{f}(\mathsf{D}, \mathsf{D}, \mathsf{D}) & \to & \mathsf{D} & \mathsf{c} & \to & \mathsf{D} \\
\mathsf{g}(\mathsf{D}, \mathsf{D}, \mathsf{D}) & \to & \mathsf{D} & \mathsf{A}(\mathsf{D}) & \to & \mathsf{D} \\
a(\mathsf{D}) & \to & \mathsf{D} & \mathsf{B}(\mathsf{D}) & \to & \mathsf{D}
\end{array}$$

Note that the only difference between $\mathcal{R}_6^1(P)$ and $\mathcal{R}_1(P)$ is the addition of an extra argument which is simply propagated.

Lemma 5.1 *The TRS $\mathcal{R}_6(P)$ is ground confluent for every PCP instance P.*

Proof In $\mathcal{R}_6^2 \subseteq \mathcal{R}_6(P)$ every ground term rewrites to D. (As usual we assume that the signature of a TRS consists of all function symbols occurring in its rewrite rules.) Hence $\mathcal{R}_6(P)$ is ground confluent. □

Lemma 5.2 *The TRS $\mathcal{R}_6(P)$ is confluent if and only if P admits a solution.*

Proof It is not difficult to see that the statements $\mathsf{A}(z) \to^*_{\mathcal{R}_6(P)} \mathsf{B}(z)$ and $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$ are equivalent. According to Lemma 2.1 we have to show that $\mathcal{R}_6(P)$ is confluent if and only if $\mathsf{A}(z) \to^*_{\mathcal{R}_6(P)} \mathsf{B}(z)$. In $\mathcal{R}_6(P)$ we have $\mathsf{A}(z) \leftarrow \mathsf{g}(\mathsf{c},\mathsf{c},z) \to \mathsf{B}(z)$ with $\mathsf{B}(z)$ in normal form. So if $\mathcal{R}_6(P)$ is confluent then necessarily $\mathsf{A}(z) \to^* \mathsf{B}(z)$.

Conversely, if $\mathsf{A}(z) \to^* \mathsf{B}(z)$ then we obtain confluence by considering the linear and strongly closed TRS $\mathcal{R}_6'(P) = \mathcal{R}_6(P) \cup \{\mathsf{A}(z) \to \mathsf{B}(z), \mathsf{f}(x,y,z) \to \mathsf{B}(z), \mathsf{g}(x,y,z) \to \mathsf{B}(z)\}$. □

Corollary 5.3 *Confluence is not co-semi-decidable for linear ground-confluent TRSs.* □

In order to show that confluence is not semi-decidable for ground confluent TRSs, we consider the TRS $\mathcal{R}_7(P) = \mathcal{R}_7^1(P) \cup \mathcal{R}_7^2$ with $\mathcal{R}_7^1(P)$ consisting of the rules

$$
\begin{array}{lrcll@{\qquad}rcl}
(1) & \mathsf{f}(\mathsf{c}(x),\mathsf{c}(y),i(z)) & \to & \mathsf{A}(x,y,z) & & \mathsf{A}(a(x),y,z) & \to & \mathsf{A}(x,y,z)\\
(2) & \mathsf{f}(x,y,i(z)) & \to & \mathsf{f}(\alpha_i(x),\beta_i(y),z) & & \mathsf{A}(i(x),y,z) & \to & \mathsf{A}(x,y,z)\\
(2) & \mathsf{f}(\alpha_i(x),\beta_i(y),z) & \to & \mathsf{f}(x,y,i(z)) & & \mathsf{A}(\mathsf{c}(x),y,z) & \to & \mathsf{A}(x,y,z)\\
(3) & \mathsf{f}(x,y,\mathsf{c}(z)) & \to & \mathsf{g}(x,y,z) & & \mathsf{A}(x,a(y),z) & \to & \mathsf{A}(x,y,z)\\
(4) & \mathsf{g}(a(x),a(y),z) & \to & \mathsf{g}(x,y,z) & & \mathsf{A}(x,i(y),z) & \to & \mathsf{A}(x,y,z)\\
 & \mathsf{g}(a(x),b(y),z) & \to & \mathsf{A}(x,y,z) & \text{if } a \neq b & \mathsf{A}(x,\mathsf{c}(y),z) & \to & \mathsf{A}(x,y,z)\\
 & \mathsf{g}(a(x),\mathsf{c}(y),z) & \to & \mathsf{A}(x,y,z) & & \mathsf{A}(x,y,a(z)) & \to & \mathsf{A}(x,y,z)\\
 & \mathsf{g}(\mathsf{c}(x),a(y),z) & \to & \mathsf{A}(x,y,z) & & \mathsf{A}(x,y,i(z)) & \to & \mathsf{A}(x,y,z)\\
 & & & & & \mathsf{A}(x,y,\mathsf{c}(z)) & \to & \mathsf{A}(x,y,z)
\end{array}
$$

and $\mathcal{R}_7^2$ of the rules

$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl@{\qquad}rcl}
\mathsf{c}(\mathsf{D}) & \to & \mathsf{D} & \mathsf{f}(\mathsf{D},y,z) & \to & \mathsf{D} & \mathsf{g}(\mathsf{D},y,z) & \to & \mathsf{D} & \mathsf{A}(\mathsf{D},y,z) & \to & \mathsf{D}\\
a(\mathsf{D}) & \to & \mathsf{D} & \mathsf{f}(x,\mathsf{D},z) & \to & \mathsf{D} & \mathsf{g}(x,\mathsf{D},z) & \to & \mathsf{D} & \mathsf{A}(x,\mathsf{D},z) & \to & \mathsf{D}\\
i(\mathsf{D}) & \to & \mathsf{D} & \mathsf{g}(x,y,\mathsf{D}) & \to & \mathsf{D} & \mathsf{f}(x,y,\mathsf{D}) & \to & \mathsf{D} & \mathsf{A}(x,y,\mathsf{D}) & \to & \mathsf{D}
\end{array}
$$

Lemma 5.4 *The TRS $\mathcal{R}_7(P)$ is ground confluent for every PCP instance P.*

Proof An easy induction proof reveals that every ground term rewrites to D. □

Lemma 5.5 *The TRS $\mathcal{R}_7^1(P)$ is confluent if P admits no solution.*

Proof The proof is similar to the "if" direction in the proof of Lemma 4.5. Suppose that $\mathcal{R}_7^1(P)$ is not confluent. Consider the many-sorted type declarations $\mathsf{f},\mathsf{g},\mathsf{A}\colon 2\times2\times2 \to 1$, $\mathsf{c}\colon 2 \to 2$, $i\colon 2 \to 2$ for all $i \in \{1,\ldots,n\}$, and $a\colon 2 \to 2$ for all $a \in \Gamma$, which is compatible with $\mathcal{R}_7^1(P)$ by assigning type 2 to every variable. From Theorem 2.4 we conclude that $\mathcal{R}_7^1(P)$ admits diverging well-sorted reductions $u \mathrel{{}^*{\leftarrow}} t \to^* v$ for which u and v do not have a common reduct.

Since well-sorted terms of type 2 are in normal form, t must have type 1. Hence the root symbol of t is f, g, or A, and t does not have any of these symbols below the root. This implies that all rewrite steps in any reduction starting from t take place at the root position. Because the TRS consisting of all g and A rules is confluent, the root symbol of t must be f and hence we may write $t = \mathsf{f}(t_1, t_2, t_3)$. Note that every t_i contains a single variable x_i and every reduct of t contains x_i in its i-th argument. We have $u \;{}^*\!\!\leftarrow u' \;{}^*\!\!\leftarrow t \to^* v' \to^* v$ such that in $u' \;{}^*\!\!\leftarrow t \to^* v'$ only rules of type (2) are used and $u' \to^* u$ and $v' \to^* v$, if non-empty, start with a rule of type (1) or rule (3). Because rules of type (2) are reversible we easily obtain $u' \to^* v'$ by using only rules of type (2). It follows that $u' = \mathsf{f}(u_1, u_2, u_3)$, $v' = \mathsf{f}(v_1, v_2, v_3)$, and either $v_1 = \alpha_{i_m} \cdots \alpha_{i_1}(u_1)$, $v_2 = \beta_{i_m} \cdots \beta_{i_1}(u_2)$, and $u_3 = i_1 \cdots i_m(v_3)$ or $u_1 = \alpha_{i_m} \cdots \alpha_{i_1}(v_1)$, $u_2 = \beta_{i_m} \cdots \beta_{i_1}(v_2)$, and $v_3 = i_1 \cdots i_m(u_3)$, for some $m \geqslant 0$ and indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Without loss of generality we consider the former. Now, if $u' \to^* u$ or $v' \to^* v$ is empty then t is confluent, contradicting the assumption. Hence both $u' \to^* u$ and $v' \to^* v$ are non-empty. If both sequences start with an application of a rule of type (1) then, due to the shape of the A rules, u' and v' reduce to the common reduct $\mathsf{A}(x_1, x_2, x_3)$. If both sequences start with rule (3) then $u_3 = \mathsf{c}(u_3')$ and $v_3 = \mathsf{c}(v_3')$, which is only possible if $m = 0$ and thus $u' = v'$. We already observed that the TRS consisting of all g and A rules is confluent and hence u' and v' have a common reduct. Consequently, one of the sequences $u' \to u'' \to^* u$ and $v' \to v'' \to^* v$ starts with a rule of type (1) and the other starts with rule (3). First suppose that $u' \to u'' \to^* u$ starts with a rule of type (1). This implies that $u'' = \mathsf{A}(u_1', u_2', u_3')$ with $u_1 = \mathsf{c}(u_1')$, $u_2 = \mathsf{c}(u_2')$, and $u_3 = i(u_3')$ for some $i \in \{1, \ldots, n\}$. Moreover, $v'' = \mathsf{g}(v_1, v_2, v_3')$ with $v_3 = \mathsf{c}(v_3')$. We have $u'' \to^* \mathsf{A}(x_1, x_2, x_3)$. Because t is non-confluent, v'' cannot reduce to $\mathsf{A}(x_1, x_2, x_3)$ which implies that only rules of type (4) can be applied to v''. It follows that $v_1 = \gamma(\mathsf{c}(v_1'))$ and $v_2 = \gamma(\mathsf{c}(v_2'))$ for some string $\gamma \in \Gamma^*$. Hence $\gamma = \alpha_{i_m} \cdots \alpha_{i_1} = \beta_{i_m} \cdots \beta_{i_1}$ with $m \geqslant 1$. So P has a solution. In the remaining case we have $u'' = \mathsf{g}(u_1, u_2, u_3')$ with $u_3 = \mathsf{c}(u_3')$ and $v'' = \mathsf{A}(v_1', v_2', v_3')$ with $v_1 = \mathsf{c}(v_1')$, $v_2 = \mathsf{c}(v_2')$, and $v_3 = i(v_3')$ for some $i \in \{1, \ldots, n\}$. This is only possible if $m = 0$ but then $v_3 = u_3$, which contradicts $v_3 = i(v_3')$ and $u_3 = \mathsf{c}(u_3')$. ☐

Lemma 5.6 *The TRS $\mathcal{R}_7(P)$ is confluent if and only if P admits no solution.*

Proof First suppose that P admits a solution $\gamma = \alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$ for some $m \geqslant 1$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$. Then we have the following diverging reductions starting from $t = \mathsf{f}(\gamma(\mathsf{c}(x)), \gamma(\mathsf{c}(y)), \mathsf{c}(z))$:

$$t \to^+ \mathsf{f}(\mathsf{c}(x), \mathsf{c}(y), i_m \cdots i_1(\mathsf{c}(z))) \to \mathsf{A}(x, y, i_{m-1} \cdots i_1(\mathsf{c}(z))) \to^+ \mathsf{A}(x, y, z)$$

and

$$t \to \mathsf{g}(\gamma(\mathsf{c}(x)), \gamma(\mathsf{c}(y)), \mathsf{c}(z)) \to^+ \mathsf{g}(\mathsf{c}(x), \mathsf{c}(y), \mathsf{c}(z)).$$

Since $\mathsf{A}(x, y, z)$ and $\mathsf{g}(\mathsf{c}(x), \mathsf{c}(y), \mathsf{c}(z))$ are different normal forms, $\mathcal{R}_7(P)$ is not confluent.

Conversely, suppose that $\mathcal{R}_7(P)$ is not confluent. If we can show that $\mathcal{R}_7^1(P)$ is not confluent then the result follows from the preceding lemma. From an inspection of the rewrite rules of $\mathcal{R}_7(P)$ we easily infer that the set of terms that contain an occurrence of D is closed under conversion. Moreover, it is easy to prove by induction on t that every such term rewrites to D. Hence, if t is a non-confluent term then t contains no occurrences of D and rules of $\mathcal{R}_7^2$ are never applied in any rewrite sequence starting from t. It follows that $\mathcal{R}_7^1(P)$ is not confluent. ☐

Corollary 5.7 *Confluence is not semi-decidable for linear ground-confluent TRSs.* □

6 CR $\Rightarrow$ NF

Let $\mathcal{R}_8(P) = \mathcal{R}_1(P) \cup \{\mathsf{B} \to \mathsf{B}\}$.

Lemma 6.1 *The TRS $\mathcal{R}_8(P)$ has the normal form property for every PCP instance P.*

Proof The set of normal forms of $\mathcal{R}_8(P)$ coincides with the set of terms that rewrite to a normal form. In other words, reducible terms in $\mathcal{R}_8(P)$ have no normal form. Hence the normal form property is trivially satisfied. □

Lemma 6.2 *The TRS $\mathcal{R}_8(P)$ is confluent if and only if P admits a solution.*

Proof Since the relations $\to^*_{\mathcal{R}_8(P)}$ and $\to^*_{\mathcal{R}_1(P)}$ coincide, $\mathcal{R}_8(P)$ is confluent if and only if $\mathcal{R}_1(P)$ is confluent. Hence the result follows from Lemma 2.3. □

Corollary 6.3 *Confluence is not co-semi-decidable for linear TRSs with the normal form property.* □

Let $\mathcal{R}_9(P)$ be the union of $\mathcal{R}_5(P)$ and the rules

$$\begin{array}{ccc} \mathsf{f}(x,y,z) \to \mathsf{f}(x,y,z) & a(x) \to a(x) & \mathsf{A} \to \mathsf{A} \\ \mathsf{g}(x,y) \to \mathsf{g}(x,y) & i(x) \to i(x) & \mathsf{c} \to \mathsf{c} \end{array}$$

Lemma 6.4 *The TRS $\mathcal{R}_9(P)$ has the normal form property for every PCP instance P.*

Proof Only variables are in normal form. Since $\mathcal{R}_9(P)$ is non-collapsing, a variable is convertible only with itself. Hence the normal form property is trivially satisfied. □

Lemma 6.5 *The TRS $\mathcal{R}_9(P)$ is confluent if and only if P admits no solution.*

Proof Since $\to^*_{\mathcal{R}_9(P)} = \to^*_{\mathcal{R}_5(P)}$, the result follows from Lemma 4.5. □

Corollary 6.6 *Confluence is not semi-decidable for linear TRSs with the normal form property.* □

7 NF $\Rightarrow$ UN

Lemma 7.1 *The TRS $\mathcal{R}_1(P)$ has unique normal forms for every PCP instance P.*

Proof Consider the confluent TRS $\mathcal{R}_1'(P)$ defined in the proof of Lemma 2.3. The relations $\leftrightarrow^*_{\mathcal{R}_1(P)}$ and $\leftrightarrow^*_{\mathcal{R}_1'(P)}$ clearly coincide. Also the normal forms of the two TRSs are the same. It follows that $\mathcal{R}_1(P)$ has unique normal forms. □

We already observed (Lemma 2.3) that the TRS $\mathcal{R}_1(P)$ has the normal form property if and only if P has a solution.

Corollary 7.2 *The normal form property is not co-semi-decidable for linear TRSs with unique normal forms.* □

Let $\mathcal{R}_{10}(P)$ be the union of $\mathcal{R}_5(P)$ and the two rules

$$\begin{aligned} \mathsf{f}(x,y,z) &\to \mathsf{f}(x,y,z) \\ \mathsf{g}(x,y) &\to \mathsf{g}(x,y) \end{aligned}$$

Lemma 7.3 *The TRS $\mathcal{R}_{10}(P)$ has unique normal forms for every PCP instance P.*

Proof Assume $n_1 \leftrightarrow^*_{\mathcal{R}_{10}(P)} n_2$ for two normal forms n_1, n_2; we have to prove $n_1 = n_2$. For a term t, let $\phi(t)$ denote the result of replacing all maximal subterms in t with root symbol f or g by A. An inspection of the rewrite rules of $\mathcal{R}_{10}(P)$ reveals that $\phi(s) = \phi(t)$ if $s \to_{\mathcal{R}_{10}(P)} t$. It follows that $\phi(n_1) = \phi(n_2)$. Since n_1, n_2 are normal forms, they do not contain the symbols f and g, and thus $n_1 = \phi(n_1) = \phi(n_2) = n_2$. $\square$

Lemma 7.4 *The TRS $\mathcal{R}_{10}(P)$ has the normal form property if and only if P admits no solution.*

Proof Assume that P admits a solution. Then we have a conversion $\mathsf{g}(\mathsf{c},\mathsf{c}) \leftrightarrow^*_{\mathcal{R}_{10}(P)} \mathsf{A}$ as in the first part of the proof of Lemma 4.5. Since $\mathsf{g}(\mathsf{c},\mathsf{c})$ does not rewrite to A and A is a normal form, we conclude that $\mathcal{R}_{10}(P)$ does not have the normal form property.

Conversely, assume that P admits no solution. Then according to Lemma 4.5 $\mathcal{R}_5(P)$ is confluent. Since $\to^*_{\mathcal{R}_{10}(P)} = \to^*_{\mathcal{R}_5(P)}$ the TRS $\mathcal{R}_{10}(P)$ is confluent and hence has the normal form property. $\square$

Corollary 7.5 *The normal form property is not semi-decidable for linear TRSs with unique normal forms.* $\square$

8 UN $\Rightarrow$ CON

Let $\mathcal{R}_{11}(P)$ be the union of $\mathcal{R}_2(P)$ and the rules

$$\begin{array}{lll} \mathsf{f}(x,y,z,w) \to \mathsf{f}(x,y,z,w) & a(x) \to a(x) & \mathsf{c} \to \mathsf{c} \\ \mathsf{g}(x,y,z) \to \mathsf{g}(x,y,z) & i(x) \to i(x) & \end{array}$$

Lemma 8.1 $\mathsf{A} \leftrightarrow^*_{\mathcal{R}_{11}(P)} \mathsf{B}$ *if and only if P admits a solution.*

Proof Immediate consequence of Lemma 2.5 as $\leftrightarrow^*_{\mathcal{R}_{11}(P)} = \leftrightarrow^*_{\mathcal{R}_2(P)}$. $\square$

Lemma 8.2 *The TRS $\mathcal{R}_{11}(P)$ is consistent for every PCP instance P.*

Proof Trivial, as $\mathcal{R}_{11}(P)$ lacks collapsing rules. $\square$

Lemma 8.3 *The TRS $\mathcal{R}_{11}(P)$ has unique normal forms if and only if P does not have a solution.*

Proof According to Lemma 8.1 we have to show that $\mathcal{R}_{11}(P)$ admits two different convertible normal forms if and only if A and B are convertible. Since A and B are normal forms, the "if" direction is trivial.

Conversely, suppose that $\mathcal{R}_{11}(P)$ admits two different convertible normal forms t_1, t_2. The only normal forms of $\mathcal{R}_{11}(P)$ are A, B, and variables. Because $\mathcal{R}_{11}(P)$ is non-collapsing, variables are convertible only to themselves. Hence $t_1 = \mathsf{A}$ and $t_2 = \mathsf{B}$ or vice-versa. $\square$

Corollary 8.4 *The property of having unique normal forms is not semi-decidable for linear consistent TRSs.* □

Theorem 8.5 *The property of having unique normal forms is co-semi-decidable for TRSs.*

Proof By enumerating all conversions we can easily find out if there exists a conversion between different normal forms. Hence the property of not having unique normal forms is semi-decidable. □

9 CON $\Rightarrow$ CON$^{\rightarrow}$

Let $\mathcal{R}_{12}(P) = \mathcal{R}_2(P) \cup \mathcal{R}_{12}^1$ with $\mathcal{R}_{12}^1$ consisting of the two rules

$$\mathsf{e}(\mathsf{A}, x) \to \mathsf{C}$$
$$\mathsf{e}(\mathsf{B}, x) \to x$$

Lemma 9.1 $\mathsf{A} \leftrightarrow^*_{\mathcal{R}_{12}(P)} \mathsf{B}$ *if and only if P admits a solution.*

Proof We show that in a shortest conversion between A and B rules of $\mathcal{R}_{12}^1$ are not used. The desired result then follows from Lemma 2.5. Suppose to the contrary that in a shortest conversion between A and B rules of $\mathcal{R}_{12}^1$ are used. It is not difficult to see that this is only possible if the conversion contains an outermost e symbol that is introduced and eliminated by a rule of $\mathcal{R}_{12}^1$. In other words, there exists a fragment

$$C[t] \leftarrow \underbrace{C[\mathsf{e}(t_1, t_2)] \leftrightarrow^* C'[\mathsf{e}(t_1', t_2')]} \to C'[t'] \tag{1}$$

such that every step in the underbraced part is of the form $C_1[\mathsf{e}(s_1, s_2)] \leftrightarrow C_2[\mathsf{e}(s_1', s_2')]$ with no occurrences of e above the displayed ones and either (i) a rewrite rule of $\mathcal{R}_2(P)$ is applied above the displayed occurrences of e, (ii) a rewrite rule is applied to a subterm of $C_1[\mathsf{e}(s_1, s_2)]$ ($C_2[\mathsf{e}(s_1', s_2')]$) disjoint from $\mathsf{e}(s_1, s_2)$ ($\mathsf{e}(s_1', s_2')$), or (iii) a rewrite rule is applied to one the arguments of the displayed occurrences of e. Because $\mathcal{R}_2(P)$ is linear and variable preserving, we can shift all (i) steps in the underbraced part in front of $C[t]$. We do the same with all (ii) steps. The result is a new fragment

$$C[t] \leftrightarrow^* C'[t] \leftarrow \underbrace{C'[\mathsf{e}(t_1, t_2)] \leftrightarrow^* C'[\mathsf{e}(t_1', t_2')]} \to C'[t'] \tag{2}$$

of the same length as (1) such that all steps in the underbraced part take place below the displayed occurrences of e. So $t_1 \leftrightarrow^* t_1'$ and $t_2 \leftrightarrow^* t_2'$. There are four possibilities:

1. $t_1 = t_1' = \mathsf{B}$, $t = t_2$, and $t' = t_2'$,
2. $t_1 = t_1' = \mathsf{A}$ and $t = t' = \mathsf{C}$,
3. $t_1 = \mathsf{B}$, $t_1' = \mathsf{A}$, $t = t_2$, and $t' = \mathsf{C}$,
4. $t_1 = \mathsf{A}$, $t_1' = \mathsf{B}$, $t = \mathsf{C}$, and $t' = t_2'$.

In the first case we obtain the shorter fragment

$$C[t] \leftrightarrow^* C'[t] = C'[t_2] \leftrightarrow^* C'[t_2'] = C'[t']$$

contradicting the fact that the given conversion between A and B is shortest. In the second case we obtain the shorter fragment

$$C[t] \leftrightarrow^* C'[t] = C'[\mathsf{C}] = C'[t'].$$

In the third case we have the shorter conversion $\mathsf{B} = t_1 \leftrightarrow^* t_1' = \mathsf{A}$ between A and B, again contradicting the fact that the given conversion between A and B is shortest. Finally, in the fourth case we obtain a contradiction in the same way. □

Lemma 9.2 *The TRS $\mathcal{R}_{12}(P)$ is consistent with respect to reduction.*

Proof If $\mathcal{R}_{12}(P)$ is not consistent with respect to reduction then there must be different variables x and y and a term t such that

$$x \leftarrow \mathsf{e}(\mathsf{B}, x) \;{}^*\!\!\leftarrow t \rightarrow^* \mathsf{e}(\mathsf{B}, y) \rightarrow y$$

because $\mathsf{e}(\mathsf{B}, x) \rightarrow x$ is the only collapsing rule in $\mathcal{R}_8(P)$. Only terms in the set

$$S_x = \{\mathsf{e}(t_1, x), \mathsf{e}(t_1, \mathsf{e}(t_2, x)), \mathsf{e}(t_1, \mathsf{e}(t_2, \mathsf{e}(t_3, x))), \cdots \mid t_i \rightarrow^* \mathsf{B}\}$$

rewrite to $\mathsf{e}(\mathsf{B}, x)$. Similarly, only terms in the set

$$S_y = \{\mathsf{e}(t_1, y), \mathsf{e}(t_1, \mathsf{e}(t_2, y)), \mathsf{e}(t_1, \mathsf{e}(t_2, \mathsf{e}(t_3, y))), \cdots \mid t_i \rightarrow^* \mathsf{B}\}$$

rewrite to $\mathsf{e}(\mathsf{B}, y)$. However, as $S_x \cap S_y = \emptyset$, term t does not exist. □

Lemma 9.3 *The TRS $\mathcal{R}_{12}(P)$ is consistent if and only if P does not have a solution.*

Proof According to Lemma 9.1 we have to show that $\mathcal{R}_{12}(P)$ is inconsistent if and only if A and B are convertible. If $\mathsf{A} \leftrightarrow^*_{\mathcal{R}_{12}(P)} \mathsf{B}$ then $\mathcal{R}_{12}(P)$ is inconsistent as

$$x \leftarrow \mathsf{e}(\mathsf{B}, x) \leftrightarrow^* \mathsf{e}(\mathsf{A}, x) \rightarrow \mathsf{C} \leftarrow \mathsf{e}(\mathsf{A}, y) \leftrightarrow^* \mathsf{e}(\mathsf{B}, y) \rightarrow y$$

for different variables x and y. If $\mathcal{R}_{12}(P)$ is inconsistent then all terms are convertible. In particular, $\mathsf{A} \leftrightarrow^*_{\mathcal{R}_{12}(P)} \mathsf{B}$. □

Corollary 9.4 *Consistency is not semi-decidable for linear terminating TRSs that are consistent with respect to reduction.* □

Theorem 9.5 *Consistency is co-semi-decidable for TRSs.*

Proof By enumerating and inspecting all conversions we easily obtain a semi-decision procedure for the problem whether there exists a conversion between different variables. Hence inconsistency is semi-decidable. □

10 UN $\Rightarrow$ UN$^{\rightarrow}$

Let $\mathcal{R}_{13}(P) = \mathcal{R}_{11}(P) \cup \mathcal{R}_{12}^1 \cup \{\mathsf{e}(x,y) \to \mathsf{e}(x,y), \mathsf{C} \to \mathsf{C}\}$.

Lemma 10.1 $\mathsf{A} \leftrightarrow^*_{\mathcal{R}_{13}(P)} \mathsf{B}$ *if and only if P admits a solution.*

Proof Immediate consequence of Lemma 9.1 as $\leftrightarrow^*_{\mathcal{R}_{13}(P)} = \leftrightarrow^*_{\mathcal{R}_{12}(P)}$. □

Lemma 10.2 *The TRS $\mathcal{R}_{13}(P)$ has unique normal forms with respect to reduction for every PCP instance P.*

Proof By induction on the structure of terms we can easily prove that every term has at most one normal form. □

Lemma 10.3 *The TRS $\mathcal{R}_{13}(P)$ has unique normal forms if and only if P does not have a solution.*

Proof Since the normal forms of $\mathcal{R}_{13}(P)$ and $\mathcal{R}_{11}(P)$ coincide, the "if" direction follows from Lemma 8.3. Suppose that $\mathcal{R}_{13}(P)$ admits two different convertible normal forms t_1, t_2. According to Lemma 10.1 it suffices to show that A and B are convertible. The only normal forms of $\mathcal{R}_{13}(P)$ are A, B, and variables. If t_1 and t_2 are different variables then we obtain a conversion between A and B by substituting A for all occurrences of t_1 and B for all occurrences of A in the conversion $t_1 \leftrightarrow^* t_2$. If one of the normal forms t_1, t_2 is a variable and the other is A (B) then we obtain a conversion between A and B by substituting B (A) for all occurrences of the variable in the conversion $t_1 \leftrightarrow^* t_2$. □

Corollary 10.4 *The property of having unique normal forms is not semi-decidable for linear TRSs that have unique normal forms with respect to reduction.* □

In Section 8 we already observed that the the property of having unique normal forms is co-semi-decidable.

11 UN$^{\rightarrow}$ $\Rightarrow$ CON$^{\rightarrow}$

Let $\mathcal{R}_{14}(P) = \mathcal{R}_1(P) \cup \{\mathsf{A} \to \mathsf{C}\}$.

Lemma 11.1 *The TRS $\mathcal{R}_{14}(P)$ is consistent with respect to reduction for every PCP instance P.*

Proof Trivial, as $\mathcal{R}_{14}(P)$ lacks collapsing rules. □

Lemma 11.2 *The TRS $\mathcal{R}_{14}(P)$ has unique normal forms with respect to reduction if and only if P does not have a solution.*

Proof If P admits a solution then $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$ and thus $\mathsf{A} \to^*_{\mathcal{R}_{14}(P)} \mathsf{B}$ by Lemma 2.1. Since $\mathsf{A} \to_{\mathcal{R}_{14}(P)} \mathsf{C}$ and both B and C are normal forms, $\mathcal{R}_{14}(P)$ does not have unique normal forms with respect to reduction.

Conversely, suppose that P does not have a solution. Then, by Lemma 2.1, $\mathsf{A} \to^*_{\mathcal{R}_1(P)} \mathsf{B}$ does not hold. This immediately implies that $\mathsf{A} \to^*_{\mathcal{R}_{14}(P)} \mathsf{B}$ does not hold. Now one easily shows by structural induction that every term has at most one normal form. Hence $\mathcal{R}_{14}(P)$ has unique normal forms with respect to reduction. □

Corollary 11.3 *The property of having unique normal forms with respect to reduction is not semi-decidable for linear TRSs that are consistent with respect to reduction.* □

Theorem 11.4 *The property of having unique normal forms with respect to reduction is co-semi-decidable for TRSs.*

Proof By enumerating and inspecting all conversions we easily obtain a semi-decision procedure for the problem whether there exists a conversion of the form $s\ {}^*\!\!\leftarrow \cdot \rightarrow^* t$ with s and t different normal forms. Hence the property of not having unique normal forms with respect to reduction is semi-decidable. □

12 WCR, GCR, CON$^{\rightarrow}$

In this section we investigate (semi-)decidability issues for the three properties in the confluence hierarchy that do not appear in an X position of an implication $X \Rightarrow Y$.

We start with local confluence. From Lemma 2.3 we obtain the following result.

Corollary 12.1 *Local confluence is not co-semi-decidable for linear TRSs.* □

Theorem 12.2 *Local confluence is semi-decidable for TRSs.*

Proof Let $\mathcal{R}$ be an arbitrary finite TRS. By enumerating and inspecting all conversions between two terms s and t we easily obtain a semi-decision procedure for the problem whether s and t are joinable. Applying this procedure in parallel to the finitely many critical pairs of $\mathcal{R}$ yields a semi-decision procedure for local confluence. □

Next we consider ground confluence. In the introduction we already mentioned that ground confluence is undecidable for terminating TRSs. Actually from the proof in [9, Theorem 3.3] it follows that ground confluence is not semi-decidable. From Lemma 2.3 we obtain the following result.

Corollary 12.3 *Ground confluence is not co-semi-decidable for linear TRSs.* □

Finally, we consider consistency with respect to reduction.

Theorem 12.4 *Consistency with respect to reduction is co-semi-decidable for TRSs.*

Proof By enumerating and inspecting all conversions we easily obtain a semi-decision procedure for the problem whether there exists a conversion of the form $s\ {}^*\!\!\leftarrow \cdot \rightarrow^* t$ with s and t different variables. Hence the property of not being consistent with respect to reduction is semi-decidable. □

Let the TRS $\mathcal{R}_{15}(P)$ consist of the rules

$$\begin{array}{rlrl}
\mathsf{f}(\alpha(x),\beta(y),z) & \rightarrow \mathsf{f}(x,y,z) & \mathsf{g}(a(x),a(y),z) & \rightarrow \mathsf{g}(x,y,z) \\
\mathsf{f}(\mathsf{c},\mathsf{c},z) & \rightarrow z & \mathsf{g}(\mathsf{c},\mathsf{c},z) & \rightarrow z \\
\mathsf{h}(x,y,z,w) & \rightarrow \mathsf{f}(x,y,z) & \mathsf{h}(a(x),a(y),z,w) & \rightarrow \mathsf{g}(x,y,w)
\end{array}$$

Note that $\mathcal{R}_{15}(P)$ is terminating for every PCP instance P.

Lemma 12.5 *The TRS $\mathcal{R}_{15}(P)$ is consistent with respect to reduction if and only P admits a solution.*

Proof If P admits a solution $\gamma = \alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$ for some $m \geqslant 1$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$ then we have the following diverging reductions in $\mathcal{R}_{15}(P)$ starting from the term $t = \mathsf{h}(\gamma(\mathsf{c}), \gamma(\mathsf{c}), x, y)$:

$$x \leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}, x) \; {}^*\!\!\leftarrow \mathsf{f}(\gamma(\mathsf{c}), \gamma(\mathsf{c}), x) \leftarrow t \rightarrow \mathsf{g}(\gamma'(\mathsf{c}), \gamma'(\mathsf{c}), y) \rightarrow^* \mathsf{g}(\mathsf{c}, \mathsf{c}, y) \rightarrow y.$$

Here γ' is the string γ minus its first symbol. Hence $\mathcal{R}_{15}(P)$ is not consistent with respect to reduction.

Conversely, assume that $\mathcal{R}_{15}(P)$ is not consistent with respect to reduction. Then $x \; {}^*\!\!\leftarrow t \rightarrow^* y$ for some term t and distinct variables x and y. Without loss of generality we assume that the size of t is minimal. This immediately implies that the root symbol of t must be f, g, or h. We show that it must be h. If $t = \mathsf{f}(t_1, t_2, t_3)$ then the reduction from t to x must be of the form $t \rightarrow^* \mathsf{f}(\mathsf{c}, \mathsf{c}, t_3') \rightarrow t_3' \rightarrow^* x$ with $t_3 \rightarrow^* t_3'$. Hence $t_3 \rightarrow^* x$. The same reasoning yields $t_3 \rightarrow^* y$, contradicting the minimality of t. A similar argument reveals that the root symbol of t cannot be g. Hence $t = \mathsf{h}(t_1, t_2, t_3, t_4)$. We must have

$$x \; {}^*\!\!\leftarrow u \; {}^*\!\!\leftarrow \mathsf{h}(t_1, t_2, t_3, t_4) \rightarrow^* u' \rightarrow^* y \tag{3}$$

with $t_3 \rightarrow^* u$ or $t_4 \rightarrow^* u$ and $t_3 \rightarrow^* u'$ or $t_4 \rightarrow^* u'$. If $t_3 \rightarrow^* u$ and $t_3 \rightarrow^* u'$ then t is not minimal as $x \; {}^*\!\!\leftarrow u \; {}^*\!\!\leftarrow t_3 \rightarrow^* u' \rightarrow y$. Likewise, t is not minimal if $t_4 \rightarrow^* u$ and $t_4 \rightarrow^* u'$. Now suppose that $t_3 \rightarrow^* u$ and $t_4 \rightarrow^* u'$. (In the remaining case $t_4 \rightarrow^* u$ and $t_3 \rightarrow^* u'$ we obtain a PCP solution in exactly the same way.) Then we obtain the divergence

$$x \; {}^*\!\!\leftarrow \mathsf{h}(t_1, t_2, x, y) \rightarrow^* y \tag{4}$$

from (3) by replacing t_3 by x and t_4 by y. We claim that t_1 and t_2 contain no occurrences of f, g, and h. The reason is that any subterm in t_1 or t_2 with one of these function symbols at its root position can be replaced by a fresh variable without affecting the possibility to perform the above diverging reductions; just drop all rewrite steps that took place at or below the replaced subterm. It follows that all steps in (4) take place at the root position. We may therefore write (4) as

$$x \leftarrow \mathsf{f}(\mathsf{c}, \mathsf{c}, x) \; {}^*\!\!\leftarrow \mathsf{f}(t_1, t_2, x) \leftarrow \mathsf{h}(t_1, t_2, x, y) \rightarrow \mathsf{g}(t_1', t_2', y) \rightarrow^* \mathsf{g}(\mathsf{c}, \mathsf{c}, y) \rightarrow y$$

with $t_1 = a(t_1')$ and $t_2 = a(t_2')$ for some $a \in \Gamma$. Now $\alpha_{i_1} \cdots \alpha_{i_m}(\mathsf{c}) = t_1 = t_2 = \beta_{i_1} \cdots \beta_{i_m}(\mathsf{c})$ for some $m \geqslant 1$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$. Hence P admits a solution. $\square$

Corollary 12.6 *Consistency with respect to reduction is not semi-decidable for linear terminating TRSs.* $\square$

13 Conclusions and Further Research

The results proved in this paper are summarized in the following tables:

$X \Rightarrow Y$	(1)	(2)	(3)	(4)
SCR $\Rightarrow$ CR		3.4^a	3.3	
CR $\Rightarrow$ WCR	4.6		4.3	
CR $\Rightarrow$ GCR	5.7		5.3	
CR $\Rightarrow$ NF	6.6		6.3	
NF $\Rightarrow$ UN	7.5		7.2	
UN $\Rightarrow$ CON	8.4			8.5
CON $\Rightarrow$ CON$^{\rightarrow}$	9.4^b			9.5
UN $\Rightarrow$ UN$^{\rightarrow}$	10.4			8.5
UN$^{\rightarrow}$ $\Rightarrow$ CON$^{\rightarrow}$	11.3			11.4

X	$(1')$	(2)	$(3')$	(4)
WCR		12.2	12.1	
GCR	$[9]^b$		12.3	
CON$^{\rightarrow}$	12.6^b			12.4

[a] only for linear TRSs
[b] even for terminating TRSs

Here (1) means that X is not a semi-decidable property of linear TRSs that satisfy property Y, (2) means that X is semi-decidable, (3) means that X is not a co-semi-decidable property of linear TRSs that satisfy property Y, (4) means that X is co-semi-decidable, and $(1')$ and $(3')$ mean the same as (1) and (3) without the qualification "that satisfy property Y".

In Section 3 we already mentioned the open problem whether strong confluence is semi-decidable for arbitrary (finite) TRSs. Even for terminating TRSs it is unknown whether strong confluence is (semi-)decidable. All other properties in the confluence hierarchy, except GCR, CON and CON$^{\rightarrow}$, are equivalent and decidable in the presence of termination. We do not know whether Corollary 12.3 can be strengthened to (linear) terminating TRSs.

In our companion paper [4] we present relative undecidability results for properties related to termination. Most of these results are obtained for TRSs consisting of a single rewrite rule. For the confluence hierarchy we do not have results for single rewrite rules.

Another problem is whether the results obtained in this paper can be strengthened to string rewriting systems. The papers [13, 16, 19] contain some kind of relative undecidability results for string rewriting. In [16] it is proved that a number of properties are undecidable for string rewriting systems for which the word problem is decidable. One of these properties is the negation of CON, meaning that all strings are convertible. In [13] it is proved that the property of having finite derivation type is undecidable for string rewriting systems for which the word problem is polynomially decidable. In [19] it is shown that confluence is undecidable for string rewriting systems which have a decidable word problem.

Besides ground confluence, one can extend the confluence hierarchy by introducing a property ground-X for every other X, except CON and CON$^{\rightarrow}$. Ground-X is obtained from X by replacing the quantification over terms by quantification over ground terms. This gives rise to many new relative undecidability questions.

Acknowledgements

We are grateful to an anonymous referee for the suggestion to expand our results by investigating semi-decidability. Aart Middeldorp is partially supported by the Grant-in-Aid for Scientific Research C(2) 11680338 of the Ministry of Education, Science, Sports and Culture of Japan.

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