Context-Freeness and Infinitary Normalization[1]

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In [2] it is shown that context-free root-normalizing reduction strategies are infinitary fair-normalizing for confluent TRSs. Lucas [1, Theorem 6.6] shows that every NV-strategy is infinitary fair-normalizing for almost orthogonal NV-sequential TRSs. Lucas remarks: "*Theorem 6.6 proves that $\mathcal{S}$ in Example 6.4 is infinitary fair-normalizing. This cannot be obtained from Middeldorp's results.*" In this note we show that Lucas' result is a trivial consequence of Middeldorp's results.

The reader is referred to [2, 3, 1] for definitions and explanations of the terminology used above. The following result is due to Middeldorp [2].

Theorem 1 *Let $\mathcal{R}$ be a confluent TRS. Context-free root-normalizing reduction strategies for $\mathcal{R}$ are infinitary fair-normalizing.* □

Lucas [1, Theorem 6.2] obtained the following result.

Theorem 2 *NV-strategies are hyper root-normalizing for almost orthogonal NV-sequential TRSs.* □

Since almost orthogonal TRSs are confluent and hyper root-normalization implies root-normalization, a proof of context-freeness is sufficient to obtain the infinitary fair-normalization of NV-strategies (for almost orthogonal NV-sequential TRSs) by Theorem 1. However, Lucas observed that general NV-strategies need not be context-free by means of the strategy $\mathcal{S}$ that contracts the leftmost (rightmost) NV-redex if the total number of redexes in the term at hand is odd (even). For instance, with respect to the (almost orthogonal NV-sequential) TRS consisting of the single rewrite rule

$$\mathsf{a} \to \mathsf{f}(\mathsf{a}, \mathsf{a})$$

all redexes are NV-redexes (i.e., occur at NV-index positions) and we have $\mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{a}),\mathsf{a}) \to^{\mathcal{S}} \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{a}),\mathsf{a}),\mathsf{a})$ and $\mathsf{f}(\mathsf{a},\mathsf{a}) \to^{\mathcal{S}} \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{a}))$, revealing that $\mathcal{S}$ is not context-free.

It is interesting to note that $\mathcal{S}$ will fail to compute any infinite normal form. For instance, the limit of the (unique) $\mathcal{S}$-rewrite sequence starting at the term a is the infinite non-normal form $\mathsf{f}(\mathsf{f}(\mathsf{f}(\cdots,\mathsf{a}),\mathsf{a}),\mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\cdots)))$. This does not contradict infinitary fair-normalization, simply because there are no *fair* $\mathcal{S}$-rewrite sequences!

Lucas [1, Theorem 6.6] proved the following result.

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Theorem 3 *NV-strategies are infinitary fair-normalizing for almost orthogonal NV-sequential TRSs.* □

In his proof ([1, Proposition 6.5]) the following well-known property of NV-indices (Oyamaguchi [3, Lemma 6.3]) is used: If pq is an NV-index in t then q is an NV-index in $t_{|p}$. Interestingly, context-freeness of NV-reduction is a direct consequence of this property.

We claim that Theorem 3 is a trivial consequence of Theorem 1. Actually, it is a special case of the following trivial consequence of Theorem 1.

Corollary 1 *Let $\mathcal{R}$ be a confluent TRS. Every reduction strategy $\mathcal{S}$ for $\mathcal{R}$ that can be extended to a context-free root-normalizing reduction strategy for $\mathcal{R}$ is infinitary fair-normalizing.*

Proof Let $\mathcal{S}'$ be a context-free root-normalizing reduction strategy for $\mathcal{R}$ that extends $\mathcal{S}$. According to Theorem 1, $\mathcal{S}'$ is infinitary fair-normalizing. If $\mathcal{S}$ is not infinitary fair-normalizing then, by definition of infinitary fair-normalization, there exists a term t with a (finite or infinite) normal form and a perpetual fair $\mathcal{S}$-rewrite sequence (i.e., an infinite fair $\mathcal{S}$-rewrite sequence whose limit is not a normal form). Since $\mathcal{S}$ is a restriction of $\mathcal{S}'$, there exists a perpetual $\mathcal{S}'$-rewrite sequence starting at t, contradicting the infinitary fair-normalization of $\mathcal{S}'$. □

Note that the above proof involves nothing more than applying the definition of infinitary fair-normalization. Further note that $\mathcal{S}$ is neither required to be context-free nor root-normalizing.

References

[1] S. Lucas, *Root-Neededness and Approximations of Neededness*, Information Processing Letters **67**(5), pp. 245–254, 1998.

[2] A. Middeldorp, *Call by Need Computations to Root-Stable Form*, Proceedings of the 24th Annual Symposium on Principles of Programming Languages, ACM Press, pp. 94–105, 1997.

[3] M. Oyamaguchi, *NV-Sequentiality: A Decidable Condition for Call-by-Need Computations in Term Rewriting Systems*, SIAM Journal on Computing **22**, pp. 114–135, 1993.