

# On the Domain and Dimension Hierarchy of Matrix Interpretations

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**Abstract.** Matrix interpretations are a powerful technique for proving termination of term rewrite systems. Depending on the underlying domain of interpretation, one distinguishes between matrix interpretations over the real, rational and natural numbers. In this paper we clarify the relationship between all three variants, showing that matrix interpretations over the reals are more powerful than matrix interpretations over the rationals, which are in turn more powerful than matrix interpretations over the natural numbers. We also clarify the ramifications of matrix dimension on termination proving power. To this end, we establish a hierarchy of matrix interpretations with respect to matrix dimension and show it to be infinite, with each level properly subsuming its predecessor.

**Keywords:** term rewriting, termination, matrix interpretations

## 1 Introduction

Since their inception in 2006, matrix interpretations have evolved into one of the most important (that is, powerful) methods for termination analysis and complexity analysis of term rewrite systems. While originally introduced by Hofbauer and Waldmann as a stand-alone method for termination proofs in the context of string rewriting [13, 14], allowing them to solve challenging termination problems like  $\{\mathbf{aa} \rightarrow \mathbf{bc}, \mathbf{bb} \rightarrow \mathbf{ac}, \mathbf{cc} \rightarrow \mathbf{ab}\}$ , problem #104 on the RTA list of open problems,<sup>1</sup> it was not long until Endrullis *et al.* [6] generalized (one particular instance of) the matrix method to term rewriting and also incorporated it into the dependency pair (DP) framework [3, 9–11, 23], the state-of-the-art framework for establishing termination of term rewrite systems.

The matrix method is based on the well-known paradigm of interpreting terms into a domain equipped with a suitable well-founded order. In the original approach of [6], the authors consider the set of vectors of natural numbers as underlying domain, together with a well-founded order that relates two vectors if and only if there is a strict decrease in the respective first components and a weak decrease in all other components. Function symbols are interpreted by suitable linear mappings represented by square matrices of natural numbers. Recently, another generalization appeared in [5] that employs matrices of natural

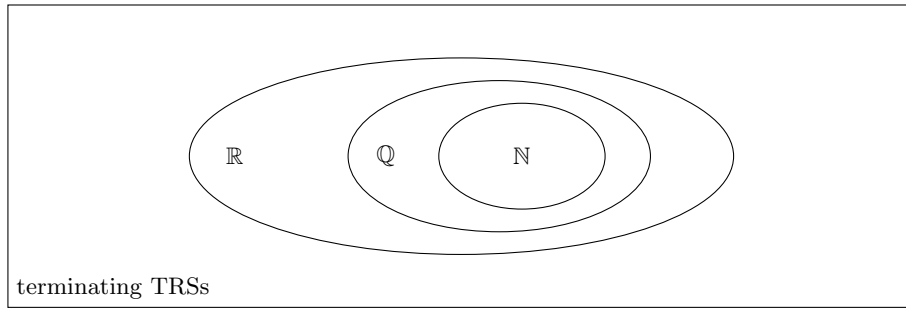
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<sup>1</sup> <http://rtaloop.mancoosi.univ-paris-diderot.fr>.

numbers as underlying domain and interprets each function symbol by a linear matrix polynomial. In principle, this approach also allows for non-linear matrix polynomials. In [1, 7, 24] the method of Endrullis *et al.* was lifted to the non-negative rational and real (algebraic) numbers using the same technique that was already used to lift polynomial interpretations from the natural numbers to the rationals and reals (cf. [12]). Thus, one distinguishes three variants of matrix interpretations, matrix interpretations over the real, rational and natural numbers. So the obvious question is: what is their relationship with regard to termination proving power?

As a starting point, it is instructive to restrict to one-dimensional matrix interpretations, that is, *linear* polynomial interpretations, for which the termination hierarchy is known (cf. [16, 18]) and can be pictured as in Figure 1. That



**Fig. 1.** Linear polynomial interpretations

is, linear polynomial interpretations over the real numbers subsume linear polynomial interpretations over the rational numbers, which in turn subsume linear polynomial interpretations over the natural numbers. Both inclusions are proper. To this end, [16] introduces the rewrite systems  $\mathcal{R}_{\mathbb{Q}}$  and  $\mathcal{R}_{\mathbb{R}}$ , the first of which can be shown terminating by a linear polynomial interpretation over the rational numbers but not over the natural numbers. Similarly, the second system can be shown terminating by a linear polynomial interpretation over the reals but not over the rationals. Unfortunately, the usefulness of both  $\mathcal{R}_{\mathbb{Q}}$  and  $\mathcal{R}_{\mathbb{R}}$  is limited to dimension one (cf. [17]) because, without restricting the dimension, both systems can be handled with 2-dimensional matrix interpretations over the natural numbers. In this context, we also mention related work appearing in [8], where a relative termination problem in the form of a string rewrite system is presented that can be handled with matrix interpretations over the rationals but not with matrix interpretations over the natural numbers. However, *relative* termination is essential in this example because the relative component is the key ingredient for precluding matrix interpretations over the natural numbers. As the latter component consists of a single non-terminating rule, the entire example does not readily generalize to (real) termination problems. Besides, there is no ev-

idence in [8] demonstrating the benefit of using irrational numbers in matrix interpretations. Thus, we conclude that new techniques are required to clarify the relationship between the aforementioned variants of matrix interpretations.

One of the main results of this paper is to show that the termination hierarchy depicted in Figure 1 does in fact extend from one-dimensional matrix interpretations to arbitrary matrix interpretations. That is, matrix interpretations over the reals are more powerful with respect to proving termination than matrix interpretations over the rationals, which are in turn more powerful than matrix interpretations over the natural numbers. In particular, we show that this relationship does not only hold in the context of direct termination (using matrix interpretations as a stand-alone method) but also in the setting of the DP framework. Moreover, our results point out the limitations of a recent attempt [17] to simulate matrix interpretations over the rationals with matrix interpretations over the natural numbers (of higher dimension).

We also investigate the ramifications of matrix dimension on termination proving power. Clearly, by increasing the dimension, one can never lose power (in theory; in practice the increased search space may prohibit finding a termination proof). But what is the exact shape of the inherent dimension hierarchy? A partial answer to this question was given in [8], where the authors show that the hierarchy is infinite. Yet no exact information is provided as to which levels are actually inhabited. We close this gap in the second part of this paper, thus giving a complete answer to the question raised above. To this end, we establish a hierarchy of matrix interpretations with respect to matrix dimension and show it to be infinite, with each level properly subsuming its predecessor. In other words, we show that matrix interpretations of dimension  $(n+1)$  are strictly more powerful for proving termination than  $n$ -dimensional matrix interpretations (for any  $n \geq 1$ ). The construction we use for this purpose is entirely different from the one proposed in [8]. Apart from the fact that it allows to infer the exact shape of the dimension hierarchy, it has the additional advantage that it produces witnesses (that is, rewrite systems) that are substantially smaller than the ones of [8]. To be precise, the construction employed in [8] gives rise to a family of string rewrite systems  $(\mathcal{S}_d)_{d \geq 2}$  having the property that any of its members  $\mathcal{S}_{2d}$  (of even index) cannot be handled with matrix interpretations of dimension  $d$  or less (as a consequence of the Amitsur-Levitzki theorem [2]), but can be handled with dimension  $d' = 2d + 3$ . Each system  $\mathcal{S}_d$  consists of the following rules over the finite alphabet  $\Sigma_d = \{\mathbf{s}, \mathbf{1}, \dots, \mathbf{d}, \mathbf{f}\}$ :  $\mathbf{s}e_k\mathbf{f} \rightarrow \mathbf{s}o_k\mathbf{f}$  for all  $1 \leq k \leq \frac{d!}{2}$ . Here,  $e_1, e_2, \dots$  ( $o_1, o_2, \dots$ ) is any enumeration of even (odd)<sup>2</sup> permutations of the symbols  $\{\mathbf{1}, \dots, \mathbf{d}\}$ . Hence, the number of rewrite rules in  $\mathcal{S}_d$  exhibits factorial growth in the dimension  $d$ . In contrast, the systems created by our approach have constant size and the dimension  $d'$  is *optimal*, i.e.,  $d' = d + 1$ .

The remainder of this paper is organized as follows. In the next section we recall preliminaries from linear algebra and term rewriting. In particular, we review the matrix method for establishing termination of term rewrite systems.

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<sup>2</sup> A permutation is called even (odd) if it can be written as a composition of an even (odd) number of transpositions.

Then, in Section 3, we show that matrix interpretations over the reals are more powerful than matrix interpretations over the rationals, which are in turn more powerful than matrix interpretations over the natural numbers. Subsequently, we present our results on the dimension hierarchy related to matrix interpretations in Section 4, before concluding with suggestions for future research in Section 5.

## 2 Preliminaries

As usual, we denote by  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  the sets of natural, integer, rational and real numbers. A real number is said to be *algebraic* if it is a root of a non-zero polynomial in one indeterminate with integer coefficients, otherwise it is said to be *transcendental*. The set of all real algebraic numbers is denoted by  $\mathbb{R}_{\text{alg}}$ . Given  $D \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\text{alg}}, \mathbb{R}\}$  and  $m \in D$ ,  $>_D$  (resp.  $>$  if  $D$  is clear from the context) denotes the natural order of the respective domain,  $\geq_D$  (resp.  $\geq$ ) its reflexive closure, and  $D_m$  abbreviates  $\{x \in D \mid x \geq m\}$ ; for example,  $\mathbb{Q}_0$  ( $\mathbb{R}_0$ ) refers to the set of all non-negative rational (real) numbers.

### 2.1 Linear Algebra

Let  $R$  be a commutative ring (e.g.,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}_{\text{alg}}$ ,  $\mathbb{R}$ ). The ring of all  $n$ -dimensional square matrices over  $R$  is denoted by  $R^{n \times n}$  and the *polynomial ring in  $n$  indeterminates*  $x_1, \dots, x_n$  by  $R[x_1, \dots, x_n]$ . In the special case  $n = 1$ , any polynomial  $p \in R[x]$  can be written as  $p(x) = \sum_{k=0}^d a_k x^k$  for some  $d \in \mathbb{N}$ . For the largest  $k$  such that  $a_k \neq 0$ , we call  $a_k x^k$  the *leading term* of  $p$ ,  $a_k$  its *leading coefficient* and  $k$  its *degree*. The polynomial  $p$  is said to be *monic* if its leading coefficient is one. It is said to be *linear*, *quadratic*, *cubic* if its degree is one, two, three.

In case  $R$  is equipped with a partial order  $\geq$ , the component-wise extension of this order to  $R^{n \times n}$  is also denoted by  $\geq$ . The  $n \times n$  *identity matrix* is denoted by  $I_n$  and the  $n \times n$  *zero matrix* by  $0_n$ . We simply write  $I$  and  $0$  if  $n$  is clear from the context. We say that a matrix  $A$  is *non-negative* if  $A \geq 0$  and denote the set of all non-negative  $n$ -dimensional square matrices of  $\mathbb{Z}^{n \times n}$  by  $\mathbb{N}^{n \times n}$ . As usual, we write  $A^T$  for the *transpose* of a matrix (vector)  $A$ .

For a square matrix  $A \in R^{n \times n}$ , the *characteristic polynomial*  $\chi_A(\lambda)$  is defined as  $\det(\lambda I_n - A)$ , where  $\det$  denotes the (matrix) determinant. It is a monic polynomial of degree  $n$  with coefficients in  $R$ . The equation  $\chi_A(\lambda) = 0$  is called the *characteristic equation* of  $A$ . The solutions of this equation, that is, the *roots* of  $\chi_A(\lambda)$ , are precisely the *eigenvalues* of  $A$ . If  $R$  is a subset of an algebraically closed field (where each polynomial of degree  $n$  with coefficients in the field is guaranteed to have exactly  $n$  roots), then  $A$  has exactly  $n$  (not necessarily distinct) eigenvalues in this field.

We say that a polynomial  $p \in R[x]$  *annihilates* a square matrix  $A \in R^{n \times n}$  if  $p(A) = 0$ . The Cayley-Hamilton theorem [21] states that  $A$  satisfies its own characteristic equation, that is,  $\chi_A$  annihilates  $A$ . Let  $R$  be a field and consider the set  $\{p \in R[x] \mid p(A) = 0\}$  of annihilating polynomials of  $A \in R^{n \times n}$ . This set is generated by the *minimal polynomial*  $m_A(x)$  of  $A$ , which is the unique

monic polynomial of minimum degree that annihilates  $A$ . Any polynomial that annihilates  $A$  is a (polynomial) multiple of  $m_A(x)$ . In other words, if  $p(A) = 0$  for  $p \in R[x]$ , then  $m_A(x)$  divides  $p(x)$ . In particular,  $m_A(x)$  divides the characteristic polynomial of  $A$ , and  $m_A(\lambda) = 0$  if and only if  $\lambda$  is an eigenvalue of  $A$  (cf. [15]).

## 2.2 Term Rewriting

We assume familiarity with the basics of term rewriting [4, 22]. Let  $\mathcal{V}$  denote a countably infinite set of variables and  $\mathcal{F}$  a *signature*, that is, a set of function symbols equipped with fixed arities. The set of *terms* over  $\mathcal{F}$  and  $\mathcal{V}$  is denoted by  $\mathcal{T}(\mathcal{F}, \mathcal{V})$ . A *rewrite rule* is a pair of terms written as  $\ell \rightarrow r$  such that  $\ell$  is not a variable and all variables of  $r$  are contained in  $\ell$ . A *term rewrite system* (TRS for short)  $\mathcal{R}$  over  $\mathcal{T}(\mathcal{F}, \mathcal{V})$  is a finite set of rewrite rules. The rewrite relation induced by  $\rightarrow$  is denoted by  $\rightarrow_{\mathcal{R}}$ . As usual,  $\rightarrow_{\mathcal{R}}^*$  denotes the reflexive transitive closure of  $\rightarrow_{\mathcal{R}}$ .

## 2.3 Monotone Algebras and Matrix Interpretations

We use the following notation for monotone algebras [6]. An  $\mathcal{F}$ -*algebra*  $\mathcal{A}$  consists of a non-empty *carrier* set  $A$  and a collection of *interpretation functions*  $f_{\mathcal{A}}: A^k \rightarrow A$  for each  $k$ -ary function symbol  $f \in \mathcal{F}$ . By  $[\alpha]_{\mathcal{A}}(\cdot)$  we denote the usual evaluation function of  $\mathcal{A}$  with respect to a variable assignment  $\alpha: \mathcal{V} \rightarrow A$ . A *weakly monotone  $\mathcal{F}$ -algebra*  $(\mathcal{A}, >, \geq)$  is an  $\mathcal{F}$ -algebra  $\mathcal{A}$  together with two binary relations  $>$  and  $\geq$  on  $A$  such that  $>$  is well-founded,  $> \cdot \geq \subseteq >$  and for each  $f \in \mathcal{F}$ ,  $f_{\mathcal{A}}$  is monotone with respect to  $\geq$  (in all arguments). If, in addition, each  $f_{\mathcal{A}}$  is monotone with respect to  $>$ , then we speak of an *extended monotone algebra*. Any monotone algebra  $(\mathcal{A}, >, \geq)$  (or just  $\mathcal{A}$  if  $>$  and  $\geq$  are clear from the context) induces the following relations on  $\mathcal{T}(\mathcal{F}, \mathcal{V})$ :

- $s >_{\mathcal{A}} t$  if and only if  $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$  for all assignments  $\alpha$ , and
- $s \geq_{\mathcal{A}} t$  if and only if  $[\alpha]_{\mathcal{A}}(s) \geq [\alpha]_{\mathcal{A}}(t)$  for all assignments  $\alpha$ .

We say that a monotone algebra  $\mathcal{A}$  is *compatible* with a rewrite rule  $\ell \rightarrow r$  if  $\ell >_{\mathcal{A}} r$ , it is said to be *weakly compatible* if  $\ell \geq_{\mathcal{A}} r$ . In the same vein, we say that  $\mathcal{A}$  is (weakly) compatible with a TRS  $\mathcal{R}$  if it is (weakly) compatible with all rewrite rules of  $\mathcal{R}$ . We use the following abbreviations:  $\mathcal{R} \subseteq >_{\mathcal{A}}$  for compatibility and  $\mathcal{R} \subseteq \geq_{\mathcal{A}}$  for weak compatibility.

It is well-known that a TRS is terminating if and only if there is an extended monotone algebra that is compatible with it (cf. [6, Theorem 2]). Moreover, extended monotone algebras facilitate *incremental termination proofs* (cf. [6, Theorem 3]). To this end, let  $\mathcal{A}$  be an extended monotone algebra and suppose  $\mathcal{R}$  is a TRS such that  $\mathcal{R} \subseteq \geq_{\mathcal{A}}$  and  $\mathcal{S} \subseteq >_{\mathcal{A}}$  for some non-empty subset  $\mathcal{S}$  of  $\mathcal{R}$ . Then, after removing all  $\mathcal{S}$ -rules from  $\mathcal{R}$ , termination of  $\mathcal{R} \setminus \mathcal{S}$  implies termination of  $\mathcal{R}$ . Thus, one is free to choose a different extended monotone algebra for the remaining rules  $\mathcal{R} \setminus \mathcal{S}$ . This process is continued until eventually all rewrite rules have been removed.

Weakly monotone algebras play an important role in the context of termination analysis in the DP framework. In this modular framework, the problem of establishing termination of a TRS is typically split into several subproblems called *DP problems*. A DP problem is a pair  $(\mathcal{P}, \mathcal{S})$ , where  $\mathcal{P}$  and  $\mathcal{S}$  are finite sets of rewrite rules such that the root symbols of the rules in  $\mathcal{P}$  neither occur in  $\mathcal{S}$  nor in proper subterms of the left- and right-hand sides of the rules in  $\mathcal{P}$ . In the sequel, we sometimes write  $(-, \mathcal{S})$  to indicate that we are only interested in the second component of a DP problem. A *DP processor* is a mapping that takes a DP problem as input and returns a set of DP problems as output. In the context of this paper, we only consider DP processors based on *reduction pairs*. Given a DP problem  $(\mathcal{P}, \mathcal{S})$ , the aim of such a processor is to return a simplified version of its input by removing rules from the  $\mathcal{P}$  component. It is well-known that weakly monotone algebras give rise to reduction pairs. One can use them to simplify DP problems as follows. Let  $\mathcal{A}$  be a weakly monotone algebra and  $(\mathcal{P}, \mathcal{S})$  a DP problem. If  $\mathcal{P} \cup \mathcal{S} \subseteq \geq_{\mathcal{A}}$  and  $\mathcal{P}' \subseteq >_{\mathcal{A}}$  for some non-empty subset  $\mathcal{P}'$  of  $\mathcal{P}$ , then one may remove all rules of  $\mathcal{P}'$  from  $\mathcal{P}$ , thus simplifying the original DP problem to the DP problem  $(\mathcal{P} \setminus \mathcal{P}', \mathcal{S})$  containing less rules. In this situation, we say that the weakly monotone algebra  $\mathcal{A}$  *succeeds on the DP problem*  $(\mathcal{P}, \mathcal{S})$ , otherwise it *fails*.

We define matrix interpretations as follows. For *matrix interpretations over*  $\mathbb{R}$ , we fix a dimension  $n \in \mathbb{N} \setminus \{0\}$ , some positive real number  $\delta$  and use the set  $\mathbb{R}_0^n$  as the carrier of an algebra  $\mathcal{M}$ , together with the orders  $>_{\delta}$  and  $\geq$  on  $\mathbb{R}_0^n$ :

$$\begin{aligned} (x_1, \dots, x_n)^{\text{T}} >_{\delta} (y_1, \dots, y_n)^{\text{T}} &\iff x_1 >_{\mathbb{R}, \delta} y_1 \wedge x_i \geq_{\mathbb{R}} y_i \text{ for } i = 2, \dots, n \\ (x_1, \dots, x_n)^{\text{T}} \geq (y_1, \dots, y_n)^{\text{T}} &\iff x_i \geq_{\mathbb{R}} y_i \text{ for } i = 1, \dots, n \end{aligned}$$

Here,  $x >_{\mathbb{R}, \delta} y$  if and only if  $x \geq_{\mathbb{R}} y + \delta$ . Each  $k$ -ary function symbol  $f$  is interpreted by a linear function of the shape

$$f_{\mathcal{M}}: (\mathbb{R}_0^n)^k \rightarrow \mathbb{R}_0^n, (\mathbf{x}_1, \dots, \mathbf{x}_k) \mapsto F_1 \mathbf{x}_1 + \dots + F_k \mathbf{x}_k + \mathbf{f}$$

where  $\mathbf{x}_1, \dots, \mathbf{x}_k$  are (column) vectors of variables,  $F_1, \dots, F_k \in \mathbb{R}_0^{n \times n}$  and  $\mathbf{f} \in \mathbb{R}_0^n$ . In this way,  $(\mathcal{M}, >_{\delta}, \geq)$  forms a weakly monotone algebra. If, in addition, the top left entry  $(F_i)_{11}$  of each matrix  $F_i$  is at least one, then we call  $\mathcal{M}$  a *monotone matrix interpretation over*  $\mathbb{R}$ , in which case  $(\mathcal{M}, >_{\delta}, \geq)$  becomes an extended monotone algebra. Note that in any case we have  $>_{\mathcal{M}} \subseteq \geq_{\mathcal{M}}$  since  $>_{\delta} \subseteq \geq$  (independently of  $\delta$ ).

We obtain matrix interpretations over  $\mathbb{R}_{\text{alg}}$  by restricting the carrier to the set of vectors of non-negative real algebraic numbers. Similarly, matrix interpretations over  $\mathbb{Q}$  operate on the carrier  $\mathbb{Q}_0^n$ . For matrix interpretations over  $\mathbb{N}$ , one uses the carrier  $\mathbb{N}^n$  and  $\delta = 1$ , such that

$$(x_1, \dots, x_n)^{\text{T}} >_{\delta} (y_1, \dots, y_n)^{\text{T}} \iff x_1 >_{\mathbb{N}} y_1 \wedge x_i \geq_{\mathbb{N}} y_i \text{ for } i = 2, \dots, n$$

According to [20], matrix interpretations over  $\mathbb{R}$  are equivalent to matrix interpretations over  $\mathbb{R}_{\text{alg}}$  with respect to proving termination. So transcendental numbers are not relevant for termination proofs based on matrix interpretations. Nevertheless, for the sake of brevity of notation, we will stick to the term “matrix interpretations over the real numbers” for the rest of this paper.

### 3 The Domain Hierarchy

In this section we show that matrix interpretations over the real numbers are more powerful with respect to proving termination than matrix interpretations over the rational numbers, which are in turn more powerful than matrix interpretations over the natural numbers. To begin with, we show that matrix interpretations over  $\mathbb{R}$  subsume matrix interpretations over  $\mathbb{Q}$ , which in turn subsume matrix interpretations over  $\mathbb{N}$ . Then, in Sections 3.1 and 3.2, both inclusions are proved to be proper.

**Lemma 1.** *Let  $\mathcal{M}$  be an  $n$ -dimensional matrix interpretation over  $\mathbb{N}$  (not necessarily monotone), and let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be finite sets of rewrite rules such that  $\mathcal{S}_1 \subseteq >_{\mathcal{M}}$  and  $\mathcal{S}_2 \subseteq \geq_{\mathcal{M}}$ . Then there exists an  $n$ -dimensional matrix interpretation  $\mathcal{N}$  over  $\mathbb{Q}$  such that  $\mathcal{S}_1 \subseteq >_{\mathcal{N}}$  and  $\mathcal{S}_2 \subseteq \geq_{\mathcal{N}}$ . Moreover,  $\mathcal{N}$  is monotone if and only if  $\mathcal{M}$  is monotone.*

*Proof.* Let  $\mathcal{F}$  denote the signature associated with  $\mathcal{S}_1 \cup \mathcal{S}_2$ . Then, by assumption,  $\mathcal{M}$  associates each  $k$ -ary function symbol  $f \in \mathcal{F}$  with a linear function  $f_{\mathcal{M}}(\mathbf{x}_1, \dots, \mathbf{x}_k) = F_1 \mathbf{x}_1 + \dots + F_k \mathbf{x}_k + \mathbf{f}$ , where  $F_1, \dots, F_k \in \mathbb{N}^{n \times n}$  and  $\mathbf{f} \in \mathbb{N}^n$ , such that  $\mathcal{S}_1 \subseteq >_{\mathcal{M}}$  and  $\mathcal{S}_2 \subseteq \geq_{\mathcal{M}}$ . Based on this interpretation, we define the matrix interpretation  $\mathcal{N}$  by letting  $\delta = 1$  and taking the same interpretation functions, i.e.,  $f_{\mathcal{N}}(\mathbf{x}_1, \dots, \mathbf{x}_k) = f_{\mathcal{M}}(\mathbf{x}_1, \dots, \mathbf{x}_k)$  for all  $f \in \mathcal{F}$ . Then  $\mathcal{N}$  is well-defined, and it is monotone if and only if  $\mathcal{M}$  is monotone.

As to compatibility of  $\mathcal{N}$  with  $\mathcal{S}_1$ , let us consider an arbitrary rewrite rule  $\ell \rightarrow r \in \mathcal{S}_1$  and show that  $\ell >_{\mathcal{M}} r$  implies  $\ell >_{\mathcal{N}} r$ , i.e.,  $[\alpha]_{\mathcal{N}}(\ell) >_{\delta} [\alpha]_{\mathcal{N}}(r)$  for all variable assignments  $\alpha$ . Because of linearity of the interpretation functions, we can write  $[\alpha]_{\mathcal{N}}(\ell) = L_1 \mathbf{x}_1 + \dots + L_m \mathbf{x}_m + \ell$  and  $[\alpha]_{\mathcal{N}}(r) = R_1 \mathbf{x}_1 + \dots + R_m \mathbf{x}_m + \mathbf{r}$ , where  $x_1, \dots, x_m$  are the variables occurring in  $\ell, r$  and  $\mathbf{x}_i = \alpha(x_i)$  for  $i = 1, \dots, m$ . Thus, it remains to show that the inequality

$$L_1 \mathbf{x}_1 + \dots + L_m \mathbf{x}_m + \ell >_{\delta} R_1 \mathbf{x}_1 + \dots + R_m \mathbf{x}_m + \mathbf{r}$$

holds for all  $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{Q}_0^n$ . This is exactly the case if  $L_i \geq R_i$  for  $i = 1, \dots, m$  and  $\ell >_{\delta} \mathbf{r}$ , i.e.,  $\ell_i \geq r_i$  for  $i = 2, \dots, n$  and  $\ell_1 \geq r_1 + \delta = r_1 + 1$ . Indeed, all these conditions follow from compatibility of  $\mathcal{M}$  with  $\ell \rightarrow r$  because, by the same reasoning as above (and since the interpretation functions of  $\mathcal{M}$  and  $\mathcal{N}$  coincide),  $\ell >_{\mathcal{M}} r$  holds in  $(\mathcal{M}, >, \geq)$  if and only if

$$L_1 \mathbf{x}_1 + \dots + L_m \mathbf{x}_m + \ell > R_1 \mathbf{x}_1 + \dots + R_m \mathbf{x}_m + \mathbf{r}$$

holds for all  $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{N}^n$ , which implies  $L_i \geq R_i$  for  $i = 1, \dots, m$  and  $\ell > \mathbf{r}$ , i.e.,  $\ell_i \geq r_i$  for  $i = 2, \dots, n$  and  $\ell_1 >_{\mathbb{N}} r_1$ , the latter being equivalent to  $\ell_1 \geq r_1 + 1$  as  $\ell, \mathbf{r} \in \mathbb{N}^n$ . This shows compatibility of  $\mathcal{N}$  with  $\mathcal{S}_1$ . Weak compatibility with  $\mathcal{S}_2$  follows in the same way.  $\square$

The essence of the proof of this lemma is that any matrix interpretation over  $\mathbb{N}$  can be conceived as a matrix interpretation over  $\mathbb{Q}$ . Likewise, any matrix interpretation over  $\mathbb{Q}$  can be conceived as a matrix interpretation over  $\mathbb{R}$ .

**Lemma 2.** *Let  $\mathcal{M}$  be an  $n$ -dimensional matrix interpretation over  $\mathbb{Q}$  (not necessarily monotone), and let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be finite sets of rewrite rules such that  $\mathcal{S}_1 \subseteq >_{\mathcal{M}}$  and  $\mathcal{S}_2 \subseteq \geq_{\mathcal{M}}$ . Then there exists an  $n$ -dimensional matrix interpretation  $\mathcal{N}$  over  $\mathbb{R}$  such that  $\mathcal{S}_1 \subseteq >_{\mathcal{N}}$  and  $\mathcal{S}_2 \subseteq \geq_{\mathcal{N}}$ . Moreover,  $\mathcal{N}$  is monotone if and only if  $\mathcal{M}$  is monotone.*

*Proof.* Similar to the proof of Lemma 1, with  $\mathcal{N}$  defined as follows:  $\delta_{\mathcal{N}} = \delta_{\mathcal{M}} = \delta$  and  $f_{\mathcal{N}}(\mathbf{x}_1, \dots, \mathbf{x}_k) = f_{\mathcal{M}}(\mathbf{x}_1, \dots, \mathbf{x}_k)$  for all  $f \in \mathcal{F}$ .  $\square$

As an immediate consequence of the previous lemmata, we obtain the following corollary stating that matrix interpretations over  $\mathbb{N}$  are no more powerful than matrix interpretations over  $\mathbb{Q}$ , which are in turn no more powerful than matrix interpretations over  $\mathbb{R}$ .

**Corollary 3.** *Let  $\mathcal{R}$  be a TRS and  $(\mathcal{P}, \mathcal{S})$  a DP problem.*

1. *If there is an (incremental) termination proof for  $\mathcal{R}$  using monotone matrix interpretations over  $\mathbb{N}$  (resp.  $\mathbb{Q}$ ), then there is also one using monotone matrix interpretations over  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ).*
2. *If a matrix interpretation over  $\mathbb{N}$  (resp.  $\mathbb{Q}$ ) succeeds on  $(\mathcal{P}, \mathcal{S})$ , then there is also a matrix interpretation over  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ) of the same dimension that succeeds on  $(\mathcal{P}, \mathcal{S})$ .*  $\square$

In the remainder of this section we show that the converse statements do not hold.

### 3.1 Matrix Interpretations over $\mathbb{Q}$

In order to show that matrix interpretations over  $\mathbb{Q}$  are indeed more powerful than matrix interpretations over  $\mathbb{N}$ , let us first consider the TRS  $\mathcal{S}$  consisting of the following rewrite rules:

$$x + \mathbf{a} \rightarrow x \tag{1}$$

$$x + \mathbf{a} \rightarrow (x + \mathbf{b}) + \mathbf{b} \tag{2}$$

$$\mathbf{a} + x \rightarrow x \tag{3}$$

$$\mathbf{a} + x \rightarrow \mathbf{b} + (\mathbf{b} + x) \tag{4}$$

This TRS will turn out to be very helpful for our purposes, not only in the current subsection but also in the subsequent one. This is due to the following property, which holds for matrix interpretations over  $\mathbb{N}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$ .

**Lemma 4.** *Let  $\mathcal{M}$  be a matrix interpretation (not necessarily monotone) with carrier set  $M$  such that  $\mathcal{S} \subseteq \geq_{\mathcal{M}}$ . Then  $+_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{v}$ ,  $\mathbf{v} \in M$ .*

*Proof.* Without loss of generality, let  $+_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = A_1\mathbf{x} + A_2\mathbf{y} + \mathbf{v}$ ,  $\mathbf{v} \in M$ . As  $\mathcal{M}$  is weakly compatible with rule (1), we obtain  $A_1 \geq I$ ; hence,  $A_1^2 \geq A_1$  due to non-negativity of  $A_1$ . Similarly, by weak compatibility with (2), we infer



$A_1 \geq A_1^2$ , which implies  $A_1^2 = A_1 \geq I$  together with the previous result. Yet this means that  $A_1$  must in fact be equal to  $I$ . To this end, we observe that  $A_1 \geq I$  implies  $(A_1 - I)^2 \geq 0$ , which simplifies to  $I \geq 2A_1 - A_1^2 = A_1$ ; hence,  $A_1 = I$ . In the same way, we obtain  $A_2 = I$  from the compatibility constraints associated with (3) and (4).  $\square$

So in any matrix interpretation that is weakly compatible with the TRS  $\mathcal{S}$  the symbol  $+$  must be interpreted by a function  $+_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{v}$  that models addition of two elements of the underlying carrier set (modulo adding a constant). The inherent possibility to count objects can be exploited to show that matrix interpretations over  $\mathbb{Q}$  are indeed more powerful than matrix interpretations over  $\mathbb{N}$ . To this end, we extend the TRS  $\mathcal{S}$  with the rules (5) and (6), calling the resulting system  $\mathcal{R}_1$ :

$$((x + x) + x) + \mathbf{a} \rightarrow \mathbf{g}(x + x) \quad (5)$$

$$\mathbf{g}(x + x) \rightarrow (x + x) + x \quad (6)$$

By construction, this TRS is not compatible, not even weakly compatible, with any matrix interpretation over  $\mathbb{N}$ .

**Lemma 5.** *Let  $\mathcal{M}$  be an  $n$ -dimensional matrix interpretation (not necessarily monotone) with carrier set  $M$  such that  $\mathcal{R}_1 \subseteq \geq_{\mathcal{M}}$ . Then  $M \neq \mathbb{N}^n$ .*

*Proof.* As  $\mathcal{M}$  is weakly compatible with  $\mathcal{R}_1$ , it is also weakly compatible with the TRS  $\mathcal{S}$ . So, by Lemma 4, the function symbol  $+$  must be interpreted by  $+_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{v}$ ,  $\mathbf{v} \in M$ . Assuming  $\mathbf{g}_{\mathcal{M}}(\mathbf{x}) = G\mathbf{x} + \mathbf{g}$  without loss of generality, we obtain  $3I \geq 2G$  from weak compatibility of  $\mathcal{M}$  with (5) and  $2G \geq 3I$  from weak compatibility with (6); hence,  $G = \frac{3}{2}I \notin \mathbb{N}^{n \times n}$ . Therefore,  $\mathcal{M}$  cannot be a matrix interpretation over  $\mathbb{N}$ .  $\square$

The previous lemma, together with the observation that the TRS  $\mathcal{R}_1$  admits a compatible matrix interpretation over  $\mathbb{Q}$ , directly leads to the main result of this subsection.

**Theorem 6.**

1. *The TRS  $\mathcal{R}_1$  is terminating. In particular,  $\mathcal{R}_1$  is compatible with a monotone matrix interpretation over  $\mathbb{Q}$ .*
2. *There cannot be an (incremental) termination proof of  $\mathcal{R}_1$  using only monotone matrix interpretations over  $\mathbb{N}$ .*
3. *No matrix interpretation over  $\mathbb{N}$  succeeds on the DP problem  $(-, \mathcal{R}_1)$ .*

*Proof.* The last two statements are immediate consequences of Lemma 5. As to the first claim, the following monotone one-dimensional matrix interpretation (i.e., linear polynomial interpretation) over  $\mathbb{Q}$  is compatible with  $\mathcal{R}_1$ :  $\delta = 1$ ,  $\mathbf{a}_{\mathcal{M}} = 2$ ,  $\mathbf{b}_{\mathcal{M}} = 0$ ,  $\mathbf{g}_{\mathcal{M}}(x) = \frac{3}{2}x + 1$  and  $+_{\mathcal{M}}(x, y) = x + y$ .  $\square$

### 3.2 Matrix Interpretations over $\mathbb{R}$

Next we show that matrix interpretations over  $\mathbb{R}$  are more powerful than matrix interpretations over  $\mathbb{Q}$ . To this end, we extend the TRS  $\mathcal{S}$  of the previous subsection with the rules (7) – (9) and call the resulting system  $\mathcal{R}_2$ :

$$(x + x) + \mathbf{a} \rightarrow \mathbf{k}(\mathbf{k}(x)) \quad (7)$$

$$\mathbf{k}(\mathbf{k}(x)) \rightarrow x + x \quad (8)$$

$$\mathbf{k}(x) \rightarrow x \quad (9)$$

By construction, this TRS admits only matrix interpretations over  $\mathbb{R}$ .

**Lemma 7.** *Let  $\mathcal{M}$  be an  $n$ -dimensional matrix interpretation (not necessarily monotone) with carrier set  $M$  such that  $\mathcal{R}_2 \subseteq \geq_{\mathcal{M}}$ . Then  $M \neq \mathbb{N}^n$  and  $M \neq \mathbb{Q}_0^n$ .*

*Proof.* As the TRS  $\mathcal{S}$  is a subsystem of  $\mathcal{R}_2$ ,  $\mathcal{R}_2 \subseteq \geq_{\mathcal{M}}$  implies  $\mathcal{S} \subseteq \geq_{\mathcal{M}}$ . Hence, by Lemma 4, the function symbol  $+$  must be interpreted by  $+_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{v}$ ,  $\mathbf{v} \in M$ . Assuming  $\mathbf{k}_{\mathcal{M}}(\mathbf{x}) = K\mathbf{x} + \mathbf{k}$  without loss of generality, the (weak) compatibility constraint associated with rule (7) implies  $2I \geq K^2$ . We also have  $K^2 \geq 2I$  by weak compatibility with (8) and  $K \geq I$  due to (9). Hence, the  $n \times n$  square matrix  $K$  must satisfy the following conditions:

$$K^2 = 2I \quad \text{and} \quad K \geq I \quad (10)$$

Clearly, for dimension  $n = 1$ , the unique solution is  $K = \sqrt{2}$ ; in particular,  $K$  is not a rational number. In fact, for any dimension  $n \geq 1$ , the unique solution turns out to be  $K = \sqrt{2}I$ . To this end, let us first show that the conditions given in (10) imply that  $K$  is a diagonal matrix. Because of  $K \geq I$ , we can write  $K = I + N$  for some non-negative matrix  $N$ . Then  $K^2 = 2I$  if and only if  $N^2 + 2N = I$ . Now non-negativity of  $N$  implies  $I \geq N$ . Hence,  $N$  is a diagonal matrix and therefore also  $K$ . So all entries of  $K^2$  are zero except its diagonal entries:  $(K^2)_{ii} = K_{ii}^2$  for  $i = 1, \dots, n$ . But then  $K_{ii}$  must be  $\sqrt{2}$  in order to satisfy  $K^2 = 2I$  and  $K \geq I$ . In other words,  $K = \sqrt{2}I \notin \mathbb{Q}_0^{n \times n}$ . Therefore,  $\mathcal{M}$  cannot be a matrix interpretation over  $\mathbb{N}$  or  $\mathbb{Q}$ .  $\square$

*Remark 8.* Rule (9) is essential for the statement of Lemma 7. Without it, the conditions given in (10) would turn into  $K^2 = 2I$  and  $K \geq 0$ , the conjunction of which is satisfiable over  $\mathbb{N}^{n \times n}$ ; for example, by choosing

$$\mathbf{a}_{\mathcal{M}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathbf{b}_{\mathcal{M}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \mathbf{k}_{\mathcal{M}}(\mathbf{x}) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad +_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y}$$

we obtain a non-monotone 2-dimensional matrix interpretation over  $\mathbb{N}$  that is compatible with the TRS  $\mathcal{R}_2 \setminus \{(9)\}$ . However, in case monotonicity of the matrix interpretation in Lemma 7 is explicitly required, rule (9) becomes superfluous because  $K_{11} \geq 1$  and  $K^2 = 2I$  imply that all entries of the first row and the first column of  $K$  are zero except  $K_{11}$  (as  $K$  must be non-negative). This means that  $(K^2)_{11} = K_{11}^2$ , so  $K_{11}$  must be equal to  $\sqrt{2}$ , hence irrational, in order to satisfy  $K^2 = 2I$ .

Lemma 7 shows that no matrix interpretation over  $\mathbb{N}$  or  $\mathbb{Q}$  is weakly compatible with the TRS  $\mathcal{R}_2$ . However,  $\mathcal{R}_2$  can be shown terminating by a compatible matrix interpretation over  $\mathbb{R}$ .

**Theorem 9.**

1. The TRS  $\mathcal{R}_2$  is terminating. In particular,  $\mathcal{R}_2$  is compatible with a monotone matrix interpretation over  $\mathbb{R}$ .
2. There cannot be an (incremental) termination proof of  $\mathcal{R}_2$  using only monotone matrix interpretations over  $\mathbb{N}$  or  $\mathbb{Q}$ .
3. No matrix interpretation over  $\mathbb{N}$  or  $\mathbb{Q}$  succeeds on the DP problem  $(-, \mathcal{R}_2)$ .

*Proof.* The last two claims are immediate consequences of Lemma 7. Finally, the first claim holds by the following monotone 1-dimensional matrix interpretation over  $\mathbb{R}$  that is compatible with  $\mathcal{R}_2$ :  $\delta = 1$ ,  $\mathbf{a}_{\mathcal{M}} = 4$ ,  $\mathbf{b}_{\mathcal{M}} = 0$ ,  $\mathbf{k}_{\mathcal{M}}(x) = \sqrt{2}x + 1$  and  $+_{\mathcal{M}}(x, y) = x + y$ .  $\square$

## 4 The Dimension Hierarchy

Unlike the previous section, where we have established a hierarchy of matrix interpretations regarding the domain of the matrix entries, the purpose of this section is to examine matrix interpretations with respect to their dimension. That is, we fix  $D \in \{\mathbb{N}, \mathbb{Q}_0, \mathbb{R}_0\}$  and consider matrix interpretations over the family of carrier sets  $(D^n)_{n \geq 1}$ . The main result is that the inherent termination hierarchy is infinite with respect to the dimension  $n$ , with each level of the hierarchy properly subsuming its predecessor. In other words,  $(n+1)$ -dimensional matrix interpretations are strictly more powerful for proving termination than  $n$ -dimensional matrix interpretations (for any  $n \geq 1$ ). We show this by constructing a family of TRSs  $(\mathcal{T}_k)_{k \geq 2}$  having the property that any of its members  $\mathcal{T}_k$  can only be handled with matrix interpretations of dimension at least  $k$ . The construction is based on the idea of encoding (i.e., specifying) the degree of the minimal polynomial  $m_A(x)$  of some matrix  $A$  occurring in a matrix interpretation in terms of rewrite rules. Thus, if  $\mathcal{M}$  is an  $n$ -dimensional matrix interpretation such that the degree of the minimal polynomial of some matrix is fixed to a value of  $k$ , then the degree of the characteristic polynomial of this matrix must be at least  $k$ , i.e.,  $n \geq k$  (since the minimal polynomial divides the characteristic polynomial whose degree is  $n$ ). In other words, the dimension  $n$  of  $\mathcal{M}$  must then be at least  $k$ . The family of TRSs  $(\mathcal{T}_k)_{k \geq 2}$  mentioned above is made up as follows. For any natural number  $k \geq 2$ ,  $\mathcal{T}_k$  denotes the union of the TRS  $\mathcal{S}$  of Section 3 and the following rewrite rules:

$$\mathbf{f}^k(x) + \mathbf{d} \rightarrow \mathbf{f}^{k-1}(x) + \mathbf{c} \tag{11}$$

$$\mathbf{f}^{k-1}(x) + \mathbf{c} \rightarrow \mathbf{f}^k(x) \tag{12}$$

$$\mathbf{h}(\mathbf{f}^{k-2}(\mathbf{h}(x))) \rightarrow \mathbf{h}(\mathbf{f}^{k-1}(\mathbf{h}(x))) + x \tag{13}$$

$$\mathbf{h}(\mathbf{f}^{k-1}(\mathbf{h}(x))) \rightarrow x \tag{14}$$

The intuition is that if  $\mathcal{M}$  is an  $n$ -dimensional matrix interpretation that is weakly compatible with all rules of  $\mathcal{T}_k$ , then the minimal polynomial  $m_F(x)$  of the matrix  $F$  associated with the interpretation of the unary function symbol  $f$  is forced to be equal to the polynomial  $p_k(x) = x^k - x^{k-1}$ , a monic polynomial of degree  $k$ . This is the purpose of the rules (11) – (14). More precisely, the first two rules ensure that  $p_k(x)$  annihilates  $F$ , whereas the latter two specify that  $p_k(x)$  is the monic polynomial of least degree having this property.

**Lemma 10.** *Let  $\mathcal{M}$  be an  $n$ -dimensional matrix interpretation (not necessarily monotone), and let  $k \geq 2$  be a natural number. Then  $\mathcal{T}_k \subseteq \geq_{\mathcal{M}}$  implies  $n \geq k$ .*

*Proof.* Let us assume  $\mathcal{T}_k \subseteq \geq_{\mathcal{M}}$ . Then we also have  $\mathcal{S} \subseteq \geq_{\mathcal{M}}$  because the TRS  $\mathcal{S}$  is contained in  $\mathcal{T}_k$ . Therefore, the function symbol  $+$  must be interpreted by  $+_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{v}$  according to Lemma 4. Assuming  $f_{\mathcal{M}}(\mathbf{x}) = F\mathbf{x} + \mathbf{f}$  and  $h_{\mathcal{M}}(\mathbf{x}) = H\mathbf{x} + \mathbf{h}$  without loss of generality, the (weak) compatibility constraint associated with rule (11) implies  $F^k \geq F^{k-1}$ . We also have  $F^{k-1} \geq F^k$  due to rule (12); hence,  $F^k = F^{k-1}$ . Next we consider the compatibility constraints associated with rule (13) and rule (14). From the former we infer  $HF^{k-2}H \geq HF^{k-1}H + I$ , which implies  $F^{k-2} \neq F^{k-1}$ , whereas the latter enforces  $HF^{k-1}H \geq I$ , which implies  $F^{k-1} \neq 0$ . Thus, the  $n \times n$  square matrix  $F$  must satisfy the following conditions:

$$F^k = F^{k-1} \quad F^{k-2} \neq F^{k-1} \quad F^{k-1} \neq 0 \quad (15)$$

These conditions imply that the minimal polynomial of  $F$  must be equal to the polynomial  $p_k(x) = x^k - x^{k-1}$ ; i.e.,  $m_F(x) = x^k - x^{k-1}$ . In order to show this, we first observe that  $F^k = F^{k-1}$  means that the polynomial  $p_k(x)$  annihilates the matrix  $F$ . So  $m_F(x)$  divides  $p_k(x)$ . Writing  $p_k(x) = (x-1)x^{k-1}$  as a product of irreducible factors, we see that if  $m_F(x) \neq p_k(x)$  (i.e.,  $m_F(x)$  is a proper divisor of  $p_k(x)$  of degree at most  $k-1$ ), then  $m_F(x)$  must divide the polynomial  $(x-1)x^{k-2}$  or the polynomial  $x^{k-1}$  (depending on whether  $(x-1)$  occurs as a factor in  $m_F(x)$  or not). As in both cases the corresponding polynomial annihilates  $F$ , we obtain  $F^{k-2} = F^{k-1}$  or  $F^{k-1} = 0$ , contradicting (15). Consequently,  $p_k(x)$  must indeed be the minimal polynomial of  $F$ , and since it divides the characteristic polynomial of  $F$ , the degree of the latter must be greater than or equal to the degree of the former, that is,  $n \geq k$ .  $\square$

*Remark 11.* If one explicitly requires monotonicity of the matrix interpretation  $\mathcal{M}$  in Lemma 10, then the condition  $F^{k-1} \neq 0$  is automatically satisfied, such that rule (14) becomes superfluous in this case.

Lemma 10 shows that no matrix interpretation of dimension less than  $k$  can be weakly compatible with the TRS  $\mathcal{T}_k$ . However,  $\mathcal{T}_k$  can be shown terminating by a compatible matrix interpretation of dimension  $k$ .

**Theorem 12.** *Let  $k \geq 2$ .*

1. *The TRS  $\mathcal{T}_k$  is terminating. In particular,  $\mathcal{T}_k$  is compatible with a monotone matrix interpretation over  $\mathbb{N}$  of dimension  $k$ .*

2. There cannot be an (incremental) termination proof of  $\mathcal{T}_k$  using only monotone matrix interpretations of dimension less than  $k$ .
3. No matrix interpretation of dimension less than  $k$  succeeds on the DP problem  $(-, \mathcal{T}_k)$ .

*Proof.* The last two claims are immediate consequences of Lemma 10. The first claim holds by the following monotone  $k$ -dimensional matrix interpretation over  $\mathbb{N}$  that is compatible with  $\mathcal{T}_k$ :

$$\begin{aligned} \mathbf{a}_{\mathcal{M}} = \mathbf{c}_{\mathcal{M}} &= (1, 0, \dots, 0)^{\top} & \mathbf{b}_{\mathcal{M}} &= 0 & \mathbf{d}_{\mathcal{M}} &= 2 \mathbf{a}_{\mathcal{M}} \\ +_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) &= \mathbf{x} + \mathbf{y} & \mathbf{f}_{\mathcal{M}}(\mathbf{x}) &= F\mathbf{x} & \mathbf{h}_{\mathcal{M}}(\mathbf{x}) &= H\mathbf{x} + \mathbf{h} \end{aligned}$$

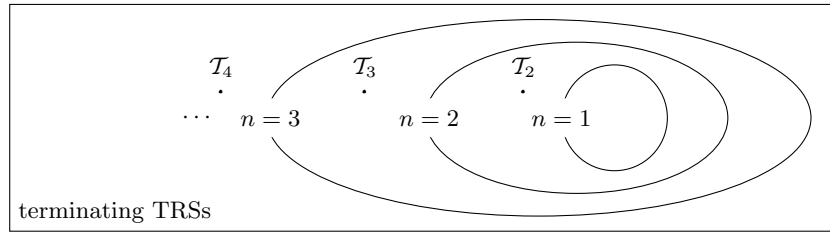
where  $\mathbf{h} = (1, \dots, 1)^{\top}$ , all rows of  $H$  have the shape  $(1, 2, 1, \dots, 1)$  and  $F$  is zero everywhere except for the entries  $F_{11}$  and  $F_{i,i+1}$ ,  $i = 1, \dots, k-1$ , which are all set to one:

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 \end{pmatrix}$$

□

## 5 Conclusion

In this paper we have established two hierarchies of matrix interpretations. On the one hand, there is the domain hierarchy stating that matrix interpretations over the real numbers are more powerful with respect to proving termination than matrix interpretations over the rational numbers, which are in turn more powerful than matrix interpretations over the natural numbers (cf. Figure 1). On the other hand, we have established a hierarchy of matrix interpretations with respect to matrix dimension, which was shown to be infinite, with each level properly subsuming its predecessor (cf. Figure 2). Both hierarchies hold in the context of direct termination (using matrix interpretations as a stand-alone termination method) as well as in the setting of the DP framework. Concerning the latter, we remark that the corresponding results in Theorems 6, 9 and 12 do not only hold for standard reduction pairs (as described in Section 2) but also for reduction pairs incorporating the *basic* version of usable rules [3], where the set of usable rules of a DP problem  $(\mathcal{P}, \mathcal{S})$  is computed as follows. First, for each defined symbol  $f$  occurring in the right-hand side of some rule of  $\mathcal{P}$ , all  $f$ -rules of  $\mathcal{S}$  are marked as usable. Then, whenever a rule is usable and its right-hand side contains a defined symbol  $g$ , all  $g$ -rules of  $\mathcal{S}$  become usable as well. In this way, all rules of the TRSs  $\mathcal{R}_1$ ,  $\mathcal{R}_2$  and  $\mathcal{T}_k$  are usable. It is an easy exercise to make our TRSs also withstand reduction pairs that incorporate usable rules with (implicit) *argument filters* [10] (induced by matrix interpretations).



**Fig. 2.** The dimension hierarchy

Our results concerning the domain hierarchy provide a definitive answer to a question raised in [17] whether *rational numbers are somehow unnecessary when dealing with matrix interpretations*. The answer is in the negative, so the attempt of [17] to simulate matrix interpretations over  $\mathbb{Q}$  with matrix interpretations over  $\mathbb{N}$  (of higher dimension) must necessarily remain incomplete.

Moreover, we remark that the results of this paper do not only apply to the standard variant of matrix interpretations of Endrullis et al. [6] (though the technical part of the paper refers to it) but also to the kinds of matrix interpretations recently introduced in [19] (which are based on various different well-founded orders on vectors of natural numbers) and extensions thereof to vectors of non-negative rational and real numbers. On the technical level, this is due to the fact that our main Lemmata 5, 7 and 10 only require weak compatibility (rather than strict) and do not demand monotonicity of the respective matrix interpretations. Also note that the interpretations given in the proofs of Theorems 6, 9 and 12 can be conceived as matrix interpretations over the base order  $>_{\Sigma}^w$ , which relates two vectors  $\mathbf{x}$  and  $\mathbf{y}$  if and only if there is a weak decrease in every single component of the vectors and a strict decrease with respect to the sum of the components of  $\mathbf{x}$  and  $\mathbf{y}$  (cf. [19]). We expect our results to carry over to the matrix interpretations of [5]. For linear interpretations, this should be possible without further ado, whereas non-linear interpretations conceivably require the addition of new rules enforcing linearity of the interpretations of *some* function symbols (e.g. by using techniques from [18]).

We conclude with a remark on future work and related work. For future work, we mention the extension of the results of this paper to more restrictive classes of TRSs like left-linear ones and SRSs. In this context we also note that the partial result of [8] showing that the dimension hierarchy is infinite applies without further ado since the underlying construction is based on SRSs in contrast to our approach of Section 4.

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