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Completeness of Combinations of Conditional Constructor Systems[1,2]

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In this paper we extend the divide and conquer technique of Middeldorp and Toyama for establishing (semi-)completeness of constructor systems to conditional constructor systems. We show that both completeness (i.e. the combination of confluence and strong normalisation) and semi-completeness (confluence plus weak normalisation) are decomposable properties of conditional constructor systems without extra variables in the conditions of the rewrite rules.

1. Introduction

A property of term rewriting systems is *modular* if it is preserved under disjoint union. In the past few years the modularity of properties of term rewriting systems has been extensively studied. The first results in this direction were obtained by Toyama. In (Toyama, 1987*a*) he showed that confluence is a modular property (see Klop *et al.* (1994) for a simplified proof) and in (Toyama, 1987*b*) he refuted the modularity of strong normalisation. His counterexample inspired many researchers to look for conditions which are sufficient to recover the modularity of strong normalisation (e.g. Rusinowitch (1987), Middeldorp (1989), Kurihara and Ohuchi (1990), and Toyama *et al.* (1989).) Recently Gramlich (1992*a*) proved an interesting theorem which generalises the results of Rusinowitch (1987), Middeldorp (1989), and Kurihara and Ohuchi (1990), in case of finitely branching term rewriting systems. Ohlebusch (1993*b*) extended Gramlich's result to arbitrary term rewriting systems. Another recent contribution to the topic of modularity is the work of Caron (1992) who investigates the decidability problem of reachability from a modularity perspective.

The disjointness requirement limits the practical applicability of the results mentioned above. The results of Rusinowitch (1987), Middeldorp (1989), and Kurihara and Ohuchi (1990) were generalised to combinations of term rewriting systems that possibly share constructors—function symbols which do not occur at the leftmost position in left-hand

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sides of rewrite rules—by Kurihara and Ohuchi (1992), Gramlich (1992*a*), and Ohlebusch (1993*b*). Middeldorp and Toyama (1993) obtained a useful divide and conquer technique for establishing (semi-)completeness of term rewriting systems that adhere to the so-called constructor discipline. In such *constructor systems* all function symbols occurring at non-leftmost positions in left-hand sides of rewrite rules are constructors. Very recently Krishna Rao (1993) generalised the completeness part of Middeldorp and Toyama (1993) to so-called *hierarchical combinations* of constructor systems. We refer to the final section of this paper for details.

Several modularity results have been extended to conditional term rewriting systems by Middeldorp (1990, 1993), Ohlebusch (1993*a*) and Gramlich (1993). In the present paper we extend the results of Middeldorp and Toyama (1993) to conditional constructor systems. The paper is organised as follows. In the next section we briefly recapitulate the basic notions of (conditional) term rewriting. Section 3 presents the results that are proved in detail in the subsequent two sections. We conclude in Section 6 with suggestions for further research.

2. Preliminaries

We start this section with a concise introduction to term rewriting. Extensive surveys can be found in Dershowitz and Jouannaud (1990) and Klop (1992).

A *signature* is a set $\mathcal{F}$ of *function symbols*. Associated with every $f \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F},\mathcal{V})$ of *terms* built from a signature $\mathcal{F}$ and a countably infinite set of *variables* $\mathcal{V}$ with $\mathcal{F} \cap \mathcal{V} = \varnothing$ is the smallest set such that $\mathcal{V} \subset \mathcal{T}(\mathcal{F},\mathcal{V})$ and if $f \in \mathcal{F}$ has arity n and $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$ then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})$. The *root symbol* of a term t is defined as follows: $root(t) = f$ if $t = f(t_1,\ldots,t_n)$ and $root(t) = t$ if $t \in \mathcal{V}$.

Let $\Box$ be a fresh constant, named *hole*. A *context* C is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\},\mathcal{V})$. The designation *term* is restricted to members of $\mathcal{T}(\mathcal{F},\mathcal{V})$. A context may contain zero, one or more holes. If C is a context with n holes and $t_1,\ldots,t_n$ are terms then $C[t_1,\ldots,t_n]$ denotes the result of replacing from left to right the holes in C by $t_1,\ldots,t_n$. A term s is a *subterm* of a term t, denoted by $s \subseteq t$, if there exists a context C such that $t = C[s]$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that its *domain* $\mathcal{D}(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. If σ is a substitution and t a term then $t\sigma$ denotes the result of applying σ to t, inductively defined as follows: $t\sigma = \sigma(t)$ if t is a variable and $t\sigma = f(t_1\sigma,\ldots,t_n\sigma)$ if $t = f(t_1,\ldots,t_n)$. We call $t\sigma$ an *instance* of t. Let $\mathcal{T}' \subseteq \mathcal{T}(\mathcal{F},\mathcal{V})$. The set of all substitutions σ with $\sigma(x) \in \mathcal{T}'$ for every $x \in \mathcal{D}(\sigma)$ is denoted by $\Sigma(\mathcal{T}')$.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F},\mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of pairs (l,r) with $l,r \in \mathcal{T}(\mathcal{F},\mathcal{V})$ such that:

(1) the left-hand side l is not a variable, and
(2) variables which occur in the right-hand side r also occur in l.

Pairs (l,r) are called *rewrite rules* and will henceforth be written as $l \to r$. A rewrite rule $l \to r$ is *left-linear* if l does not contain multiple occurrences of the same variable. A *left-linear* TRS contains only left-linear rewrite rules.

The *rewrite relation* $\to_\mathcal{R}$ is defined as follows: $s \to_\mathcal{R} t$ if there exists a rewrite rule $l \to r \in \mathcal{R}$, a context C, and a substitution σ such that $s = C[l\sigma]$ and $t = C[r\sigma]$. The subterm $l\sigma$ of s is called a *redex* and we say that s rewrites to t by *contracting* redex $l\sigma$. We call $s \to_\mathcal{R} t$ a *rewrite step*. The transitive-reflexive closure of $\to_\mathcal{R}$ is denoted by

$\to^*_{\mathcal{R}}$. If $s \to^*_{\mathcal{R}} t$ we say that s *reduces* to t. The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to^+_{\mathcal{R}}$. We write $s \leftarrow_{\mathcal{R}} t$ if $t \to_{\mathcal{R}} s$; likewise for $s \leftarrow^*_{\mathcal{R}} t$. The transitive-reflexive-symmetric closure of $\to_{\mathcal{R}}$ is called *conversion* and denoted by $\leftrightarrow^*_{\mathcal{R}}$. If $s \leftrightarrow^*_{\mathcal{R}} t$ then s and t are *convertible*. Two terms t_1, t_2 are *joinable*, denoted by $t_1 \downarrow_{\mathcal{R}} t_2$, if there exists a term t_3 such that $t_1 \to^*_{\mathcal{R}} t_3 \leftarrow^*_{\mathcal{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. When no confusion can arise, we omit the subscript $\mathcal{R}$.

A term s is a *normal form* if there is no term t with $s \to t$. A TRS is *weakly normalising* if every term reduces to a normal form. We write $s \to^! t$ if $s \to^* t$ and t is a normal form. A TRS is *strongly normalising* if there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \cdots$. A TRS is *locally confluent* if for all terms s, t_1, t_2 with $t_1 \leftarrow s \to t_2$ we have $t_1 \downarrow t_2$. A TRS is *confluent* or has the *Church-Rosser* property if for all terms s, t_1, t_2 with $t_1 \leftarrow^* s \to^* t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 \leftrightarrow^* t_2 \Rightarrow t_1 \downarrow t_2$). The renowned Newman's Lemma states that every locally confluent and strongly normalising TRS is confluent (Newman, 1942). A *complete* TRS is confluent and strongly normalising. A *semi-complete* TRS is confluent and weakly normalising. Each term in a (semi-)complete TRS has a unique normal form. The above properties of TRSs specialise to terms in the obvious way.

Let $(\mathcal{F}, \mathcal{R})$ be an arbitrary TRS. A function symbol $f \in \mathcal{F}$ is called a *defined symbol* if there exists a rewrite rule $l \to r \in \mathcal{R}$ such that $f = root(l)$. Function symbols in $\mathcal{F}$ that are not defined symbols are called *constructors*. The subset of $\mathcal{F}$ consisting of all defined symbols is denoted by $\mathcal{D}$ and the set of all constructors by $\mathcal{C}$ ($= \mathcal{F} - \mathcal{D}$). A *constructor system* (CS for short) is a TRS $(\mathcal{F}, \mathcal{R})$ with the property that every left-hand side $f(t_1, \ldots, t_n)$ of a rewrite rule of $\mathcal{R}$ satisfies $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$. To emphasise the partition of $\mathcal{F}$ into $\mathcal{D}$ and $\mathcal{C}$ we write $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ instead of $(\mathcal{F}, \mathcal{R})$ and $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is denoted by $\mathcal{T}(\mathcal{D}, \mathcal{C}, \mathcal{V})$.

We now turn our attention to conditional rewriting. The rules of a *conditional term rewriting system* (CTRS for short) have the form $l \to r \Leftarrow c$. Here the conditional part c is a (possibly empty) sequence $s_1 = t_1, \ldots, s_n = t_n$ of equations. We assume that l is not a variable and that variables occurring in r and c also occur in l. In other words, we do not allow *extra* variables in the conditions. The reasons for excluding conditional rewrite rules with extra variables will be explained later. A rewrite rule without conditions will be written as $l \to r$. The rewrite relation associated with a CTRS $\mathcal{R}$ is obtained by interpreting the equality signs in the conditional part of a rewrite rule as joinability. Formally, $\to_{\mathcal{R}}$ is the smallest (w.r.t. inclusion) relation $\to$ with the property that $C[l\sigma] \to C[r\sigma]$ whenever there exist a rewrite rule $l \to r \Leftarrow c$ in $\mathcal{R}$, a context C, and a substitution σ such that $s = C[l\sigma]$, $t = C[r\sigma]$, and $c_1\sigma \downarrow c_2\sigma$ for every equation $c_1 = c_2$ in c. The existence of $\to_{\mathcal{R}}$ is easily proved (see e.g. Kaplan (1984) or Giovannetti and Moiso (1986)). For every CTRS $\mathcal{R}$ we inductively define TRSs $\mathcal{R}_n$ ($n \geqslant 0$) as follows:

$$\begin{aligned} \mathcal{R}_0 &= \varnothing, \\ \mathcal{R}_{n+1} &= \{\, l\sigma \to r\sigma \mid l \to r \Leftarrow c \in \mathcal{R} \text{ and } c_1\sigma \downarrow_{\mathcal{R}_n} c_2\sigma \text{ for all } c_1 = c_2 \text{ in } c \,\}. \end{aligned}$$

Observe that $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ for all $n \geqslant 0$. We have $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}_n} t$ for some $n \geqslant 0$. The minimum such n is called the *depth* of $s \to t$. The depth of a reduction $s \to^*_{\mathcal{R}} t$ is the minimum n such that $s \to^*_{\mathcal{R}_n} t$. The depth of a 'valley' $s \downarrow_{\mathcal{R}} t$ is similarly defined. All notions previously defined for TRSs extend to CTRSs.

A CTRS $(\mathcal{F}, \mathcal{R})$ is a *conditional constructor system* (CCS for short) if the TRS obtained from $\mathcal{R}$ by omitting all conditions is a CS. Observe that we put no (further) limitations on the conditions of the rules in a CCS.

EXAMPLE 2.1. Consider the CCS $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ with $\mathcal{D} = \{\mathit{even}, \mathit{odd}\}$, $\mathcal{C} = \{0, S, \mathit{true}, \mathit{false}\}$ and

$$\mathcal{R} = \begin{cases} \mathit{even}(0) & \to & \mathit{true} \\ \mathit{even}(S(x)) & \to & \mathit{odd}(x) \\ \mathit{odd}(x) & \to & \mathit{true} & \Leftarrow & \mathit{even}(x) = \mathit{false} \\ \mathit{odd}(x) & \to & \mathit{false} & \Leftarrow & \mathit{even}(x) = \mathit{true}. \end{cases}$$

The first step in the reduction sequence $\mathit{even}(S(0)) \to_{\mathcal{R}} \mathit{odd}(0) \to_{\mathcal{R}} \mathit{false}$ has depth 1 and the second step has depth 2.

3. Decomposability

The following definition originates from Middeldorp and Toyama (1993). It expresses a natural way to divide a large (conditional) constructor system into smaller, not necessarily disjoint, parts.

DEFINITION 3.1.

(1) Let $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ be a CCS and $\mathcal{D}'$ a set of function symbols. The subset of $\mathcal{R}$ consisting of all rewrite rules $l \to r \Leftarrow c$ that satisfy $\mathit{root}(l) \in \mathcal{D}'$ is denoted by $\mathcal{R} \mid \mathcal{D}'$.
(2) Two CCSs $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$ are *composable* if $\mathcal{D}_1 \cap \mathcal{C}_2 = \mathcal{D}_2 \cap \mathcal{C}_1 = \varnothing$ and $\mathcal{R}_1 \mid \mathcal{D}_2 = \mathcal{R}_2 \mid \mathcal{D}_1$. The second requirement states that both CCSs should contain all rewrite rules which 'define' a defined symbol whenever that symbol is shared. The union of pairwise composable CCSs $\mathcal{CCS}_1, \ldots, \mathcal{CCS}_n$ is denoted by $\mathcal{CCS}_1 + \cdots + \mathcal{CCS}_n$ and we say that $\mathcal{CCS}_1, \ldots, \mathcal{CCS}_n$ is a *decomposition* of $\mathcal{CCS}_1 + \cdots + \mathcal{CCS}_n$.
(3) A property $\mathcal{P}$ of CCSs is said to be *decomposable* if for all pairwise composable CCSs $\mathcal{CCS}_1, \ldots, \mathcal{CCS}_n$ with the property $\mathcal{P}$ we have that $\mathcal{CCS}_1 + \cdots + \mathcal{CCS}_n$ has the property $\mathcal{P}$.

Note that CCSs without common function symbols are trivially composable. Thus every decomposable property of CCSs is also a modular property of CCSs.

PROPOSITION 3.2. *Let $\mathcal{P}$ be a property of CCSs. The following statements are equivalent:*

(1) *$\mathcal{P}$ is decomposable;*
(2) *for all composable CCSs $\mathcal{CCS}_1$ and $\mathcal{CCS}_2$ with the property $\mathcal{P}$ we have that $\mathcal{CCS}_1 + \mathcal{CCS}_2$ has the property $\mathcal{P}$.*

PROOF. Straightforward. □

In this paper we extend the main results of Middeldorp and Toyama (1993) from CSs to CCSs. That is, we will show that both completeness and semi-completeness are decomposable properties of CCSs. It should be noted that neither strong normalisation nor confluence are decomposable properties of CCSs. This is already the case for CSs (see Middeldorp and Toyama (1993) for the easy to construct counterexamples).

Example 3.3. Consider the CCS $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ with defined symbols $\mathcal{D} = \{+, \times, \mathit{even}, \mathit{odd}\}$, constructors $\mathcal{C} = \{0, S, \mathit{true}, \mathit{false}\}$ and rewrite rules

$$\mathcal{R} = \left\{ \begin{array}{lclcl} 0 + x & \to & x & & \\ S(x) + y & \to & S(x + y) & & \\ 0 \times x & \to & 0 & & \\ S(x) \times y & \to & x \times y + y & & \\ \mathit{even}(0) & \to & \mathit{true} & & \\ \mathit{even}(S(x)) & \to & \mathit{odd}(x) & & \\ \mathit{odd}(x) & \to & \mathit{true} & \Leftarrow & \mathit{even}(x) = \mathit{false} \\ \mathit{odd}(x) & \to & \mathit{false} & \Leftarrow & \mathit{even}(x) = \mathit{true} \end{array} \right\}.$$

We can decompose $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ into the CCSs

$$\mathcal{R}_1 = \left\{ \begin{array}{lcl} 0 + x & \to & x \\ S(x) + y & \to & S(x + y) \\ 0 \times x & \to & 0 \\ S(x) \times y & \to & x \times y + y \end{array} \right\}$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{lclcl} \mathit{even}(0) & \to & \mathit{true} & & \\ \mathit{even}(S(x)) & \to & \mathit{odd}(x) & & \\ \mathit{odd}(x) & \to & \mathit{true} & \Leftarrow & \mathit{even}(x) = \mathit{false} \\ \mathit{odd}(x) & \to & \mathit{false} & \Leftarrow & \mathit{even}(x) = \mathit{true} \end{array} \right\}$$

with $\mathcal{D}_1 = \{+, \times\}$, $\mathcal{D}_2 = \{\mathit{even}, \mathit{odd}\}$, $\mathcal{C}_1 = \{0, S\}$ and $\mathcal{C}_2 = \{0, S, \mathit{true}, \mathit{false}\}$. Standard rewriting techniques show the completeness of $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ and the strong normalisation of $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$. For (local) confluence of $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$ we have to show that there exists no term t that satisfies both $\mathit{even}(t) \downarrow_{\mathcal{R}_2} \mathit{false}$ and $\mathit{even}(t) \downarrow_{\mathcal{R}_2} \mathit{true}$, which follows by an easy induction argument. Our main result now yields the completeness of $(\mathcal{D}, \mathcal{C}, \mathcal{R})$. We would like to stress that the problem of showing the unsatisfiability of $\mathit{even}(x) \downarrow_{\mathcal{R}} \mathit{false} \wedge \mathit{even}(x) \downarrow_{\mathcal{R}} \mathit{true}$ is more complicated than showing the unsatisfiability of $\mathit{even}(x) \downarrow_{\mathcal{R}_2} \mathit{false} \wedge \mathit{even}(x) \downarrow_{\mathcal{R}_2} \mathit{true}$.

Unlike the extension of modularity results from TRSs to CTRSs in (Middeldorp, 1990, 1993), we do not make use of the decomposability of (semi-)completeness for CSs in our proofs. Rather, we extend the proof ideas in (Middeldorp and Toyama, 1993) to conditional systems.[3] This is not entirely a routine matter. For instance, the proof of the decomposability of completeness for CSs in (Middeldorp and Toyama, 1993) employs the decomposability of local confluence. That result, however, does not hold for CCSs as the following example shows. (This example is an easy adjustment of the counterexample in (Middeldorp, 1990, 1993) against the modularity of local confluence for CTRSs.)

Example 3.4. Consider the disjoint and hence composable CCSs

$$\mathcal{R}_1 = \left\{ \; a \longleftarrow b \rightleftarrows c \longrightarrow d \; \right\}$$

[3] The reader familiar with (Middeldorp and Toyama, 1993) will observe that the specialisation of the proofs in this paper to CSs yields slightly simpler proofs of the main results of (Middeldorp and Toyama, 1993). (Gramlich, 1992*b*) contains a simpler proof of the decomposability of completeness for CSs. It remains to be seen whether his approach can be extended to CCSs.

and

$$\mathcal{R}_2 = \left\{ \begin{array}{lclcl} f(x,y,z) & \to & x & \Leftarrow & x = y,\ y = z \\ f(x,y,z) & \to & z & \Leftarrow & x = y,\ y = z. \end{array} \right\}.$$

The CS $\mathcal{R}_1$ is clearly locally confluent. Let $\mathcal{R}$ be the CS consisting of the single rewrite rule $f(x,x,x) \to x$. Clearly $s \to_{\mathcal{R}} t$ implies $s \to_{\mathcal{R}_2} t$. Conversely, if $s \to_{\mathcal{R}_2} t$ then we obtain $s \leftrightarrow^*_{\mathcal{R}} t$ by a straightforward induction on the depth of $s \to_{\mathcal{R}_2} t$. Because $\mathcal{R}$ is confluent, a routine argument now shows that $\mathcal{R}_2$ is confluent and hence locally confluent. However, the union of $\mathcal{R}_1$ and $\mathcal{R}_2$ is not locally confluent: we have $a \leftarrow f(a,b,d) \to d$ since both $a \downarrow b$ and $b \downarrow d$, but a and d have no common reduct.

4. Marked Reduction

The proofs in (Middeldorp and Toyama, 1993) heavily depend on the notion of *marked reduction*. In this section we extend this notion to conditional systems and we show that the key properties of marked reduction as established in (Middeldorp and Toyama, 1993) are still valid in the conditional case. Throughout this section we will be dealing with the union $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ of two composable CCSs $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$.

DEFINITION 4.1.

(1) The set $\mathcal{D}^* = \{f^* \mid f \in \mathcal{D}_1\}$ consists of *marked* defined symbols. Terms in $\mathcal{T}(\mathcal{D}^* \cup \mathcal{D}, \mathcal{C}, \mathcal{V})$ are called *marked terms*. An *unmarked term* belongs to $\mathcal{T}(\mathcal{D}, \mathcal{C}, \mathcal{V})$. Observe that we do not mark symbols in $\mathcal{D}_2 - \mathcal{D}_1$. In the following we abbreviate $\mathcal{T}(\mathcal{D}, \mathcal{C}, \mathcal{V})$ to $\mathcal{T}$.

(2) If t is a marked term then $e(t) \in \mathcal{T}$ denotes the term obtained from t by erasing all marks and t^* denotes the term obtained from t by marking every unmarked defined symbol in t that belongs to $\mathcal{D}_1$. These notions are extended to other syntactic objects like substitutions and sequences of equations in the obvious way. The set $\{l^* \to r^* \Leftarrow c^* \mid l \to r \Leftarrow c \in \mathcal{R}_1\}$ of *marked rewrite rules* is denoted by $\mathcal{M}$.

(3) Two marked terms s and t are *similar*, denoted by $s \approx t$, if $e(s) = e(t)$. If s and t are similar then their *intersection* is the unique term $s \wedge t$ such that $s \wedge t \approx s \approx t$ and a defined symbol occurrence in $s \wedge t$ is marked if and only if the corresponding occurrences in s and t are marked. Since $\wedge$ is easily shown to be associative and commutative, we can extend it to sets of pairwise similar terms in the obvious way, i.e. if $S = \{s_1, \dots, s_n\}$ is a set of pairwise similar terms, then $\wedge S$ denotes $s_1 \wedge \dots \wedge s_n$.

In (Middeldorp and Toyama, 1993) symbols in $\mathcal{D}_2 - \mathcal{D}_1$ are also marked. Since our main results are symmetrical in $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$, it suffices to mark only defined symbols belonging to a single constructor system. This results in a few simplifications compared to (Middeldorp and Toyama, 1993).

DEFINITION 4.2. If $t = C[t_1, \dots, t_n]$ such that all defined symbols in C are marked[4] and every t_i $(i = 1, \dots, n)$ is unmarked then we call t a *capped* term. Furthermore, if $root(t_i) \in \mathcal{D}$ for $i = 1, \dots, n$ then we write $t = C[\![t_1, \dots, t_n]\!]$. Note that every capped term can be uniquely written in this way. (If $n = 0$, i.e. $t \in \mathcal{T}(\mathcal{D}^*, \mathcal{C}, \mathcal{V})$, then $t[\![\,]\!]$ is the unique representation of t as a capped term. If $t \in \mathcal{T}$ and $root(t) \in \mathcal{D}$ then $\Box[\![t]\!]$ is the

[4] This implies that C doesn't contain symbols from $\mathcal{D}_2 - \mathcal{D}_1$.

unique representation of t as a capped term.) Adopting the terminology from Kurihara and Ohuchi (1990, 1992), the subterms $t_1, \ldots, t_n$ of $t = C[\![t_1, \ldots, t_n]\!]$ will be called *aliens* of t. The set of all capped terms is denoted by $\hat{\mathcal{T}}$ and the set of all aliens in a capped term t is denoted by $\mathit{aliens}(t)$.

Definition 4.3. A set $T' \subseteq \hat{\mathcal{T}}$ is said to be *alien defined* by a property $\mathcal{P}$ if a term t belongs to T' if and only if every alien of t satisfies $\mathcal{P}$.

The set $\hat{\mathcal{T}}$ is alien defined by the property $t \in \mathcal{T}$. Observe that every alien defined subset T' of $\hat{\mathcal{T}}$ contains $\mathcal{T}(\mathcal{D}^*, \mathcal{C}, \mathcal{V})$ since terms in $\mathcal{T}(\mathcal{D}^*, \mathcal{C}, \mathcal{V})$ do not have aliens. It is also not difficult to see that every alien defined set T' satisfies the following closure properties:

(1) if $s, t \in T'$ are similar then $s \wedge t \in T'$,
(2) if $t \in T'$ and $s \in \mathit{aliens}(t)$ then $s \in T'$,
(3) if $t \in T'$, $s \subseteq t$, and $s \notin T$ then $s \in T'$,
(4) if $t \in \mathcal{T}(\mathcal{D}^*, \mathcal{C}, \mathcal{V})$ and $\sigma \in \Sigma(T')$ then $t\sigma \in T'$.

Definition 4.4. Let $s \in \hat{\mathcal{T}}$.

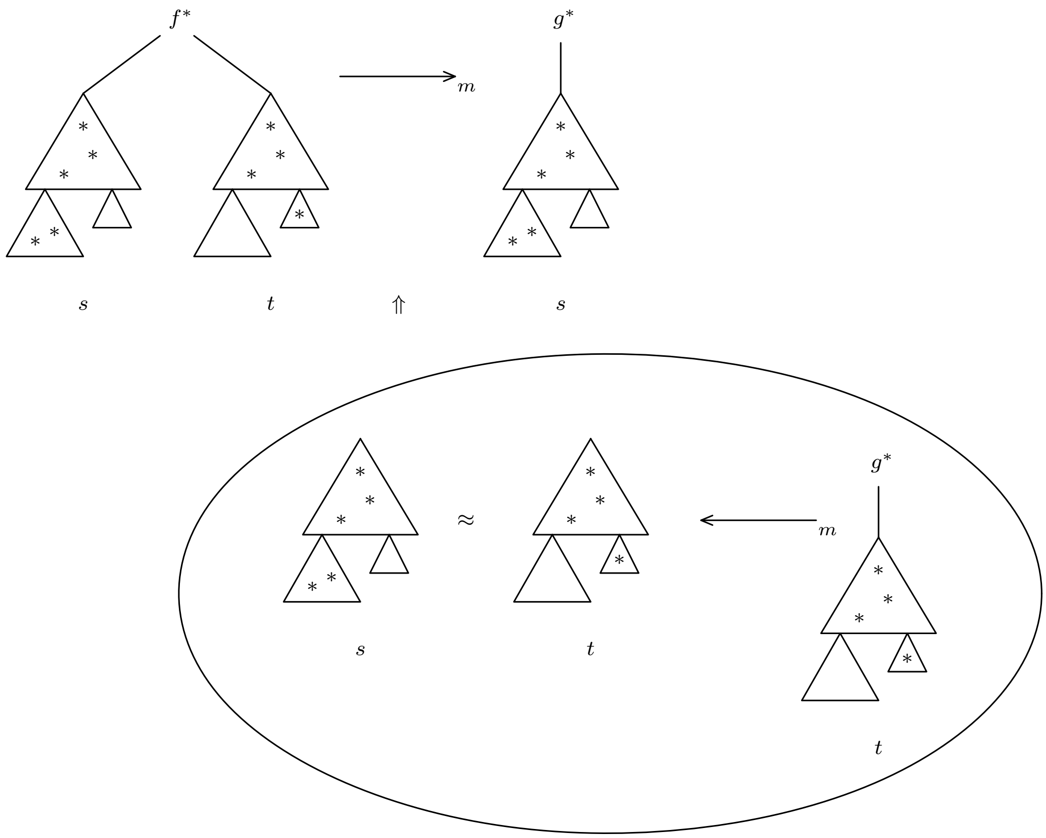
(1) We write $s \to_m t$ if there exists a context C, a rewrite rule $C_1[x_1, \ldots, x_n] \to r \Leftarrow c$ in $\mathcal{M}$ (with all variables occurring in its left-hand side displayed), and terms $s_1, \ldots, s_n$ such that the following four conditions are satisfied:
   (a) $s = C[C_1[s_1, \ldots, s_n]]$,
   (b) $s_i \approx s_j$ whenever $x_i = x_j$ for $1 \leqslant i < j \leqslant n$,
   (c) $t = C[r\sigma]$,
   (d) $c_1\sigma \downarrow_m^{\approx} c_2\sigma$ for every equation $c_1 = c_2$ in c.
   Here σ is the substitution induced by l and s, i.e.
$$\sigma(x) = \begin{cases} \wedge\{s_i \mid x_i = x\} & \text{if } x \in \{x_1, \ldots, x_n\}, \\ x & \text{otherwise,} \end{cases}$$
   and $\downarrow_m^{\approx}$ denotes joinability with respect to $\to_m$ modulo $\approx$, i.e. the relation $\to_m^* \cdot \approx \cdot \leftarrow_m^*$. The relation $\to_m$ is called *marked reduction*.
(2) We write $s \to_u t$ if $s \to_{\mathcal{R}} t$. The relation $\to_u$ is called *unmarked reduction*. Clearly $s \to_u t$ if and only if one of the aliens in s is rewritten. In the following we restrict the use of $\to_{\mathcal{R}}$ to unmarked terms.

The notion of marked reduction is illustrated in Figure 1. Specialising the above definition to CSs yields the relations $\to_m^o$ (for $\to_m$) and $\to_m^i$ (for $\to_u$) of (Middeldorp and Toyama, 1993). The well-definedness of $\to_m$ and the closure of $\hat{\mathcal{T}}$ under $\to_m$ easily follow from the alien definedness of $\hat{\mathcal{T}}$. In particular, we have the following fact.

Proposition 4.5. *Let $s \in \hat{\mathcal{T}}$. If $s \to_m t$ then $\mathit{aliens}(t) \subseteq \mathit{aliens}(s)$.*

Proof. Routine. □

As a matter of fact, the above proposition shows that any alien defined subset of $\hat{\mathcal{T}}$ is closed under marked reduction. Closure of $\hat{\mathcal{T}}$ under $\to_u$ is obvious. The crucial point in the definition of marked reduction is that we allow terms in an instantiated condition



$$\left\{ \begin{array}{llllll} f(x,y) & \to & g(x) & \Leftarrow & x = g(y) \\ g(x) & \to & x & & \end{array} \right\} \subseteq \mathcal{R}_1$$

Figure 1.

to be joinable with respect to marked reduction *up to similarity*. In (Middeldorp and Toyama, 1993) the concept of similarity was introduced to cope with non-left-linear rewrite rules. However, in a conditional system non-left-linearity can be hidden in the conditions. A conditional rewrite rule $f(x, y) \to c \Leftarrow x = y$ resembles in many respects the unconditional but non-left-linear rule $f(x, x) \to c$.

PROPOSITION 4.6. *Let* $s \in \hat{\mathcal{T}}$.

(1) *If* $s \to_m t$ *then* $e(s) \to_{\mathcal{R}_1} e(t)$.
(2) *If* $s \to_u t$ *then* $e(s) \to_{\mathcal{R}} e(t)$.

PROOF. The first part is obtained by a straightforward induction on the depth of $s \to_m t$. The second part is trivial. □

Consider a capped term s and a reduction step $e(s) \to_{\mathcal{R}} t$. In the unconditional case (i.e. if $\mathcal{R}$ is a CS) it is easy to see that we can lift $e(s) \to_{\mathcal{R}} t$ to $s \to_m t'$ or $s \to_u t'$ for some term $t' \in \hat{\mathcal{T}}$ with $e(t') = t$ $(*)$. Together with Proposition 4.6 this constitutes a very clear relationship between $\to_{\mathcal{R}}$-reduction on $\mathcal{T}$ and $\to_m$ and $\to_u$-reduction on $\hat{\mathcal{T}}$. In the conditional case this nice correspondence is lost, as shown in the next example.

Example 4.7. Consider the CCS

$$\mathcal{R}_1 = \begin{cases} f(x) & \to & x & \Leftarrow & x = b \\ a & \to & b \end{cases}$$

with $\mathcal{D}_1 = \{f, a, b\}$ and $\mathcal{C}_1 = \varnothing$. We have $f(a) \to_{\mathcal{R}_1} a$, but $f^*(a)$ is a normal form with respect to marked reduction since $a \downarrow_m^{\approx} b^*$ does not hold.

Fortunately, we do not really need this one-to-one correspondence between ordinary reduction and reduction ($\to_m$ and $\to_u$) on capped terms. The relationship expressed in Lemma 4.18 below is sufficient for our purposes. The next few propositions pave the way for Lemma 4.18. First we show that $(*)$ does hold for "inside normalised" capped terms.

Definition 4.8. A term in $\hat{\mathcal{T}}$ is called *inside normalised* if it is a normal form with respect to $\to_u$. In other words, a term in $\hat{\mathcal{T}}$ is inside normalised if and only if its aliens are normal forms. The subset of $\hat{\mathcal{T}}$ consisting of all inside normalised terms is denoted by $\hat{\mathcal{T}}_{in}$.

Observe that $\hat{\mathcal{T}}_{in}$ is alien defined by the property of being in normal form. Hence $\hat{\mathcal{T}}_{in}$ is closed under marked reduction.

Example 4.9. Consider the CCS of Example 4.7. We have $f(b) \to_{\mathcal{R}_1} b$. The term $f^*(b)$ is inside normalised, and since $b \downarrow_m^{\approx} b^*$ we can indeed lift the step $f(b) \to_{\mathcal{R}_1} b$ to $f^*(b) \to_m b$.

Proposition 4.10. *Let $s \in \hat{\mathcal{T}}_{in}$. If $e(s) \to_{\mathcal{R}} t$ then there exists a term $t' \in \hat{\mathcal{T}}_{in}$ such that $s \to_m t'$ and $e(t') = t$; see Figure 2.*

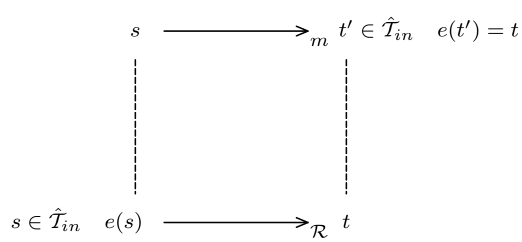


Figure 2.

Proof. We use induction on the depth of $e(s) \to_{\mathcal{R}} t$. In case of zero depth there is nothing to prove. Suppose the depth of $e(s) \to_{\mathcal{R}} t$ equals $n+1$ ($n \geqslant 0$). By definition there exists a context C, a rewrite rule $l \to r \Leftarrow c \in \mathcal{R}$, and a substitution σ such that $e(s) = C[l\sigma]$, $t = C[r\sigma]$, and, for every equation $c_1 = c_2$ in c, $c_1\sigma \downarrow_{\mathcal{R}} c_2\sigma$ with depth at most n. Write $l = C_1[x_1, \ldots, x_n]$ such that all variables in l are displayed. Since there are no extra variables in the rule $l \to r \Leftarrow c$, we may assume that $\mathcal{D}(\sigma) \subseteq \{x_1, \ldots, x_n\}$. We have $s = C'[C_1^*[s_1, \ldots, s_n]]$ for some context C' and terms $s_1, \ldots, s_n \in \hat{\mathcal{T}}_{in}$ such that $e(C') = C$ and $e(s_i) = x_i\sigma$ for $i = 1, \ldots, n$. Let τ be the substitution induced by l and s. Note that $\tau \in \Sigma(\hat{\mathcal{T}}_{in})$ and $e(\tau) = \sigma$. Let $c_1 = c_2$ be an equation in c. We will show that $c_1^*\tau \downarrow_m^{\approx} c_2^*\tau$. We know that there exists a valley $c_1\sigma \to_{\mathcal{R}}^* u \leftarrow_{\mathcal{R}}^* c_2\sigma$ in which the depth of every step is at most n. Since $e(c_i^*\tau) = c_i\sigma$ and $c_i^*\tau \in \hat{\mathcal{T}}_{in}$, a straightforward induction argument shows the existence of a term u_i such that $c_i^*\tau \to_m^* u_i$ and $e(u_i) = u$, for $i = 1, 2$. Clearly $u_1 \approx u_2$. Hence $c_1^*\tau \downarrow_m^{\approx} c_2^*\tau$. From the inside normalisation of s we infer that $l \to r \Leftarrow c \in \mathcal{R}_1$ and hence $l^* \to r^* \Leftarrow c^* \in \mathcal{M}$. Define $t' = C'[r^*\tau]$. We have $s \to_m t'$. Clearly $e(t') = C[r\sigma] = t$. □

Proposition 4.11. *Let $s \in \hat{\mathcal{T}}_{in}$. If $e(s) \to_{\mathcal{R}}^! t$ then there exists a term $t' \in \hat{\mathcal{T}}_{in}$ such that $s \to_m^! t'$ and $e(t') = t$.*

Proof. Repeated application of Proposition 4.10 yields a term $t' \in \hat{\mathcal{T}}_{in}$ such that $s \to_m^* t'$ and $e(t') = t$. It remains to show that t' is a normal form with respect to $\to_m$. This follows from the assumption that t is a normal form, by means of Proposition 4.6(1). □

In the sequel we are mainly interested in capped terms whose aliens are semi-complete. The set of all such terms is denoted by $\hat{\mathcal{T}}_{sc}$. Clearly $\hat{\mathcal{T}}_{in} \subseteq \hat{\mathcal{T}}_{sc} \subseteq \hat{\mathcal{T}}$. Note that $\hat{\mathcal{T}}_{sc}$ is alien defined by the property of being semi-complete. Hence $\hat{\mathcal{T}}_{sc}$ is closed under marked reduction. Closure under $\to_u$ follows from Propositions 4.16 and 4.17 below.

Proposition 4.12. *Unmarked reduction is semi-complete on $\hat{\mathcal{T}}_{sc}$.*

Proof. Obvious. □

So every term $t \in \hat{\mathcal{T}}_{sc}$ has a unique normal form in $\hat{\mathcal{T}}_{in}$ with respect to $\to_u$, which will be denoted by $\psi(t)$. Consider similar terms $s, t \in \hat{\mathcal{T}}_{sc}$. In general $\psi(s)$ and $\psi(t)$ are not similar. The next proposition states that they can be made similar by applying certain 'balancing' marked reduction steps.

Proposition 4.13. *Let $s, t \in \hat{\mathcal{T}}_{sc}$. If $s \approx t$ then there exist similar terms $s', t' \in \hat{\mathcal{T}}_{sc}$ such that $\psi(s) \to_m^* s'$, $\psi(t) \to_m^* t'$, and $\psi(s \wedge t) = s' \wedge t'$.*

Proof. We may write $s \wedge t = C[\![s_1 \wedge t_1, \ldots, s_n \wedge t_n]\!]$, $s = C[s_1, \ldots, s_n]$, and $t = C[t_1, \ldots, t_n]$. We will define similar terms s_i' and t_i' such that $\psi(s_i) \to_m^* s_i'$, $\psi(t_i) \to_m^* t_i'$, and $\psi(s_i \wedge t_i) = s_i' \wedge t_i'$, for every $i \in \{1, \ldots, n\}$. Fix i. We have $e(s_i) = t_i$ or $s_i = e(t_i)$. Assume without loss of generality the former. By definition $s_i \to_u^! \psi(s_i)$. Proposition 4.6 yields $t_i \to_u^* e(\psi(s_i))$ and since t_i is semi-complete we obtain $e(\psi(s_i)) \to_u^! \psi(t_i)$. According to Proposition 4.11 $\psi(s_i)$ has a normal from s_i' with respect to $\to_m$ such that $e(s_i') = \psi(t_i)$. Define $t_i' = \psi(t_i)$. We clearly have $\psi(s_i \wedge t_i) = \psi(t_i) = s_i' \wedge \psi(t_i) = s_i' \wedge t_i'$. Now that we have defined $s_1', t_1' \ldots, s_n', t_n'$, let $s' = C[s_1', \ldots, s_n']$ and $t' = C[t_1', \ldots, t_n']$. Clearly $s' \approx t'$, $\psi(s) = C[\psi(s_1), \ldots, \psi(s_n)] \to_m^* s'$, $\psi(t) = C[\psi(t_1), \ldots, \psi(t_n)] \to_m^* t'$, and $\psi(s \wedge t) = C[\psi(s_1 \wedge t_1), \ldots, \psi(s_n \wedge t_n)] = C[s_1' \wedge t_1', \ldots, s_n' \wedge t_n'] = s' \wedge t'$. □

The above proposition easily generalises to the case of $n \geqslant 2$ pairwise similar terms.

Proposition 4.14. *Let* $s_1, \ldots, s_n \in \hat{\mathcal{T}}_{sc}$ *be pairwise similar terms. There exist pairwise similar terms* $t_1, \ldots, t_n \in \hat{\mathcal{T}}_{sc}$ *such that* $\psi(s_i) \to_m^* t_i$ *for every* $i \in \{1, \ldots, n\}$ *and* $\psi(\wedge\{s_i \mid 1 \leqslant i \leqslant n\}) = \wedge\{t_i \mid 1 \leqslant i \leqslant n\}$.

Proof. Straightforward induction on n, using Proposition 4.13. □

Definition 4.15. We define a relation $\rhd$ on terms in $\hat{\mathcal{T}}$ as follows: $s \rhd t$ if for every alien $t' \subseteq t$ there exists an alien $s' \subseteq s$ and a context C such that $s' \to_{\mathcal{R}}^* C[t']$ and t' is an alien in $C[t']$. Since $C[t']$ doesn't contain marked symbols, the latter condition simply means that the context C doesn't contain any defined symbols *above* its hole □. The relation $\rhd$ is easily seen to be transitive. In words, $s \rhd t$ if every alien of t can be 'traced back' to some alien in s.

Proposition 4.16. *Let* $s \in \hat{\mathcal{T}}_{sc}$. *If* $s \rhd t$ *then* $t \in \hat{\mathcal{T}}_{sc}$.

Proof. We have to show that every alien t' in t is semi-complete. By definition there exists an alien s' in s and a context C such that $s' \to_{\mathcal{R}}^* C[t']$ and $t' \in \mathit{aliens}(C[t'])$. From the semi-completeness of s' we infer the semi-completeness of $C[t']$. This implies that all aliens in $C[t']$ are semi-complete. □

Proposition 4.17. *Let* $s \in \hat{\mathcal{T}}$. *If* $s \to_m t$ *or* $s \to_u t$ *then* $s \rhd t$.

Proof. If $s \to_m t$ then the result follows from Proposition 4.5. Suppose $s \to_u t$. Let $s = C⁅s_1, \ldots, s_n⁆$. We have $t = C[s_1, \ldots, t_i, \ldots, s_n]$ for some term t_i with $s_i \to_{\mathcal{R}} t_i$. Let a be an alien in t. Either $a = s_j$ for some $j \neq i$ or a is an alien in t_i. In the former case a is an alien in s. In the latter case we have $s_i \to_{\mathcal{R}} t_i = C'[a]$ for some context C'. □

Lemma 4.18. *Let* $s \in \hat{\mathcal{T}}_{sc}$. *If* $e(s) \to_{\mathcal{R}} t$ *then* $s \to_u t'$ *or* $s \to_u^! \cdot \to_m^+ \cdot \leftarrow_u^! t'$ *for some term* $t' \in \hat{\mathcal{T}}_{sc}$ *such that* $e(t') = t$. *Moreover, we may assume that* $s \rhd t'$.

Proof. We use induction on the depth of $e(s) \to_{\mathcal{R}} t$. The case of zero depth is trivial. Suppose the depth of $e(s) \to_{\mathcal{R}} t$ equals $n+1$ ($n \geqslant 0$). By definition there exists a context C, a rewrite rule $l \to r \Leftarrow c \in \mathcal{R}$, and a substitution σ such that $e(s) = C[l\sigma]$, $t = C[r\sigma]$, and, for every equation $c_1 = c_2$ in c, $c_1\sigma \downarrow_{\mathcal{R}} c_2\sigma$ with depth at most n. We distinguish two cases.

(1) If the 'redex occurrence' $l\sigma$ is not marked in s then we may write $s = C'[l\sigma]$ for some context C' with $e(C') = C$. In this case we clearly have $s \to_u C'[r\sigma]$ and $e(C'[r\sigma]) = t$. Moreover, $s \rhd C'[r\sigma]$ by the preceding proposition.
(2) Suppose $l\sigma$ is marked in s. Write $l = C_1[x_1, \ldots, x_n]$ such that C_1 is variable-free. Without loss of generality we assume that $\mathcal{D}(\sigma) \subseteq \{x_1, \ldots, x_n\}$. We have $s = C'[C_1^*[s_1, \ldots, s_n]]$ for some context C' and terms $s_1, \ldots, s_n \in \hat{\mathcal{T}}_{sc}$ such that $e(C') = C$ and $e(s_i) = x_i\sigma$ for $i = 1, \ldots, n$. Let τ be the substitution induced by l^* and s, and define a substitution υ by $\upsilon(x) = \psi(x\tau)$ for every $x \in \mathcal{V}$. Note that υ is well-defined since $\tau \in \Sigma(\hat{\mathcal{T}}_{sc})$. Let $c_1 = c_2$ be an equation in c. We will show that $c_1^*\upsilon \downarrow_m^{\approx} c_2^*\upsilon$; see Figure 3. We know the existence of a valley $c_1\sigma \to_{\mathcal{R}}^* u \leftarrow_{\mathcal{R}}^* c_2\sigma$ in which the depth of every step is at most n. Since $e(c_i^*\tau) = c_i\sigma$ and $c_i^*\tau \in \hat{\mathcal{T}}_{sc}$, a straightforward induction argument shows the existence of a term u_i such that $\psi(c_i^*\tau) \to_m^* \psi(u_i)$

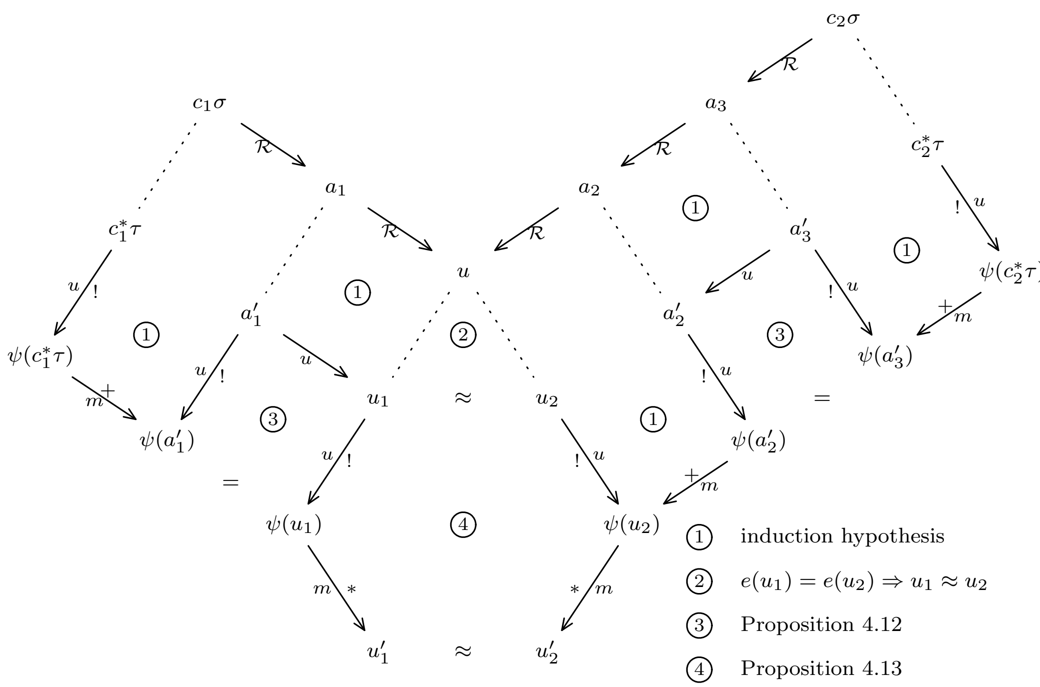


Figure 3.

and $e(u_i) = u$, for $i = 1, 2$. Since $u_1 \approx u_2$, Proposition 4.13 yields $\psi(u_1) \downarrow_m^{\approx} \psi(u_2)$. Hence $c_1^* \upsilon = \psi(c_1^* \tau) \downarrow_m^{\approx} \psi(c_2^* \tau) = c_2^* \upsilon$. We have $\psi(s) = \psi(C')[C_1^*[\psi(s_1), \ldots, \psi(s_n)]]$. With help of Proposition 4.14 we can find terms $t_1, \ldots, t_n$ such that $\psi(s_i) \rightarrow_m^* t_i$ for $i \in \{1, \ldots, n\}$ and $\psi(\wedge\{s_i \mid x_i = x\}) = \wedge\{t_i \mid x_i = x\}$ for every $x \in \{x_1, \ldots, x_n\}$. Clearly $\psi(s) \rightarrow_m^* \psi(C')[C_1^*[t_1, \ldots, t_n]]$. Note that $\wedge\{t_i \mid x_i = x\} = x\upsilon$ for every $x \in \{x_1, \ldots, x_n\}$. Hence $\psi(C')[C_1^*[t_1, \ldots, t_n]] \rightarrow_m \psi(C')[r^* \upsilon]$. Define $t' = C'[r^* \tau]$. Clearly $t \in \hat{\mathcal{T}}_{sc}$ and $\psi(t') = \psi(C')[\psi(r^* \tau)] = \psi(C')[r^* \upsilon]$. Therefore $\psi(s) \rightarrow_m^+ \psi(t')$, i.e., $s \rightarrow_u^! \cdot \rightarrow_m^+ \cdot \leftarrow_u^! t'$. One easily shows that $e(t') = t$ and $s \rhd t'$.

□

Observe that the first alternative in Lemma 4.18 implies $\psi(s) = \psi(t')$ and that the second alternative is equivalent to $\psi(s) \rightarrow_m^+ \psi(t')$.

Corollary 4.19. *Let $s \in \hat{\mathcal{T}}_{sc}$. If $e(s) \rightarrow_{\mathcal{R}}^* t$ then there exists a term $t' \in \hat{\mathcal{T}}_{sc}$ such that $e(t') = t$ and $\psi(s) \rightarrow_m^* \psi(t')$.*

In the proof of our main results we transform reduction sequences in the union $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ of two composable CCSs $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$ having a certain property $\mathcal{P}$ by means of the previous results into $\mathcal{R}_1$-reduction sequences. In order to employ the fact that $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ satisfies $\mathcal{P}$, we have to get rid of subterms that do not belong to $\mathcal{T}(\mathcal{D}_1, \mathcal{C}_1, \mathcal{V})$. Actually, constructors in $\mathcal{C}_2 - \mathcal{C}_1$ do not bother us: if $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ satisfies $\mathcal{P}$ then $(\mathcal{D}_1, \mathcal{C}, \mathcal{R}_1)$ also satisfies $\mathcal{P}$, for all interesting properties $\mathcal{P}$. So we only have to get rid of maximal subterms with root symbol belonging to $\mathcal{D}_2 - \mathcal{D}_1$.

DEFINITION 4.20.

(1) A set of pairs $\phi = \{\langle s_1, x_1\rangle, \dots, \langle s_n, x_n\rangle\}$ is a *garbage collector* if $x_1, \dots, x_n$ are mutually distinct variables and $s_1, \dots, s_n$ are mutually distinct unmarked terms without occurrences of $x_1, \dots, x_n$ such that $root(s_i) \in \mathcal{D}_2 - \mathcal{D}_1$ for $i = 1, \dots, n$.[5] Let $t = C[t_1, \dots, t_m]$ such that all maximal occurrences of unmarked subterms in t with root symbol in $\mathcal{D}_2 - \mathcal{D}_1$ are displayed. We say that ϕ is *applicable* to t, denoted by $\phi \ltimes t$, if $x_1, \dots, x_n$ do not occur in t and $\{t_1, \dots, t_m\} \subseteq \{s_1, \dots, s_n\}$. In this case we may write $t = C[s_{i_1}, \dots, s_{i_m}]$ with all occurrences of $s_1, \dots, s_n$ in t displayed and we define $\phi(t) = C[x_{i_1}, \dots, x_{i_m}]$.
(2) Let $\phi = \{\langle s_1, x_1\rangle, \dots, \langle s_n, x_n\rangle\}$ be a garbage collector. If $t = C[x_{i_1}, \dots, x_{i_m}]$ such that all occurrences of the variables $x_1, \dots, x_n$ in t are displayed, then the term $C[s_{i_1}, \dots, s_{i_m}]$ is denoted by $\phi^{-1}(t)$.

PROPOSITION 4.21. *Every capped term t has an applicable garbage collector ϕ.*

PROOF. Let $\{t_1, \dots, t_n\}$ be the set of all maximal subterms of t with root symbol in $\mathcal{D}_2 - \mathcal{D}_1$. Choose fresh variables $x_1, \dots, x_n$ and define $\phi = \{\langle t_1, x_1\rangle, \dots, \langle t_n, x_n\rangle\}$. By construction ϕ is a garbage collector with $\phi \ltimes t$. □

PROPOSITION 4.22. *The relations $\rightarrow_m$, $\rightarrow_{\mathcal{R}_1}$, and $\rightarrow_{\mathcal{R}_2}$ are closed under ϕ^{-1}, for all garbage collectors ϕ.*

PROOF. Since the mapping ϕ^{-1} can be considered as a substitution in $\Sigma(\mathcal{T})$, the result follows from the closure of $\rightarrow_m$, $\rightarrow_{\mathcal{R}_1}$, and $\rightarrow_{\mathcal{R}_2}$ under such substitutions, which is easily shown by an induction on the depth of steps $s \rightarrow_m t$, $s \rightarrow_{\mathcal{R}_1} t$, and $s \rightarrow_{\mathcal{R}_2} t$. □

PROPOSITION 4.23. *Let t be a capped term and suppose that ϕ is a garbage collector.*

(1) *If $\phi \ltimes t$ then $\phi^{-1}(\phi(t)) = t$.*
(2) *We have $\phi \ltimes t$ if and only if $\phi \ltimes e(t)$. Moreover, if $\phi \ltimes t$ then $e(\phi(t)) = \phi(e(t))$.*
(3) *If t does not contain symbols in $\mathcal{D}_2 - \mathcal{D}_1$ then $\phi \ltimes \phi^{-1}(t)$ and $\phi(\phi^{-1}(t)) = t$.*

PROOF. Straightforward. □

PROPOSITION 4.24. *Let s and t be similar capped terms. If $\phi \ltimes s$ then $\phi \ltimes t$ and $\phi(s) \approx \phi(t)$.*

PROOF. We obtain $\phi \ltimes t$ by two applications of Proposition 4.23(2). The similarity of $\phi(s)$ and $\phi(t)$ also follows from Proposition 4.23(2). □

PROPOSITION 4.25. *Let ϕ be a garbage collector that is applicable to a capped term s. If $s \rightarrow_m t$ then $\phi \ltimes t$ and $\phi(s) \rightarrow_m \phi(t)$.*

PROOF. Since every unmarked subterm of t with root symbol in $\mathcal{D}_2 - \mathcal{D}_1$ occurs in some alien in t, we obtain $\phi \ltimes t$ from Proposition 4.5. We prove that $\phi(s) \rightarrow_m \phi(t)$ by induction

[5] The notion of garbage collector is simpler than the corresponding notion of $\mathcal{D}'$-*replacement* in (Middeldorp and Toyama, 1993).

on the depth of $s \to_m t$. In case of zero depth we have nothing to prove. Suppose the depth of $s \to_m t$ equals $n+1$ ($n \geqslant 0$). By definition there exists a context C, a rewrite rule $l = C_1[x_1,\dots,x_n] \to r \Leftarrow c$ in $\mathcal{M}$ with C_1 variable-free, and terms $s_1,\dots,s_n$ such that $s = C[C_1[s_1,\dots,s_n]]$, $s_i \approx s_j$ whenever $x_i = x_j$, $t = C[r\sigma]$, and, for every equation $c_1 = c_2$ in c, $c_1\sigma \downarrow_m^{\approx} c_2\sigma$ with depth at most n. Here σ is the substitution induced by l and s. We have $\phi(s) = \phi(C)[C_1[\phi(s_1),\dots,\phi(s_n)]]$. Let τ be the substitution induced by l and $\phi(s)$. Proposition 4.24 yields $x\tau = \phi(x\sigma)$, hence $\phi(t) = \phi(C)[r\tau]$. Let $c_1 = c_2$ be an equation in c. From $c_1\sigma \downarrow_m^{\approx} c_2\sigma$ we obtain $c_1\tau = \phi(c_1\sigma) \downarrow_m^{\approx} \phi(c_2\sigma) = c_2\tau$ by a routine induction argument. Therefore $\phi(s) \to_m \phi(t)$. □

5. Main Results

In this section we present our main results. We prove the decomposability of weak normalisation, semi-completeness, and completeness, in that order. The proof of the decomposability of completeness for CSs in (Middeldorp and Toyama, 1991) does not depend on the decomposability of semi-completeness. This is possible since the decomposability of local confluence—which holds in the unconditional case—enables one to circumvent a direct proof of confluence. However, local confluence is not a decomposable property of CCSs. Hence we show the decomposability of semi-completeness before we tackle completeness.

Proposition 5.1. *Suppose $(\mathcal{D}_1,\mathcal{C}_1,\mathcal{R}_1)$ and $(\mathcal{D}_2,\mathcal{C}_2,\mathcal{R}_2)$ are composable CCSs and let $s \in \mathcal{T}(\mathcal{D}_i,\mathcal{C}_i,\mathcal{V})$ for some $i \in \{1,2\}$. If $s \to_{\mathcal{R}_1\cup\mathcal{R}_2} t$ then $s \to_{\mathcal{R}_i} t$ and thus $t \in \mathcal{T}(\mathcal{D}_i,\mathcal{C}_i,\mathcal{V})$.*

Proof. The proposition is easily proved by induction on the depth of the step $s \to_{\mathcal{R}_1\cup\mathcal{R}_2} t$, using the equality $\mathcal{R}_1 \mid \mathcal{D}_2 = \mathcal{R}_2 \mid \mathcal{D}_1$. □

Lemma 5.2. *Weak normalisation is a decomposable property of CCSs.*

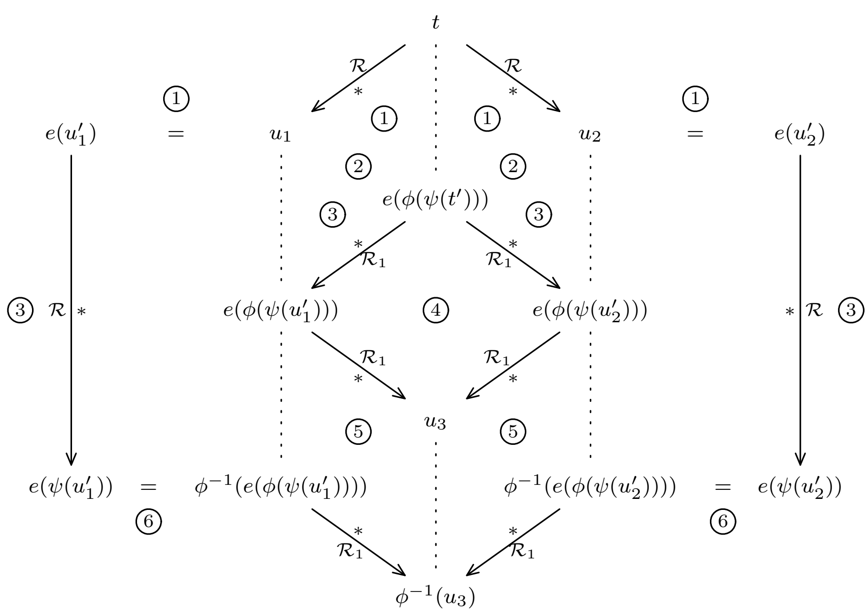
Proof. Let $(\mathcal{D},\mathcal{C},\mathcal{R})$ be the union of two weakly normalising and composable CCSs $(\mathcal{D}_1,\mathcal{C}_1,\mathcal{R}_1)$ and $(\mathcal{D}_2,\mathcal{C}_2,\mathcal{R}_2)$. By induction on the structure of t we will show that every term $t \in \mathcal{T}$ has a normal form with respect to $\mathcal{R}$. The case $t \in \mathcal{C} \cup \mathcal{V}$ is trivial. If t is a defined constant then t belongs to some $\mathcal{D}_i$ and the result follows from the weak normalisation of $(\mathcal{D}_i,\mathcal{C}_i,\mathcal{R}_i)$ and Proposition 5.1. For the induction step, suppose that $t = f(t_1,\dots,t_n)$ with $t_1,\dots,t_n$ weakly normalising. Let s_i be an $\mathcal{R}$-normal form of t_i for $i = 1,\dots,n$. Clearly $t \to_{\mathcal{R}}^* f(s_1,\dots,s_n)$. If $f \in \mathcal{C}$ then $f(s_1,\dots,s_n)$ is an $\mathcal{R}$-normal form of t. Suppose $f \in \mathcal{D}$. Without loss of generality we may assume that $f \in \mathcal{D}_1$. Let $t' = f^*(s_1,\dots,s_n)$. From Proposition 4.21 we obtain a garbage collector ϕ which is applicable to t'. Note that $e(\phi(t'))$ belongs to $\mathcal{T}(\mathcal{D}_1,\mathcal{C},\mathcal{V})$. Since $(\mathcal{D}_1,\mathcal{C},\mathcal{R}_1)$ inherits weak normalisation from $(\mathcal{D}_1,\mathcal{C}_1,\mathcal{R}_1)$,[6] there exists an $\mathcal{R}_1$-normal form n such that $e(\phi(t')) \to_{\mathcal{R}_1}^! n$. According to Proposition 5.1 n is also normal form with respect to $\mathcal{R}_2$. Using Proposition 4.22, we easily infer $\phi(t') \in \hat{\mathcal{T}}_{in}$ from $t' \in \hat{\mathcal{T}}_{in}$. From Proposition 4.11 we obtain a term $n' \in \hat{\mathcal{T}}_{in}$ with $e(n') = n$ and $\phi(t') \to_m^! n'$. Propositions 4.22 and 4.23(1) yield $t' \to_m^* \phi^{-1}(n')$. The inside normalisation of $\phi^{-1}(n')$ follows from the

[6] This follows e.g. from the more general Corollary 5.4 in (Middeldorp, 1993).

inside normalisation of t' by means of Proposition 4.5. From Proposition 4.6(1) we obtain $f(s_1,\ldots,s_n) = e(t') \to^*_{\mathcal{R}_1} e(\phi^{-1}(n'))$. Clearly $e(\phi^{-1}(n')) = \phi^{-1}(n)$. We conclude the proof by showing that $\phi^{-1}(n)$ is an $\mathcal{R}$-normal form. Suppose to the contrary that $\phi^{-1}(n) \to_{\mathcal{R}} u$ for some term u. Since $\phi^{-1}(n') \in \hat{\mathcal{T}}_{in}$, we can apply Proposition 4.10, which yields a term $u' \in \hat{\mathcal{T}}_{in}$ such that $e(u') = u$ and $\phi^{-1}(n') \to_m u'$. Since n' does not contain function symbols in $\mathcal{D}_2 - \mathcal{D}_1$, we obtain $\phi \ltimes \phi^{-1}(n')$ and $\phi(\phi^{-1}(n')) = n'$ from Proposition 4.23(3). Proposition 4.25 now yields $n' \to_m \phi(u')$, contradicting the fact that n' is a normal form with respect to $\to_m$. □

THEOREM 5.3. *Semi-completeness is a decomposable property of CCSs.*

PROOF. Let $(\mathcal{D},\mathcal{C},\mathcal{R})$ be the union of semi-complete and composable CCSs $(\mathcal{D}_1,\mathcal{C}_1,\mathcal{R}_1)$ and $(\mathcal{D}_2,\mathcal{C}_2,\mathcal{R}_2)$. From Lemma 5.2 we infer the weak normalisation of $(\mathcal{D},\mathcal{C},\mathcal{R})$. By induction on the structure of t we will show that every term $t \in \mathcal{T}$ is confluent. The case $t \in \mathcal{D} \cup \mathcal{C} \cup \mathcal{V}$ is easy. Suppose $t = f(t_1,\ldots,t_n)$ such that $t_1,\ldots,t_n$ are confluent and thus semi-complete. If f is a constructor then t is easily shown to be confluent. Suppose $f \in \mathcal{D}$. Without loss of generality we assume that $f \in \mathcal{D}_1$. Let $t' = f^*(t_1,\ldots,t_n)$. Clearly $t' \in \hat{\mathcal{T}}_{sc}$. Proposition 4.21 yields a garbage collector ϕ which is applicable to $\psi(t')$. After these preliminary definitions, we consider diverging reductions $u_1 \leftarrow^*_{\mathcal{R}} t \to^*_{\mathcal{R}} u_2$. Figure 4 shows how to obtain a common reduct of u_1 and u_2. Therefore t is confluent. □



① Corollary 4.19 ③ Proposition 4.6 ⑤ Proposition 4.22

② Proposition 4.25 ④ confluence of $(\mathcal{D}_1,\mathcal{C},\mathcal{R}_1)$[7] ⑥ Proposition 4.23(1,2)

Figure 4.

[7] $(\mathcal{D}_1,\mathcal{C},\mathcal{R}_1)$ inherits confluence from $(\mathcal{D}_1,\mathcal{C}_1,\mathcal{R}_1)$: Theorem 3.17 in (Middeldorp, 1993) states a more general result.

Theorem 5.4. *Completeness is a decomposable property of CCSs.*

Proof. Suppose $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$ are complete and composable CCSs. From Theorem 5.3 we obtain the semi-completeness of their union $(\mathcal{D}, \mathcal{C}, \mathcal{R})$. Hence it remains to show that $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ is strongly normalising. This will be established by induction on the structure of terms $t \in \mathcal{T}$. The base case is easy. Let $t = f(t_1, \ldots, t_n)$ such that $t_1, \ldots, t_n$ are strongly normalising and thus complete. If f is a constructor then t is clearly strongly normalising. So assume without loss of generality that $f \in \mathcal{D}_1$. If t is not strongly normalising then there exists an infinite reduction sequence

$$t = s_1 \rightarrow_{\mathcal{R}} s_2 \rightarrow_{\mathcal{R}} s_3 \rightarrow_{\mathcal{R}} \cdots .$$

Let $t' = s'_1 = f^*(t_1, \ldots, t_n)$. Clearly $t' \in \hat{\mathcal{T}}_{sc}$. Repeated application of Lemma 4.18 yields terms $s'_i \in \hat{\mathcal{T}}_{sc}$ $(i > 1)$ with $e(s'_i) = s_i$ and, for all $i \geqslant 1$, $s'_i \rhd s'_{i+1}$, and either $s'_i \rightarrow_u s'_{i+1}$ or $\psi(s'_i) \rightarrow_m^+ \psi(s'_{i+1})$; see Figure 5. We now show that the second possibility

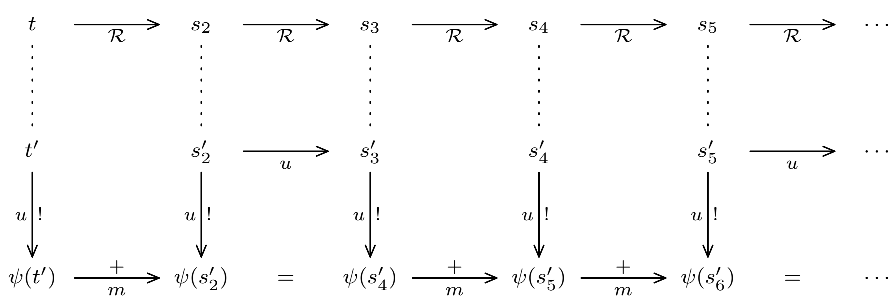


Figure 5.

occurs infinitely often. Suppose to the contrary that there exists an index N such that $s'_i \rightarrow_u s'_{i+1}$ for all $i \geqslant N$. By the pigeon-hole principle there exists an alien a in s'_N with an infinite $\mathcal{R}$-reduction. Since $t' \rhd s'_N$ there exists an alien a' in t' such that $a' \rightarrow_{\mathcal{R}}^* C[a]$ for some context C. Hence a' is not strongly normalising. However, since a' is an unmarked subterm of $t' = f^*(t_1, \ldots, t_n)$, it must be a subterm of one of the strongly normalising terms $t_1, \ldots, t_n$. This is impossible. From the semi-completeness of $\rightarrow_u$ on $\hat{\mathcal{T}}_{sc}$ we infer that $\psi(s'_i) = \psi(s'_{i+1})$ whenever $s'_i \rightarrow_u s'_{i+1}$. Hence we obtain a sequence

$$\psi(t') = \psi(s'_1) \rightarrow_m^* \psi(s'_2) \rightarrow_m^* \psi(s'_3) \rightarrow_m^* \cdots$$

containing infinitely many reduction steps. From Proposition 4.21 we obtain a garbage collector ϕ which is applicable to $\psi(t')$. According to Proposition 4.25 the term $\phi(\psi(t'))$ has an infinite reduction sequence

$$\phi(\psi(t')) = \phi(\psi(s'_1)) \rightarrow_m^* \phi(\psi(s'_2)) \rightarrow_m^* \phi(\psi(s'_3)) \rightarrow_m^* \cdots .$$

Employing Proposition 4.6, the erasure of all markers in the last sequence yields an infinite $\mathcal{R}_1$-reduction sequence starting from the term $e(\phi(\psi(t'))) \in \mathcal{T}(\mathcal{D}_1, \mathcal{C}, \mathcal{V})$. Since $(\mathcal{D}_1, \mathcal{C}, \mathcal{R}_1)$ inherits completeness from $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$,[8] this is impossible. □

[8] This follows e.g. from the more general Theorem 4.31 in (Middeldorp, 1993).

6. Conclusion

The results presented in this paper should be extended to CCSs with extra variables in the conditions of the rewrite rules. This will not be an easy matter. First of all, the results on marked reduction in Section 4 do not extend to such CCSs (after an obvious adjustment of the substitution σ in Definition 4.4). However, all counterexamples (against Proposition 4.10 and Lemma 4.18) that we could think of involved a non-confluent CCS. Another problem is that Proposition 5.1 and Lemma 5.2 do not allow this relaxation in the distribution of variables. Consider for instance the following disjoint CCSs from Middeldorp (1990, 1993):

$$\mathcal{R}_1 = \left\{ \begin{array}{lllll} f(x,x) & \to & d & & \\ f(x,y) & \to & f(x,y) & \Leftarrow & x = z,\ z = y \end{array} \right\},$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{lll} a & \to & b \\ a & \to & c \end{array} \right\}.$$

One easily shows that $\mathcal{R}_1$ is confluent. From this we obtain the weak normalisation of $\mathcal{R}_1$ by a routine argument. Clearly $\mathcal{R}_2$ is weakly normalising. However, $\mathcal{R}_1 \cup \mathcal{R}_2$ is not weakly normalising since the term $f(b,c)$ reduces only to itself. Observe that $\mathcal{R}_2$ lacks confluence. We strongly believe that both semi-completeness and completeness are decomposable properties of CCSs with extra variables in the conditions.

Using different proof techniques, Krishna Rao (1993) recently extended the completeness part of (Middeldorp and Toyama, 1993)—the decomposability of completeness for CSs, that is—to *hierarchical* systems. Hierarchical systems are CSs that can be partitioned into two CSs such that one of the systems may use a defined symbol of the other system in the right-hand sides of its rewrite rules without importing the rules defining that symbol, see Krishna Rao (1993) for a precise formulation. The typical example is the extension of

$$\mathcal{R}_1 = \left\{ \begin{array}{lll} 0 + x & \to & x \\ S(x) + y & \to & S(x + y) \end{array} \right\}$$

with

$$\mathcal{R}_2 = \left\{ \begin{array}{lll} 0 \times x & \to & 0 \\ S(x) \times y & \to & x \times y + y \end{array} \right\}.$$

Observe that $\mathcal{R}_1$ and $\mathcal{R}_2$ are not composable as $\mathcal{R}_1 \mid \{+\} \neq \mathcal{R}_2 \mid \{+\}$. It goes without saying that it should be investigated whether Krishna Rao's result can be carried over to CCSs.

A CTRS $\mathcal{R}$ is called *level-confluent* if every $\mathcal{R}_n$ is confluent. Level-confluence (Giovannetti and Moiso, 1986) is a key property for ensuring completeness of languages that amalgamate the logic and functional programming paradigms and whose operational semantics is conditional narrowing, see e.g. (Giovannetti and Moiso, 1986) and (Middeldorp and Hamoen, 1992). At present very few techniques are available for establishing level-confluence. It would be interesting to investigate whether our results can be extended to level-complete (i.e. level-confluent and strongly normalising) and level-semi-complete CCSs. The confluence part of our proof of the decomposability of semi-completeness (see Figure 4) doesn't yield level-confluence, since the depth of the reduction sequence from $e(u_i')$ to $e(\psi(u_i'))$ may very well exceed the depth of the peak $u_1 \leftarrow^*_{\mathcal{R}} t \to^*_{\mathcal{R}} u_2$. A possible approach is to reduce u'_i only to its $\mathcal{R}_n$-normal form, where n is the depth of $u_1 \leftarrow^*_{\mathcal{R}} t \rightarrow^*_{\mathcal{R}} u_2$, but so far we haven't been able to put this idea into a proof.

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