A Simple Proof to a Result of Bernhard Gramlich

Note distributed by Aart Middeldorp at the 5th Term Rewriting Meeting
in Tsukuba, February 3 and 4, 1994.

In this note we present a simple proof to the following result of Bernhard Gramlich: $\mathrm{WCR}(\mathcal{R})$ & $\mathrm{OS}(\mathcal{R}) \Rightarrow \forall t\ [\mathrm{SIN}(t) \Rightarrow \mathrm{SN}(t)]$ (in a locally confluent overlay system, every strongly innermost normalizing term is strongly normalizing). This result appeared in *Relating Innermost, Weak, Uniform and Modular Termination of Term Rewriting Systems*, Proceedings of the Conference on Logic Programming and Automated Reasoning, St. Petersburg, Lecture Notes in Artificial Intelligence **624**, pp. 285–296, 1992.

Throughout the following we assume that we are dealing with a locally confluent system $\mathcal{R}$, i.e., $\mathrm{WCR}(\mathcal{R})$. First some easy definitions. Every term t can be (uniquely) written as $C[t_1, \ldots, t_n]$ where $t_1, \ldots, t_n$ are the maximal complete subterms of t. We define $\phi(t)$ as $C[t_1{\downarrow}, \ldots, t_n{\downarrow}]$. Here $t_i{\downarrow}$ denotes the unique normal form of t_i. Clearly $t \to^* \phi(t)$. Because of the global WCR assumption, a term is complete if and only if it is strongly normalizing. (This follows from the following localized variant of Newman's Lemma: $\mathrm{WCR}(\mathcal{R}) \Rightarrow \forall t\ [\mathrm{SN}(t) \Rightarrow \mathrm{CR}(t)]$.) Hence a term is complete if and only if all its subterms are complete. Let $s \to t$ be an arbitrary reduction step. We write $s \to_c t$ if the contracted redex is complete, and $s \to_{nc} t$ if the contracted redex is not complete. Clearly every reduction step can be written as either $\to_c$ or $\to_{nc}$.

Lemma 1. *The relation $\to_c$ is terminating.*

Proof. Straightforward. □

Lemma 2. *Suppose $\neg\mathrm{SN}(\mathcal{R})$. Every infinite reduction sequence contains infinitely many $\to_{nc}$ steps.*

Proof. Immediate consequence of Lemma 1. □

Lemma 3. *If $s \to_c t$ then $\phi(s) \to^* \phi(t)$.*

Proof. Clearly $t \to^* \phi(s)$ by performing only reductions in complete subterms of t. Hence $\phi(s) \to^* \phi(t)$. □

Observe that in general we do not have $\phi(s) = \phi(t)$ in Lemma 3: take for instance $\mathcal{R}_1 = \{a \to b,\ f(a) \to f(a),\ f(b) \to c\}$ and consider the step $s = f(a) \to_c f(b) = t$. The analogous statement (of Lemma 3) for $\to_{nc}$ does not hold, consider for instance the TRS $\mathcal{R}_2 = \{a \to b,\ f(a) \to g(a),\ g(x) \to f(x)\}$ and the step $s = f(a) \to_{nc} g(a) = t$; we have $\phi(s) = f(b)$ and $\phi(t) = g(b)$. Note that $\mathcal{R}_2$ is not an overlay system. This is essential:

Lemma 4. *Suppose $\mathrm{OS}(\mathcal{R})$. If $s \to_{nc} t$ then $\phi(s) \to^+ \phi(t)$.*

Proof. Suppose $s \to_{nc} t$ by applying rewrite rule $l \to r$ at position p with substitution σ, so $s_{|p} = l\sigma$ and $t = s[r\sigma]_p$. Because $s_{|p}$ is not complete, p is a position in $\phi(s)$. The crucial observation is that $\phi(s)_{|p}$ is still an instance of l. This follows from the OS assumption by a routine argument. Actually we know which instance: $\phi(s)_{|p} = l\tau$ with substitution τ defined by $\tau(x) = \phi(\sigma(x))$ for all variables x. Hence $\phi(s) \to \phi(s)[r\tau]_p$. Clearly $t \to^* \phi(s)[r\tau]_p$ by performing only reductions in complete subterms of t. Therefore $\phi(s)[r\tau]_p \to^* \phi(t)$. We conclude that $\phi(s) \to^+ \phi(t)$. □

The above lemma is the most difficult part of the whole proof. Only here we make (explicit) use of the OS restriction.

Lemma 5. *Suppose OS($\mathcal{R}$). If SN($\phi(t)$) then SN(t).*

Proof. Simply combine Lemma's 3, 4, and 5. □

Theorem 6. *Suppose OS($\mathcal{R}$). If SIN(t) then SN(t).*

Proof. For a proof by contradiction, suppose t admits an infinite reduction sequence. Every infinite reduction sequence starting from t must contain a non-innermost step, due to SIN(t). We consider an infinite reduction sequence D starting from t that has the property that the first non-innermost step is essential: selecting any innermost redex at that point would result in a term with the property SN. Write

$$D\colon t = t_0 \to t_1 \to \cdots \to t_n \to t_{n+1} \to \cdots$$

where $t_n \to t_{n+1}$ is the first non-innermost step. By assumption, contracting an innermost redex in t_n yields a strongly normalizing term. This implies that every innermost redex in t_n is complete. Since there is at least one innermost redex in t_n, we conclude that SN($\phi(t_n)$). Since we also have $\neg$SN(t), this contradicts Lemma 5. □