Innermost Termination
of Context-Sensitive Rewriting

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Abstract. Context-sensitive rewriting is a restriction of term rewriting used to model evaluation strategies in functional programming and in programming languages like OBJ. For example, under certain conditions termination of an OBJ program is equivalent to innermost termination of the corresponding context-sensitive rewrite system [18]. To prove termination of context-sensitive rewriting, several methods have been proposed in the literature which transform context-sensitive rewrite systems into ordinary rewrite systems such that termination of the transformed ordinary system implies termination of the original context-sensitive system. Most of these transformations are not very satisfactory when it comes to proving *innermost* termination. We investigate the relationship between termination and innermost termination of context-sensitive rewriting and we examine the applicability of the different transformations for innermost termination proofs. Finally, we present a simple transformation which is both sound and complete for innermost termination.

1   Introduction

Evaluation in functional languages is often guided by specific evaluation strategies. For example, in the program consisting of the rules

$$\mathsf{from}(x) \to x : \mathsf{from}(\mathsf{s}(x)) \quad \mathsf{nth}(0, x : y) \to x \quad \mathsf{nth}(\mathsf{s}(n), x : y) \to \mathsf{nth}(n, y)$$

the term $\mathsf{nth}(\mathsf{s}(0), \mathsf{from}(0))$ admits a finite reduction to $\mathsf{s}(0)$ as well as infinite reductions. The infinite reductions can for instance be avoided by always contracting the outermost redex. Context-sensitive rewriting [16,17] provides an alternative way of solving the non-termination problem and of dealing with infinite data objects. Here, every n-ary function symbol f is equipped with a *replacement map* $\mu(f) \subseteq \{1, \ldots, n\}$ which indicates which arguments of f may be evaluated and a contraction of a redex is allowed only if it does not take place in a forbidden argument of a function symbol somewhere above it. So by

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defining $\mu(:) = \{1\}$, contractions in the argument t of a term $s : t$ are forbidden. Now in the example infinite reductions are no longer possible while normal forms can still be computed. (See [20] for the relationship between normalization under ordinary rewriting and under context-sensitive rewriting.) Context-sensitive rewriting can also model the usual evaluation strategy for conditionals.

Example 1.

$$0 \leqslant y \to \mathsf{true} \qquad\qquad\qquad \mathsf{p}(0) \to 0$$
$$\mathsf{s}(x) \leqslant 0 \to \mathsf{false} \qquad\qquad\qquad \mathsf{p}(\mathsf{s}(x)) \to x$$
$$\mathsf{s}(x) \leqslant \mathsf{s}(y) \to x \leqslant y \qquad\qquad \mathsf{if}(\mathsf{true}, x, y) \to x$$
$$x - y \to \mathsf{if}(x \leqslant y, 0, \mathsf{s}(\mathsf{p}(x) - y)) \qquad \mathsf{if}(\mathsf{false}, x, y) \to y$$

Because of the "$-$"-rule, this system is not terminating. But in functional languages if's first argument is evaluated first and either the second or third argument is evaluated afterwards. Again, this can easily be modeled with context-sensitive rewriting by the replacement map $\mu(\mathsf{if}) = \{1\}$ which forbids all reductions in the arguments t_2 and t_3 of $\mathsf{if}(t_1, t_2, t_3)$.

In programming languages like OBJ [4,5,12], the user can supply strategy annotations to control the evaluation [6,21,22]. For every n-ary symbol f, a (positive) strategy annotation is a list $\varphi(f)$ of numbers $(i_1, \ldots, i_k)$ from $\{0, 1, \ldots, n\}$. When reducing a term $f(t_1, \ldots, t_n)$ one first has to evaluate the i_1-th argument of f (if $i_1 > 0$), then one evaluates the i_2-th argument (if $i_2 > 0$), and so on, until a 0 is encountered. At this point one tries to evaluate the whole term $f(\ldots)$ at its root position. So in order to enforce the desired evaluation strategy for if in Example 1, it has to be equipped with the strategy annotation $(1, 0)$.

Context-sensitive rewriting can simulate OBJ's evaluation strategy. A strategy is *elementary* if for every defined[1] symbol f, $\varphi(f)$ contains a single occurrence of 0, at the end. Lucas [18] showed that for elementary strategies, the OBJ program terminates iff the corresponding context-sensitive rewrite system (CSRS) is *innermost* terminating[2]. Here $\mu(f) = \{i \in \varphi(f) \mid i > 0\}$. For example, the program with the rules $\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a})$ and $\mathsf{a} \to \mathsf{b}$ is terminating if $\varphi(\mathsf{f}) = (1, 0)$ and $\varphi(\mathsf{a}) = (0)$. The corresponding CSRS with $\mu(\mathsf{f}) = \{1\}$ is not terminating, but *innermost* terminating. Thus, to simulate OBJ evaluations with CSRSs, we have to restrict ourselves to innermost reductions where (allowed) arguments to a function are evaluated before evaluating the function.

Because of this connection to OBJ and also because innermost termination is easier to prove automatically than termination [1], it is worthwhile to investigate innermost termination of context-sensitive rewriting. (An alternative approach to prove termination of OBJ-like programs by direct induction proofs is proposed in [8].) Termination of CSRSs has been studied in a number of papers (e.g., [3,7,9,11,15,16,17,20,23]). Apart from a direct semantic characterization [23] and some recent extensions of standard termination methods for term rewriting to context-sensitive rewriting [3,15], all other proposed methods transform CSRSs into ordinary term rewrite systems (TRSs) such that termination of the transformed TRS implies termination of the original CSRS (i.e., all

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[1] Every symbol on the root position of a left-hand side of a rule is called *defined*.

[2] The "if" direction even holds without the restriction to elementary strategies [18].

these transformations are *sound*). Direct approaches to termination analysis of CSRSs and transformational approaches both have their advantages. Techniques for proving termination of ordinary term rewriting have been studied extensively and the main advantage of the transformational approach is that in this way, all termination techniques for ordinary TRSs including future developments can be used to infer termination of CSRSs. For instance, the methods of [3,15] are unable to handle systems like Example 1.

After introducing the termination problem of context-sensitive rewriting in Section 2, in Section 3 we review the results of Lucas [18] on innermost termination of CSRSs and we show that the two transformations Θ_1 and Θ_2 of [9] are sound for innermost termination as well. Despite its soundness Θ_2 is not very useful for proving innermost termination, because termination and innermost termination coincide for the TRSs it produces. In Section 4 we show that for the class of orthogonal CSRSs, innermost termination already implies termination. This result is independent from the transformation framework and is of general interest when investigating the termination behavior of CSRSs. A consequence of this result is that for this particular class, Θ_1 is complete for innermost termination. In Section 5 we present a new transformation Θ_3 which is both sound and complete for innermost termination, for arbitrary CSRSs. Surprisingly, such a transformation can be obtained by just a small modification of Θ_1. In spite of the similarity between the two transformations, the new completeness proof is non-trivial. We make some remarks on a possible simplification of Θ_3 and on *ground* innermost termination in Section 6. In Section 7 we show that Θ_3 is equally powerful as Θ_1 when it comes to (non-innermost) termination. Due to lack of space, many proofs have been omitted. They can be found in [10].

2   Termination of Context-Sensitive Rewriting

Familiarity with the basics of term rewriting [2] is assumed. We require that every signature $\mathcal{F}$ contains a constant. A function $\mu\colon \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* if $\mu(f)$ is a subset of $\{1, \ldots, \mathrm{arity}(f)\}$ for all $f \in \mathcal{F}$. A *CSRS* $(\mathcal{R}, \mu)$ is a TRS $\mathcal{R}$ over a signature $\mathcal{F}$ equipped with a replacement map μ. The context-sensitive rewrite relation $\to_{\mathcal{R},\mu}$ is defined as the restriction of the usual rewrite relation $\to_{\mathcal{R}}$ to contractions of redexes at *active* positions. A position π in a term t is active if $\pi = \epsilon$ (the root position), or $t = f(t_1, \ldots, t_n)$, $\pi = i\pi'$, $i \in \mu(f)$, and π' is active in t_i. So $s \to_{\mathcal{R},\mu} t$ iff there is a rule $l \to r$ in $\mathcal{R}$, a substitution σ, and an active position π in s such that $s|_\pi = l\sigma$ and $t = s[r\sigma]_\pi$. If all active arguments of $l\sigma$ are in μ-normal form, then the reduction step is *innermost* and we write $s \xrightarrow{\mathrm{i}}_{\mathcal{R},\mu} t$. Here a μ-normal form is a normal form w.r.t. $\to_{\mathcal{R},\mu}$. We abbreviate $\to_{\mathcal{R},\mu}$ to $\to_\mu$ and $\xrightarrow{\mathrm{i}}_{\mathcal{R},\mu}$ to $\xrightarrow{\mathrm{i}}_\mu$ if $\mathcal{R}$ is clear from the context. A CSRS $(\mathcal{R}, \mu)$ is *left-linear* if the left-hand sides of the rewrite rules in $\mathcal{R}$ are linear terms (i.e., they do not contain multiple occurrences of the same variable). Let $l \to r$ and $l' \to r'$ be renamed versions of rewrite rules of $\mathcal{R}$ such that they have no variables in common and suppose $l|_\pi$ and l' are unifiable with most general unifier σ for some non-variable active position π in l. The pair of terms $\langle l[r']_\pi\sigma, r\sigma \rangle$ is a

critical pair of $(\mathcal{R}, \mu)$, except when $l \to r$ and $l' \to r'$ are renamed versions of the same rewrite rule and $\pi = \epsilon$. A *non-overlapping* CSRS has no critical pairs and an *overlay* CSRS has no critical pairs with $\pi \neq \epsilon$. A CSRS is *orthogonal* if it is left-linear and non-overlapping. Notions like "termination" for a CSRS $(\mathcal{R}, \mu)$ always concern the relation $\to_\mu$ (i.e., they correspond to "μ-termination" in [17]).

To prove termination of CSRSs, several transformations from CSRSs to ordinary TRSs were suggested. We recall Giesl & Middeldorp's transformation Θ_1 [9]. It uses new unary symbols active and mark to indicate active positions in a term on the object level. If $l \to r$ is a rule in the CSRS then the transformed TRS contains the rule $\mathsf{active}(l) \to \mathsf{mark}(r)$. The symbol mark is used to traverse a term top-down in order to place the symbol active at all active positions.

Definition 2 (Θ_1). *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. The TRS $\mathcal{R}_\mu^1$ over the signature $\mathcal{F}_1 = \mathcal{F} \cup \{\mathsf{active}, \mathsf{mark}\}$ has the following rules:*

$$(\sharp) \qquad \mathsf{active}(l) \to \mathsf{mark}(r) \qquad\qquad \forall l \to r \in \mathcal{R}$$
$$(\sharp) \qquad \mathsf{mark}(f(x_1, \ldots, x_n)) \to \mathsf{active}(f([x_1]^f, \ldots, [x_n]^f)) \qquad \forall f \in \mathcal{F}$$
$$\mathsf{active}(x) \to x$$

Here $[x_i]^f = \mathsf{mark}(x_i)$ if $i \in \mu(f)$ and $[x_i]^f = x_i$ otherwise. The transformation $(\mathcal{R}, \mu) \mapsto \mathcal{R}_\mu^1$ is denoted by Θ_1 and we shorten $\to_{\mathcal{R}_\mu^1}$ to $\to_1$.

Because every infinite reduction of a term t in the original CSRS would correspond to an infinite reduction of $\mathsf{mark}(t)$ in the transformed TRS, Θ_1 is *sound* for termination: Termination of the transformed TRS implies termination of the original CSRS. The second transformation Θ_2 of [9,11], Θ_{L} of Lucas [16], Θ_{Z} of Zantema [23], and Θ_{FR} of Ferreira & Ribeiro [7] are also sound for termination[3]. However, only Θ_2 is *complete*, i.e., the other transformations do not transform every terminating CSRS into a terminating TRS. Nevertheless, Θ_2 does not render the other transformations superfluous, since in practice, termination of $\Theta_2(\mathcal{R}, \mu)$ can be harder to show than termination of the TRSs resulting from the other transformations.

Example 3 ([9]). The non-terminating TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{b}, \mathsf{c}, x) \to \mathsf{f}(x, x, x), \mathsf{d} \to \mathsf{b}, \mathsf{d} \to \mathsf{c}\}$ demonstrates the incompleteness of Θ_1. If $\mu(\mathsf{f}) = \{3\}$ then the CSRS is terminating because the cyclic reduction of $\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d})$ to $\mathsf{f}(\mathsf{d}, \mathsf{d}, \mathsf{d})$ and further to $\mathsf{f}(\mathsf{b}, \mathsf{c}, \mathsf{d})$ cannot be done, as one would have to reduce the first and second argument of f. However, the TRS $\mathcal{R}_\mu^1$

$$\mathsf{active}(\mathsf{f}(\mathsf{b}, \mathsf{c}, x)) \to \mathsf{mark}(\mathsf{f}(x, x, x)) \qquad \mathsf{mark}(\mathsf{f}(x, y, z)) \to \mathsf{active}(\mathsf{f}(x, y, \mathsf{mark}(z)))$$
$$\mathsf{active}(\mathsf{d}) \to \mathsf{mark}(\mathsf{b}) \qquad\qquad\qquad \mathsf{mark}(\mathsf{b}) \to \mathsf{active}(\mathsf{b})$$
$$\mathsf{active}(\mathsf{d}) \to \mathsf{mark}(\mathsf{c}) \qquad\qquad\qquad \mathsf{mark}(\mathsf{c}) \to \mathsf{active}(\mathsf{c})$$
$$\mathsf{active}(x) \to x \qquad\qquad\qquad\qquad \mathsf{mark}(\mathsf{d}) \to \mathsf{active}(\mathsf{d})$$

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[3] The interested reader is referred to [11] for definitions and a comparison of these transformations.

is not terminating:

$$\mathsf{mark}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d})) \to_1 \mathsf{active}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{mark}(\mathsf{d}))) \to_1 \mathsf{active}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{active}(\mathsf{d})))$$

$$\to_1 \mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}),\mathsf{active}(\mathsf{d}))) \to_1^+ \mathsf{mark}(\mathsf{f}(\mathsf{mark}(\mathsf{b}),\mathsf{mark}(\mathsf{c}),\mathsf{d}))$$

$$\to_1^+ \mathsf{mark}(\mathsf{f}(\mathsf{active}(\mathsf{b}),\mathsf{active}(\mathsf{c}),\mathsf{d})) \to_1^+ \mathsf{mark}(\mathsf{f}(\mathsf{b},\mathsf{c},\mathsf{d}))$$

Note that in the third step the 'active' subterm $\mathsf{active}(\mathsf{d})$ is copied to the first and second argument positions of f, which are inactive according to $\mu(\mathsf{f})$. This can only happen if the reduction step is non-innermost.

3   Innermost Termination of Context-Sensitive Rewriting

Now we examine the usefulness of the five transformations for *innermost* termination of CSRSs. Lucas [18] showed that Θ_{L} and Θ_{Z} are unsound[4] for innermost termination, i.e., innermost termination of the transformed TRS does not imply innermost termination of the original CSRS. The example showing the latter ([18, Example 12]) also demonstrates that Θ_{FR} is unsound for innermost termination. Moreover, none of these transformations is complete for innermost termination. The following new result shows that Θ_1 is sound for innermost termination[5].

Theorem 4. *Let $(\mathcal{R},\mu)$ be a CSRS. If $\mathcal{R}^1_\mu$ is innermost terminating then $(\mathcal{R},\mu)$ is innermost terminating.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$ and let c be an arbitrary constant in $\mathcal{F}$. We show that every innermost reduction step $s \xrightarrow{\mathsf{i}}_\mu t$ in $(\mathcal{R},\mu)$ corresponds to an innermost reduction sequence $\mathsf{mark}(s\theta){\downarrow}_{\mathcal{M}} \xrightarrow{\mathsf{i}}_1^+ \mathsf{mark}(t\theta){\downarrow}_{\mathcal{M}}$ in $\mathcal{R}^1_\mu$. Here $\mathcal{M}$ consists of all rules in $\mathcal{R}^1_\mu$ of the form

$$\mathsf{mark}(f(x_1,\ldots,x_n)) \to \mathsf{active}(f([x_1]^f,\ldots,[x_n]^f))$$

and θ is the substitution that maps all variables to c [6]. Note that since $\mathcal{M}$ is confluent and terminating, every term u has a unique $\mathcal{M}$-normal form $u{\downarrow}_{\mathcal{M}}$. First we show $\mathsf{mark}(u\theta){\downarrow}_{\mathcal{M}} \xrightarrow{\mathsf{i}}_1^* \mathsf{active}(u\theta)$ by induction on $u \in \mathcal{T}(\mathcal{F},\mathcal{V})$. If u is a variable then $u\theta = c$ and thus $\mathsf{mark}(u\theta){\downarrow}_{\mathcal{M}} = \mathsf{active}(u\theta)$. If $u = f(u_1,\ldots,u_n)$ then $\mathsf{mark}(u\theta){\downarrow}_{\mathcal{M}} = \mathsf{active}(f(u'_1,\ldots,u'_n))$ with $u'_i = \mathsf{mark}(u_i\theta){\downarrow}_{\mathcal{M}}$ if $i \in \mu(f)$ and $u'_i = u_i\theta$ if $i \notin \mu(f)$. Let $i \in \mu(f)$. The induction hypothesis yields $u'_i \xrightarrow{\mathsf{i}}_1^* \mathsf{active}(u_i\theta)$. Since $u_i\theta$ is an $\mathcal{R}^1_\mu$-normal form, $\mathsf{active}(u_i\theta) \xrightarrow{\mathsf{i}}_1 u_i\theta$ and thus $u'_i \xrightarrow{\mathsf{i}}_1^* u_i\theta$. It follows that $\mathsf{mark}(u\theta){\downarrow}_{\mathcal{M}} \xrightarrow{\mathsf{i}}_1^* \mathsf{active}(f(u_1\theta,\ldots,u_n\theta)) = \mathsf{active}(u\theta)$.

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[4] Θ_{L} is sound for the subclass of left-linear CSRSs with the property that all function symbols in the left-hand sides are on active positions [18].

[5] The same claim is made in [18, Theorem 11]. However, Lucas only proved the soundness of Θ_1 and Θ_2 for *ground* innermost termination (cf. Section 6) and later claimed that Θ_1 and Θ_2 are unsound for innermost termination [19].

[6] It is interesting to note that the instantiated context-sensitive reduction step $s\theta \to_\mu t\theta$ need not be innermost.

Now let π be the position of the redex contracted in the reduction step $s \xrightarrow{\mathrm{i}}_\mu t$. We prove the lemma by induction on π. If $\pi = \epsilon$ then $s \to t$ and thus also $s\theta \to t\theta$ is an instance of a rule in $\mathcal{R}$. We have $\mathsf{mark}(s\theta){\downarrow}_{\mathcal{M}} \xrightarrow{\mathrm{i}}_1^* \mathsf{active}(s\theta)$ by the above observation. Moreover, $\mathsf{active}(s\theta) \xrightarrow{\mathrm{i}}_1 \mathsf{mark}(t\theta)$ since $\mathsf{active}(s\theta) \to \mathsf{mark}(t\theta)$ is an instance of a rule in $\mathcal{R}_\mu^1$. We also have $\mathsf{mark}(t\theta) \xrightarrow{\mathrm{i}}_1^* \mathsf{mark}(t\theta){\downarrow}_{\mathcal{M}}$. Combining all reductions yields $\mathsf{mark}(s\theta){\downarrow}_{\mathcal{M}} \xrightarrow{\mathrm{i}}_1^+ \mathsf{mark}(t\theta){\downarrow}_{\mathcal{M}}$.

If $\pi = i\pi'$ then $s = f(s_1, \ldots, s_i, \ldots, s_n)$ and $t = f(s_1, \ldots, t_i, \ldots, s_n)$ with $s_i \xrightarrow{\mathrm{i}}_\mu t_i$. Note that we have $i \in \mu(f)$ due to the definition of context-sensitive rewriting. For $1 \leqslant j \leqslant n$ define $s_j' = \mathsf{mark}(s_j\theta){\downarrow}_{\mathcal{M}}$ if $j \in \mu(f)$ and $s_j' = s_j\theta$ if $j \notin \mu(f)$. The induction hypothesis yields $s_i' \xrightarrow{\mathrm{i}}_1^+ \mathsf{mark}(t_i\theta){\downarrow}_{\mathcal{M}}$. The result follows since $\mathsf{mark}(s\theta){\downarrow}_{\mathcal{M}} = \mathsf{active}(f(s_1', \ldots, s_i', \ldots, s_n'))$ and $\mathsf{mark}(t\theta){\downarrow}_{\mathcal{M}} = \mathsf{active}(f(s_1', \ldots, \mathsf{mark}(t_i\theta){\downarrow}_{\mathcal{M}}, \ldots, s_n'))$. □

Not surprisingly, Θ_1 is incomplete for innermost termination.

Example 5 ([18]). Consider $(\mathcal{R}, \mu)$ with $\mathcal{R} = \{\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}), \mathsf{a} \to \mathsf{b}\}$ and $\mu(\mathsf{f}) = \{1\}$. The CSRS $(\mathcal{R}, \mu)$ is innermost terminating but $\mathcal{R}_\mu^1$

$$\mathsf{active}(\mathsf{f}(\mathsf{a})) \to \mathsf{mark}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{mark}(\mathsf{f}(x)) \to \mathsf{active}(\mathsf{f}(\mathsf{mark}(x)))$$
$$\mathsf{active}(\mathsf{a}) \to \mathsf{mark}(\mathsf{b}) \qquad \mathsf{mark}(\mathsf{a}) \to \mathsf{active}(\mathsf{a})$$
$$\mathsf{active}(x) \to x \qquad \mathsf{mark}(\mathsf{b}) \to \mathsf{active}(\mathsf{b})$$

is not an innermost terminating TRS: $\mathsf{active}(\mathsf{f}(\mathsf{a})) \xrightarrow{\mathrm{i}}_1 \mathsf{mark}(\mathsf{f}(\mathsf{a})) \xrightarrow{\mathrm{i}}_1 \mathsf{active}(\mathsf{f}(\mathsf{mark}(\mathsf{a}))) \xrightarrow{\mathrm{i}}_1 \mathsf{active}(\mathsf{f}(\mathsf{active}(\mathsf{a}))) \xrightarrow{\mathrm{i}}_1 \mathsf{active}(\mathsf{f}(\mathsf{a}))$. Note that applying the rule $\mathsf{active}(\mathsf{a}) \to \mathsf{mark}(\mathsf{b})$ instead of $\mathsf{active}(x) \to x$ in the fourth step would break the cycle. So the rule $\mathsf{active}(x) \to x$ can delete innermost redexes, causing non-innermost active redexes of the underlying CSRS to become innermost. We come back to this in Section 5.

Transformation Θ_2 is sound for innermost termination as well. However, it is also incomplete and (in contrast to Θ_1) rather useless for innermost termination. These observations are consequences of the following new result. In particular, Θ_2 cannot prove innermost termination of non-terminating CSRSs.

Theorem 6. *Let $(\mathcal{R}, \mu)$ be a CSRS. The TRS $\mathcal{R}_\mu^2$ resulting from transformation Θ_2 is innermost terminating iff it is terminating.*

Soundness of Θ_2 for innermost termination is an immediate consequence of Theorem 6 and the soundness of Θ_2 for termination.

So Θ_1 is the only sound and useful transformation for innermost termination of CSRSs so far. The next theorem shows that it is complete for an important subclass of CSRSs. More precisely, while in general termination of a CSRS $(\mathcal{R}, \mu)$ does not imply termination of the transformed TRS $\mathcal{R}_\mu^1$ (as demonstrated by Example 3), it at least implies *innermost* termination of $\mathcal{R}_\mu^1$.

Theorem 7. *Let $(\mathcal{R}, \mu)$ be a CSRS. If $(\mathcal{R}, \mu)$ is terminating then $\mathcal{R}_\mu^1$ is innermost terminating.*

Theorem 7 implies that for subclasses of CSRSs where innermost termination is equivalent to termination, Θ_1 is complete for innermost termination. In the next section we show that this subclass contains all orthogonal systems (e.g., CSRSs like Example 1).

4   Termination versus Innermost Termination

There are two motivations for studying innermost termination of CSRSs. First, innermost context-sensitive rewriting models evaluation in OBJ and thus, innermost termination analysis of CSRSs can be used for termination proofs of OBJ programs. But second, innermost termination analysis of CSRSs can also be helpful for (non-innermost) termination proofs of CSRSs. This is similar to the situation with ordinary TRSs: Proving innermost termination is much easier than proving termination, cf. [1]. There are classes of TRSs where innermost termination already implies termination and therefore for such systems, one should rather use innermost termination techniques for investigating their termination behavior.

In order to use a corresponding approach for context-sensitive rewriting, in this section we examine the connection between termination and innermost termination for CSRSs. In general, termination implies innermost termination, but not vice versa as demonstrated by Example 5. For ordinary TRSs, being *non-overlapping* suffices to ensure that innermost termination is equivalent to termination [13]. Unfortunately, as noted by Lucas [19], this criterion cannot be extended to CSRSs. However, we show the new result that the desired equivalence between innermost and full termination at least holds for *orthogonal* CSRSs. Thus, this includes all CSRSs which correspond to typical functional programs like Example 1. Theorem 9 states that for such systems we only have to prove innermost termination in order to verify their termination.

In order to prove the theorem, we need some preliminaries. For non-overlapping CSRSs $(\mathcal{R}, \mu)$ the relation $\xrightarrow{i}_{\mu}$ is confluent. Hence, for every term s there is at most one μ-normal form reachable by innermost reductions. We call this term the *innermost μ-normal form* of s and denote it by $s{\downarrow}^{i}_{\mu}$. Now for any term s, let $\nabla(s)$ be the set of those terms which result from repeatedly replacing subterms of s by their innermost μ-normal form (if it exists). Here, one may also consider subterms on inactive positions. However, the replacement must go "from the inside to the outside" (i.e., after replacing at position π one cannot replace at positions below π any more). Moreover, one may only perform replacements on such positions π where the original term $s|_{\pi}$ is terminating.

Definition 8. *Let $(\mathcal{R}, \mu)$ be a non-overlapping CSRS. For any term s we define non-empty sets $\nabla(s)$ and $\nabla'(s)$ as follows. If s is terminating, then $\nabla(s) = \nabla'(s) \cup \{u{\downarrow}^{i}_{\mu} \mid u \in \nabla'(s)$ is innermost terminating\}. Otherwise, we have $\nabla(s) = \nabla'(s)$. Moreover, $\nabla'(s) = \{f(u_1, \ldots, u_n) \mid u_i \in \nabla(s_i)\}$ if $s = f(s_1, \ldots, s_n)$ and $\nabla'(s) = \{s\}$ if s is a variable.*

Theorem 9. *An orthogonal CSRS* $(\mathcal{R}, \mu)$ *is terminating iff it is* innermost *terminating.*

Proof. The "only if" direction is trivial. We prove the "if" direction. Let $s \to_\mu t$ where the contracted redex is either terminating or a minimal non-terminating term (i.e., all proper subterms of the redex on active positions are terminating). We prove the following statements for all innermost terminating $s' \in \nabla(s)$:

(1) There exists a $t' \in \nabla(t)$ such that $s' \overset{\mathsf{i}}{\to}{}^*_\mu t'$.
(2) If the contracted redex in $s \to_\mu t$ is not terminating, then there even exists a $t' \in \nabla(t)$ such that $s' \overset{\mathsf{i}}{\to}{}^+_\mu t'$.

With (1) and (2) one can prove the theorem: If $(\mathcal{R}, \mu)$ is not terminating, then there is an infinite reduction $s_0 \to_\mu s_1 \to_\mu \ldots$ in which only terminating or minimal non-terminating redexes are contracted. Assume that $(\mathcal{R}, \mu)$ is innermost terminating. Then all $\nabla(s_i)$ contain only innermost terminating terms and since $s_0 \in \nabla(s_0)$, we can construct an infinite innermost reduction $s_0 \overset{\mathsf{i}}{\to}{}^*_\mu t_1 \overset{\mathsf{i}}{\to}{}^*_\mu t_2 \overset{\mathsf{i}}{\to}{}^*_\mu \ldots$ with $t_i \in \nabla(s_i)$. However, since the reduction contains infinitely many steps of type (2), this gives rise to an infinite innermost reduction, contradicting our assumption.

Now we prove (1) and (2) by structural induction on s. Since $s \to_\mu t$, s must have the form $f(s_1, \ldots, s_n)$. We first regard the case where $s \to_\mu t$ is not a root reduction step. Then we have $t = f(s_1, \ldots, t_i, \ldots, s_n)$ with $s_i \to_\mu t_i$ for some $i \in \mu(f)$. Let $s' \in \nabla(s)$ be innermost terminating. First, let $s' = f(u_1, \ldots, u_n)$ with $u_j \in \nabla(s_j)$ for all j. Because $i \in \mu(f)$, u_i is innermost terminating. Hence by the induction hypothesis, $u_i \in \nabla(s_i)$ implies that there exists a $v_i \in \nabla(t_i)$ such that $u_i \overset{\mathsf{i}}{\to}{}^*_\mu v_i$. Therefore, we also have $s' = f(u_1, \ldots, u_i, \ldots, u_n) \overset{\mathsf{i}}{\to}{}^*_\mu f(u_1, \ldots, v_i, \ldots, u_n) \in \nabla(t)$. Moreover, if the contracted redex in $s \to_\mu t$ and hence, in $s_i \to_\mu t_i$ is not terminating, then by the induction hypothesis we even have $u_i \overset{\mathsf{i}}{\to}{}^+_\mu v_i$ and therefore $s' \overset{\mathsf{i}}{\to}{}^+_\mu f(u_1, \ldots, v_i, \ldots, u_n) \in \nabla(t)$.

Now let $s' = f(u_1, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu$ with $u_j \in \nabla(s_j)$ for all j. Hence, s is terminating and thus, we only have to prove (1). As before, there is a $v_i \in \nabla(t_i)$ such that $u_i \overset{\mathsf{i}}{\to}{}^*_\mu v_i$ and $f(u_1, \ldots, v_i, \ldots, u_n) \in \nabla(t)$. Since innermost reduction is confluent, we have $s' = f(u_1, \ldots, u_i, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu = f(u_1, \ldots, v_i, \ldots, u_n){\downarrow}^{\mathsf{i}}_\mu \in \nabla(t)$, since t inherits termination from s.

Finally, we regard the case where $s = f(s_1, \ldots, s_n)$ and $s \to_\mu t$ is a root reduction step. Hence, there must be a rule $l \to r \in \mathcal{R}$ with $l = f(l_1, \ldots, l_n)$ and a substitution σ such that $s_i = l_i\sigma$ and $t = r\sigma$. First let $s' = f(u_1, \ldots, u_n)$ with $u_i \in \nabla(s_i)$ for all i. Since $(\mathcal{R}, \mu)$ is orthogonal and since $s_i = l_i\sigma$, there must be a substitution σ' such that $u_i = l_i\sigma'$ for all i [7]. Because s' is innermost terminating, $x\sigma'$ must also be innermost terminating for all variables x which occur on active positions of l. Let σ'' be the substitution where $x\sigma'' = x\sigma'{\downarrow}^{\mathsf{i}}_\mu$ for all x in active positions of l and $x\sigma'' = x\sigma'$ for all other x. Then we have the innermost reduction $s' = f(l_1\sigma', \ldots, l_n\sigma') \overset{\mathsf{i}}{\to}{}^*_\mu f(l_1\sigma'', \ldots, l_n\sigma'') \to_\mu r\sigma''$. We claim that $r\sigma'' \in \nabla(t) = \nabla(r\sigma)$. To this end, it suffices to show that $x\sigma'' \in \nabla(x\sigma)$

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[7] A formal proof of this observation can be found in [10].

for all variables x in r, because in the construction of ∇ arbitrary subterms q can be replaced by terms from $\nabla(q)$. Each variable x occurs in some l_i and we have $l_i\sigma' \in \nabla(l_i\sigma)$. It follows that $x\sigma' \in \nabla(x\sigma)$ for all variables x.[7] If x is on an inactive position of l, then $x\sigma'' = x\sigma' \in \nabla(x\sigma)$. If x is on an active position of l, then $x\sigma'' = x\sigma'\!\downarrow_\mu^i \in \nabla(x\sigma)$, since $x\sigma'$ is innermost terminating and because in this case, $x\sigma$ is terminating due to the fact that s is either a terminating or a *minimal* non-terminating term.

Now let $s' = f(u_1, \ldots, u_n)\!\downarrow_\mu^i$ with $u_i \in \nabla(s_i)$ for all i. Hence, s is terminating and thus we only have to prove (1). As before, $u_i = l_i\sigma'$ and $f(l_1\sigma', \ldots, l_n\sigma') \xrightarrow{i}_\mu^*$ $f(l_1\sigma'', \ldots, l_n\sigma'') \xrightarrow{i}_\mu r\sigma''$ with $r\sigma'' \in \nabla(t)$. Since innermost reduction is confluent and t inherits termination from s, $s' = f(u_1, \ldots, u_n)\!\downarrow_\mu^i = r\sigma''\!\downarrow_\mu^i \in \nabla(t)$. $\qquad\square$

Very recently, Gramlich and Lucas [14] showed that termination and innermost termination coincide for locally confluent overlay CSRSs with the additionally property that variables that occur at an active position in a left-hand side l of a rewrite rule $l \to r$ do not occur at inactive positions in l or r. The latter condition is quite restrictive, e.g., it is not satisfied by the CSRS of Example 1.

5   A Sound and Complete Transformation

In Section 3 we showed that the existing transformations are incomplete for innermost termination and that only Θ_1 and Θ_2 are sound. Because of Theorem 6, Θ_2 cannot distinguish innermost termination from termination. So when developing a sound and complete transformation for innermost termination, we take Θ_1 as starting point. As observed in Example 5, we must make sure that in innermost reductions, rules of the form $\mathsf{active}(l) \to \mathsf{mark}(r)$ get preference over the rule $\mathsf{active}(x) \to x$, because then this counterexample no longer works. Hence, we modify the rule $\mathsf{active}(x) \to x$ such that the innermost reduction strategy ensures that $\mathsf{active}(l) \to \mathsf{mark}(r)$ is applied with higher preference. In the modification, $\mathsf{active}(l) \to \mathsf{mark}(r)$ no longer overlaps with the *root position* of $\mathsf{active}(x) \to x$, but with a non-root position of the new modified rule(s).

Definition 10 (Θ_3). *Let $(\mathcal{R}, \mu)$ be a CSRS over a signature $\mathcal{F}$. The TRS $\mathcal{R}_\mu^3$ over the signature $\mathcal{F}_1$ consists of all ($\sharp$)-marked rewrite rules of $\mathcal{R}_\mu^1$ together with the rewrite rules*

$$f(x_1, \ldots, \mathsf{active}(x_i), \ldots, x_n) \to f(x_1, \ldots, x_i, \ldots, x_n)$$
$$(\flat) \qquad f(x_1, \ldots, \mathsf{mark}(x_i), \ldots, x_n) \to f(x_1, \ldots, x_i, \ldots, x_n)$$

for all $f \in \mathcal{F}$ and $1 \leqslant i \leqslant \mathrm{arity}(f)$. We denote the transformation $(\mathcal{R}, \mu) \mapsto \mathcal{R}_\mu^3$ by Θ_3 and we abbreviate $\to_{\mathcal{R}_\mu^3}$ to $\to_3$ and $\xrightarrow{i}_{\mathcal{R}_\mu^3}$ to $\xrightarrow{i}_3$.

For the CSRS $(\mathcal{R}, \mu)$ of Example 5, $\mathcal{R}_\mu^3$ differs from $\mathcal{R}_\mu^1$ in two respects: $\mathsf{active}(x) \to x$ is replaced by $\mathsf{f}(\mathsf{active}(x)) \to \mathsf{f}(x)$ and moreover, the rule $\mathsf{f}(\mathsf{mark}(x)) \to \mathsf{f}(x)$ is added. As a consequence, the cycle $\mathsf{active}(\mathsf{f}(\mathsf{a})) \xrightarrow{i}^+ \mathsf{active}(\mathsf{f}(\mathsf{a}))$ can no longer be obtained with $\mathcal{R}_\mu^3$, since $\mathsf{active}(\mathsf{f}(\mathsf{active}(\mathsf{a}))) \to \mathsf{active}(\mathsf{f}(\mathsf{a}))$ is not an *innermost* rewrite step in $\mathcal{R}_\mu^3$. Indeed, $\mathcal{R}_\mu^3$ is innermost terminating and in general, Θ_3 is sound and complete for innermost termination.

Theorem 11. *A CSRS $(\mathcal{R}, \mu)$ is innermost terminating iff $\mathcal{R}_\mu^3$ is innermost terminating.*

In [10] we show for several CSRSs $(\mathcal{R}, \mu)$ including Example 1 how innermost termination of $\mathcal{R}_\mu^3$ can be proved with *dependency pairs* [1].

With the new rules $f(x_1, \ldots, \mathsf{active}(x_i), \ldots, x_n) \to f(x_1, \ldots, x_n)$ we can remove almost every active-symbol, compensating to a large extent the lack of the rule $\mathsf{active}(x) \to x$. The $(\flat)$-marked rules can never be used in an innermost reduction if x_i is instantiated to a non-variable term from $\mathcal{T}(\mathcal{F}, \mathcal{V})$. However, they are required if x_i is instantiated by a variable or by terms containing the symbols mark and active. As a matter of fact, the transformation without the $(\flat)$-marked rules is neither sound nor complete for innermost termination (see [10] for counterexamples).

6   Ground Innermost Termination

Unlike for termination, to conclude innermost termination it is not sufficient to prove that all ground terms are innermost terminating.

Example 12. The TRS $\{f(f(x)) \to f(f(x)), f(a) \to a\}$ is not innermost terminating but ground innermost terminating over the signature $\{f, a\}$, i.e., all ground terms permit only finite innermost reductions.

It is well known that innermost termination of a TRS $\mathcal{R}$ over a signature $\mathcal{F}$ is equivalent to ground innermost termination of $\mathcal{R}$ over the signature $\mathcal{F} \cup \{c, h\}$ where c is a fresh constant and h is a fresh unary function symbol. The reason is that a term t with the variables $x_1, \ldots, x_n$ starts an infinite innermost reduction iff the ground term $t\sigma$ starts an infinite innermost reduction where $\sigma(x_i) = h^i(c)$. So the fresh symbols c and h are needed to create arbitrarily many different ground terms (in order to handle non-linear rewrite rules). A similar correspondence holds for innermost *context-sensitive* reductions with $\mu(h) = \varnothing$ or $\mu(h) = \{1\}$.

The following result states that Θ_1 and Θ_2 cannot distinguish ground innermost termination from innermost termination and thus they are sound but incomplete for ground innermost termination as well.

Theorem 13. *Let $(\mathcal{R}, \mu)$ be a CSRS. For $i \in \{1, 2\}$, the TRS $\mathcal{R}_\mu^i$ is ground innermost terminating iff it is innermost terminating.*

On the other hand, the proof of Theorem 11 can easily be adapted to show that Θ_3 is sound and complete for ground innermost termination.

Theorem 14. *A CSRS $(\mathcal{R}, \mu)$ is ground innermost terminating iff $\mathcal{R}_\mu^3$ is ground innermost terminating.*

One might think that the $(\flat)$-marked rules in Definition 10 are not needed to obtain a sound and complete transformation for ground innermost termination. While soundness is easily proved, completeness does *not* hold, as shown in the following example.

Example 15. Consider the (ground) innermost terminating CSRS $(\mathcal{R}, \mu)$ with $\mathcal{R} = \{f(x,x) \rightarrow b, g(f(x,y)) \rightarrow g(f(y,y))\}$ and $\mu(f) = \mu(g) = \{1\}$. The transformed TRS without the $(\flat)$-marked rules however is not ground innermost terminating as can be seen from the following cycle, with $t = \mathsf{mark}(\mathsf{active}(b))$:

$$\mathsf{mark}(g(f(t,t))) \xrightarrow{i}{}^{+} \mathsf{active}(g(\mathsf{active}(f(\mathsf{mark}(t),t))))$$
$$\xrightarrow{i} \mathsf{active}(g(f(\mathsf{mark}(t),t))) \xrightarrow{i} \mathsf{mark}(g(f(t,t)))$$
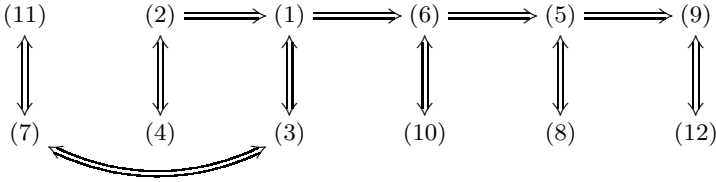
As explained above, a transformation that is sound for ground innermost termination can also be used for innermost termination analysis by adding fresh function symbols to the signature. However, for completeness the situation is different. Here, it is desirable that the transformation is not only complete for ground, but also for full innermost termination. The reason is that while there do exist techniques to analyze ground innermost termination [8], the best-known technique for automated innermost termination analysis [1] really checks full (non-ground) innermost termination of TRSs. A complete transformation for innermost termination transforms every innermost terminating CSRS into an innermost terminating TRS and hence, innermost termination of this TRS can potentially be checked by every technique for innermost termination analysis of ordinary TRSs. But if the transformed TRS is only ground innermost terminating, (full) innermost termination analysis techniques for TRSs cannot be applied successfully.

7   Conclusion and Comparison

Figure 1 contains a summary of the soundness and completeness results covered in the preceding sections. The negative results for ground innermost termination for Θ_L, Θ_Z, and Θ_{FR} are shown by the same examples used to demonstrate the corresponding results for innermost termination, cf. the first paragraph of Section 3. Of the existing transformations, only Θ_1 and Θ_2 from [9] are sound for innermost termination of CSRSs. We showed that Θ_2 is not very useful for proving innermost termination, but that termination of a CSRS $(\mathcal{R}, \mu)$ already implies innermost termination of $\Theta_1(\mathcal{R}, \mu)$. So for classes of CSRSs where termination and innermost termination are equivalent, Θ_1 is already complete for

	termination		ground innermost termination		innermost termination	
	sound	complete	sound	complete	sound	complete
Θ_L	✓	✗	✗	✗	✗	✗
Θ_Z	✓	✗	✗	✗	✗	✗
Θ_{FR}	✓	✗	✗	✗	✗	✗
Θ_1	✓	✗	✓	✗	✓	✗
Θ_2	✓	✓	✓	✗	✓	✗
Θ_3	✓	✗	✓	✓	✓	✓

Fig. 1. Summary.

Fig. 2. Comparison.

innermost termination. We proved that this equivalence holds for the class of orthogonal CSRSs. While in general Θ_1 is still incomplete, we developed a new transformation Θ_3 which is sound and complete for innermost termination. The results on termination for Θ_3 follow from Theorem 16 below.

In order to assess the power of our transformations, Figure 2 illustrates the relationship between the following twelve properties ($i = 1, 2, 3$):

$$
\begin{array}{ll}
(1) & (\mathcal{R}, \mu) \text{ is terminating} \\
(5) & (\mathcal{R}, \mu) \text{ is innermost terminating} \\
(9) & (\mathcal{R}, \mu) \text{ is ground innermost terminating} \\[4pt]
(1+i) & \mathcal{R}^i_\mu \text{ is terminating} \\
(5+i) & \mathcal{R}^i_\mu \text{ is innermost terminating} \\
(9+i) & \mathcal{R}^i_\mu \text{ is ground innermost terminating}
\end{array}
$$

Implication $(2) \Rightarrow (1)$ is the soundness of transformation Θ_1 for termination [9], implication $(1) \Rightarrow (6)$ is Theorem 7, implication $(6) \Rightarrow (5)$ is Theorem 4, and implication $(5) \Rightarrow (9)$ is trivial.

Equivalence $(1) \Leftrightarrow (3)$ is the soundness and completeness of Θ_2 for termination [9], equivalence $(3) \Leftrightarrow (7)$ is Theorem 6, equivalence $(5) \Leftrightarrow (8)$ is Theorem 11, equivalences $(10) \Leftrightarrow (6)$ and $(11) \Leftrightarrow (7)$ are Theorem 13, and equivalence $(9) \Leftrightarrow (12)$ is Theorem 14. Equivalence $(2) \Leftrightarrow (4)$ means that Θ_1 and Θ_3 are equally powerful when proving termination. This may not come as a surprise, but the proof is surprisingly difficult.

Theorem 16. *Let $(\mathcal{R}, \mu)$ be a CSRS. The TRS $\mathcal{R}^3_\mu$ is terminating iff $\mathcal{R}^1_\mu$ is terminating.*

None of the missing implications in Figure 2 hold, except those that follow by transitivity: $(1) \not\Rightarrow (2)$ and $(5) \not\Rightarrow (6)$ are the incompleteness of Θ_1 for termination (Example 3) and innermost termination (Example 5). Moreover, $(6) \not\Rightarrow (1)$ follows by using $\mu(f) = \{1, 2, 3\}$ in Example 3 and $(9) \not\Rightarrow (5)$ follows from Example 12 with $\mu(f) = \{1\}$.

To conclude, with our new transformation Θ_3, innermost termination of context-sensitive rewriting can be reduced to innermost termination of ordinary rewriting. Moreover, for orthogonal CSRSs innermost termination already suffices for termination. So for such systems, innermost termination of the transformed TRS even implies termination of the CSRS. Hence, our result now enables the use of powerful methods for *innermost* termination analysis of TRSs for (innermost) termination of context-sensitive rewriting.

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