Commutation via Relative Termination*

Nao Hirokawa
JAIST, Japan
hirokawa@jaist.ac.jp

Aart Middeldorp
University of Innsbruck, Austria
aart.middeldorp@uibk.ac.at

Abstract

We present a generalisation of a commutation criterion by Rosen (1973), the development closedness theorem by van Oostrom (1994), and a confluence criterion by Hirokawa and Middeldorp (2011).

1 Introduction

This note is about sufficient conditions for the *commutation* of two *left-linear* TRSs $\mathcal{R}$ and $\mathcal{S}$. Commutation means that the inclusion

$$\xleftarrow[\mathcal{R}]{*} \cdot \xrightarrow[\mathcal{S}]{*} \;\subseteq\; \xrightarrow[\mathcal{S}]{*} \cdot \xleftarrow[\mathcal{R}]{*}$$

holds. When $\mathcal{R} = \mathcal{S}$, commutation amounts to confluence. Furthermore, commutation is a sufficient condition for the confluence of the union of two confluent TRSs. It is well-known that $\mathcal{R}$ and $\mathcal{S}$ commute when there are no critical pairs between the rules of $\mathcal{R}$ and the rules of $\mathcal{S}$. Toyama [6] generalized this result of Rosen [4] (and Raoult and Vuillemin [3, Proposition 10]) by admitting critical pairs between $\mathcal{R}$ and $\mathcal{S}$ but restricting the way in which they are joined. More precisely, Toyama showed that $\mathcal{R}$ and $\mathcal{S}$ commute if the following two conditions are satisfied:

$$\xleftarrow[\mathcal{R}]{} \rtimes \xrightarrow[\mathcal{S}]{} \;\subseteq\; \mathrel{\underset{\mathcal{S}}{\overset{}{\;+\!\!\!\!\!\longrightarrow}}} \cdot \xleftarrow[\mathcal{R}]{*} \qquad \text{and} \qquad \xleftarrow[\mathcal{S}]{>\epsilon} \rtimes \xrightarrow[\mathcal{R}]{} \;\subseteq\; \mathrel{\underset{\mathcal{R}}{\;+\!\!\!\!\!\longrightarrow}}$$

As observed in [7, Corollaries 24 and 28] and [1, Proposition 7], Toyama's result is further generalized by adopting multi-steps:

$$\xleftarrow[\mathcal{R}]{} \rtimes \xrightarrow[\mathcal{S}]{} \;\subseteq\; \mathrel{\underset{\mathcal{S}}{\circ\!\!\!\!\!\longrightarrow}} \cdot \xleftarrow[\mathcal{R}]{*} \qquad \text{and} \qquad \xleftarrow[\mathcal{S}]{>\epsilon} \rtimes \xrightarrow[\mathcal{R}]{} \;\subseteq\; \mathrel{\underset{\mathcal{R}}{\circ\!\!\!\!\!\longrightarrow}}$$

We present a generalisation of these results by incorporating the confluence result based on relative termination [2].

2 Preliminaries

We assume familiarity with the conversion version of the decreasing diagrams technique for proving commutation, due to van Oostrom [8, Theorem 3]. Familiarity with proof terms witnessing multi-steps ([5, Chapter 8], [2]) is also helpful. Below we recall some basic definitions and recast a measure used in [7] as a measure on co-initial proof terms.

Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$. For each rule $\ell \to r \in \mathcal{R}$ we introduce a fresh *rule symbol* $\underline{\ell \to r}$ whose arity is given by the number of variables in ℓ. *Proof terms* are terms over functions in $\mathcal{F}$ and rule symbols. We write $\sqsubseteq$ for the smallest rewrite preorder on proof terms such that $\ell \sqsubseteq \underline{\ell \to r}(x_1, \ldots, x_n)$ for all rules $\ell \to r \in \mathcal{R}$ with $\mathsf{var}(\ell) = (x_1, \ldots, x_n)$. Here $\mathsf{var}(\ell)$ denotes a sequence consisting of all variables in $\mathcal{V}\mathsf{ar}(\ell)$ in some fixed order. Given co-initial

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proof terms A and B, the (partial) residual operation is the proof term $A \setminus B$ defined by the following clauses (here $\langle A_1, \ldots, A_n \rangle_\ell$ stands for the substitution $\{x_i \mapsto A_i \mid 1 \leqslant i \leqslant n\}$):

$$\begin{aligned} x \setminus x &= x \\ f(A_1, \ldots, A_n) \setminus f(B_1, \ldots, B_n) &= f(A_1 \setminus B_1, \ldots, A_n \setminus B_n) \\ \underline{\ell \to r}(A_1, \ldots, A_n) \setminus \underline{\ell \to r}(B_1, \ldots, B_n) &= r\langle A_1 \setminus B_1, \ldots, A_n \setminus B_n \rangle_\ell \\ \underline{\ell \to r}(A_1, \ldots, A_n) \setminus \ell\langle B_1, \ldots, B_n \rangle_\ell &= \underline{\ell \to r}(A_1 \setminus B_1, \ldots, A_n \setminus B_n) \\ \ell\langle A_1, \ldots, A_n \rangle_\ell \setminus \underline{\ell \to r}(B_1, \ldots, B_n) &= r\langle A_1 \setminus B_1, \ldots, A_n \setminus B_n \rangle_\ell \end{aligned}$$

When we use $A \setminus B$ below, it is guaranteed to be defined. (If $A \sqsubseteq B$ then $A \setminus B$ is a proof term without rule symbols.) A *redex* is a proof term that contains exactly one rule symbol. For a redex Δ we write $\Delta \in A$ if $\Delta \sqsubseteq A$. Given a proof term A and a redex $\Delta \in A$, we find it convenient to write $A - \Delta$ for the proof term that is obtained from A by replacing the subterm occurrence $\underline{\ell \to r}(A_1, \ldots, A_n)$ corresponding to Δ by $\ell\langle A_1, \ldots, A_n \rangle_\ell$.

For redexes $\Delta_1, \Delta_2 \in A$ we write $\Delta_1 \geqslant \Delta_2$ if the rule symbol of Δ_1 appears at or below that of Δ_2 in A. The set $\blacktriangle(A)$ of positions is inductively defined on A as follows:

$$\begin{aligned} \blacktriangle(x) &= \varnothing \\ \blacktriangle(f(A_1, \ldots, A_n)) &= \{iq \mid 1 \leqslant i \leqslant n \text{ and } q \in \blacktriangle(A_i)\} \\ \blacktriangle(\underline{\ell \to r}(A_1, \ldots, A_n)) &= \mathcal{P}\mathsf{os}_\mathcal{F}(\ell) \cup \{pq \mid p \in \mathcal{P}\mathsf{os}_{\{x_i\}}(\ell) \text{ and } q \in \blacktriangle(A_i)\} \end{aligned}$$

where $x \in \mathcal{V}$, $f \in \mathcal{F}$, and $\mathsf{var}(\ell) = (x_1, \ldots, x_n)$. So $\blacktriangle(A)$ consists of all function positions of the redex patterns contracted in the multi-step of A. We write $\blacktriangle(A, B)$ for $\blacktriangle(A) \cap \blacktriangle(B)$. The set $\blacktriangle(A, B)$ is used below as a measure for the amount of overlap between the proof terms A and B. A pair (Δ_1, Δ_2) of co-initial redexes is an *overlap* if $\blacktriangle(\Delta_1, \Delta_2) \neq \varnothing$.

Lemma 2.1. *Let A and B be proof terms of left-linear TRSs $\mathcal{R}$ and $\mathcal{S}$ respectively. If $\blacktriangle(A, B) \neq \varnothing$ then there exist a trivial or critical peak $t \xleftarrow[\mathcal{R}]{\Delta_1} s \xrightarrow[\mathcal{S}]{\Delta_2} u$, a context $\mathbb{C}$, and a substitution σ such that $\mathbb{C}[\Delta_1\sigma] \in A$ and $\mathbb{C}[\Delta_2\sigma] \in B$. If $\blacktriangle(A, B) = \varnothing$ then $\xleftarrow[\mathcal{R}]{A}\!\!\!\circ \cdot \circ\!\!\!\xrightarrow[\mathcal{S}]{B} \subseteq \circ\!\!\!\xrightarrow[\mathcal{S}]{B \setminus A} \cdot \xleftarrow[\mathcal{R}]{A \setminus B}\!\!\!\circ$.* □

3 Commutation by Relative Termination

Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs. The set of critical peak steps of $\mathcal{S}$ for $\mathcal{R}$ is defined as follows:

$$\mathsf{CPS}_\mathcal{R}(\mathcal{S}) = \{s \to u \mid t \;{}_\mathcal{R}\!\leftarrow s \to_\mathcal{S} u \text{ is a critical peak}\}$$

Lemma 3.1. *Let $\mathcal{R}$ and $\mathcal{S}$ be left-linear TRSs. If $t \;{}_\mathcal{R}\!\leftarrow\!\!\circ\; s \circ\!\!\!\to_\mathcal{S} u$ then $t \circ\!\!\!\to_\mathcal{S} \cdot {}_\mathcal{R}\!\leftarrow\!\!\circ\; u$ or $t \;{}_\mathcal{R}\!\leftarrow\!\!\circ \cdot {}_{\mathsf{CPS}_\mathcal{S}(\mathcal{R})}\!\leftarrow s \to_{\mathsf{CPS}_\mathcal{R}(\mathcal{S})} \cdot \circ\!\!\!\to_\mathcal{S} u$.* □

Theorem 3.2. *Left-linear locally commuting TRSs $\mathcal{R}$ and $\mathcal{S}$ commute if $\mathsf{CPS}_\mathcal{S}(\mathcal{R}) \cup \mathsf{CPS}_\mathcal{R}(\mathcal{S})$ is relatively terminating over $\mathcal{R} \cup \mathcal{S}$.*

Proof. We use decreasing diagrams with the predecessor labeling. To this end we define $t \circ\!\!\!\to_{\mathcal{R},s} u$ if and only if $s \to^*_{\mathcal{R}\cup\mathcal{S}} t \circ\!\!\!\to_\mathcal{R} u$. The labeled relation $\circ\!\!\!\to_{\mathcal{S},s}$ is defined similarly. We write $>$ to denote $\to^+_{(\mathsf{CPS}_\mathcal{R}(\mathcal{S})\cup\mathsf{CPS}_\mathcal{S}(\mathcal{R}))/(\mathcal{R}\cup\mathcal{S})}$. By the relative termination condition, $>$ is a well-founded order. We show that $(\{\circ\!\!\!\to_{\mathcal{R},s}\}_{s\in\mathcal{T}}, \{\circ\!\!\!\to_{\mathcal{S},s}\}_{s\in\mathcal{T}})$ is decreasing with respect to $>$. Suppose $t \;{}_{\mathcal{R},s_1}\!\leftarrow\!\!\circ\; s \circ\!\!\!\to_{\mathcal{S},s_2} u$. By Lemma 3.1 and local commutation there are terms v and w such that (a) or (b) of Figure 1 holds. Note that $s_i > v, w$ holds in case (b) because $s_i \to^*_{\mathcal{R}\cup\mathcal{S}} s > v, w$. Therefore, in both cases decreasingness is established. Hence $\mathcal{R}$ and $\mathcal{S}$ commute. □

Theorem 3.2 subsumes Rosen's commutativity criterion [4], as mutual orthogonality implies $\mathsf{CPS}_\mathcal{S}(\mathcal{R}) \cup \mathsf{CPS}_\mathcal{R}(\mathcal{S}) = \varnothing$ as well as local commutation.

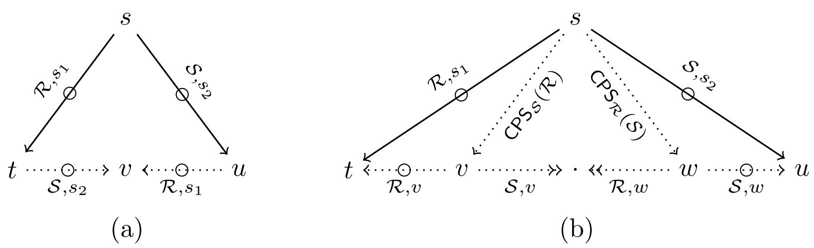


Figure 1: Decreasingness of $(\xrightarrow{\circ}_{\mathcal{R}}, \xrightarrow{\circ}_{\mathcal{S}})$.

4 Incorporating Development Closedness

We extend the last theorem in order to subsume the development closedness theorem of van Oostrom [7]. Let $t \;{}_{\mathcal{R}}\!\xleftarrow{p}\; s \xrightarrow{q}_{\mathcal{S}} u$ be a critical peak. By definition p or q must be the root position. We say that the peak is $(\mathcal{R},\mathcal{S})$-*closed* if $t \;{}_{\mathcal{R}}\!\xleftarrow{\circ}\; u$ whenever $p = \epsilon$ and $t \xrightarrow{\circ}_{\mathcal{S}} u$ whenever $q = \epsilon$. The set of all non-closed critical peak steps of $\mathcal{S}$ for $\mathcal{R}$ is defined as follows:

$$\mathsf{CPS}'_{\mathcal{R}}(\mathcal{S}) = \{s \to u \mid t \;{}_{\mathcal{R}}\!\leftarrow s \to_{\mathcal{S}} u \text{ is a critical peak which is not } (\mathcal{R},\mathcal{S})\text{-closed}\}$$

Lemma 4.1. *Let $\mathcal{R}$ and $\mathcal{S}$ be left-linear TRSs. If $t \;{}_{\mathcal{R}}\!\xleftarrow{\circ}\; s \xrightarrow{\circ}_{\mathcal{S}} u$ then (a) $t \xrightarrow{\circ}_{\mathcal{S}} \cdot \;{}_{\mathcal{R}}\!\xleftarrow{\circ}\; u$ or (b) $t \;{}_{\mathcal{R}}\!\xleftarrow{\circ}\; \cdot \;{}_{\mathsf{CPS}'_{\mathcal{S}}(\mathcal{R})}\!\leftarrow s' \to_{\mathsf{CPS}'_{\mathcal{R}}(\mathcal{S})} \cdot \xrightarrow{\circ}_{\mathcal{S}} u$ and $s \to^{*}_{\mathcal{R}\cup\mathcal{S}} s'$ for some s'.*

Lemma 4.1 is a proper generalization of [2, Lemma 4]. In order to prove the lemma we introduce some more notions. Whenever $\blacktriangle(A,B)$ is non-empty, there exists an innermost overlap. An overlap (Δ_1,Δ_2) in $A \times B$ is called *innermost* if there are no other overlaps (Δ_1',Δ_2') in $A \times B$ such that $\Delta_1' \geqslant \Delta_1$ and $\Delta_2' \geqslant \Delta_2$. We say that (Δ_1,Δ_2) is $(\mathcal{R},\mathcal{S})$-closed if the corresponding critical peak is $(\mathcal{R},\mathcal{S})$-closed. The following is the key lemma. In order to pave the way for formalisation of advanced confluence and commutation results, we formalise the proof technique introduced in [7] using proof terms.

Lemma 4.2. *Suppose (Δ_1,Δ_2) is an innermost overlap in $A \times B$ with $\Delta_1 \geqslant \Delta_2$. If (Δ_1,Δ_2) is $(\mathcal{R},\mathcal{S})$-closed then $\blacktriangle(A,B) \supsetneq \blacktriangle(A \setminus \Delta_1, C)$ for some proof term C that is co-initial with $A \setminus \Delta_1$.*

Proof. Write $t \xleftarrow[\mathcal{R}]{A \setminus \Delta_1 \,\circ} t_1 \xleftarrow[\mathcal{R}]{\Delta_1} s \xrightarrow[\mathcal{S}]{\Delta_2} u_1 \xrightarrow[\mathcal{S}]{B \setminus \Delta_2 \,\circ} u$ and let

$$t_1' \xleftarrow[\mathcal{R}]{\Delta_1'} s' \xrightarrow[\mathcal{S}]{\Delta_2'} u_1'$$

be the corresponding critical peak. Since $s \trianglerighteq s'$, there exist a context $\mathbb{C}$ and a substitution σ such that $s = \mathbb{C}[s'\sigma]$, $t_1 = \mathbb{C}[t_1'\sigma]$, $u_1 = \mathbb{C}[u_1'\sigma]$, $\Delta_1 = \mathbb{C}[\Delta_1'\sigma]$, and $\Delta_2 = \mathbb{C}[\Delta_2'\sigma]$. We have $t_1' \xrightarrow{\circ}_{\mathcal{R}} u_1'$ because the critical peak is $(\mathcal{R},\mathcal{S})$-closed. Let D' be the proof term witnessing this multi-step and define $D = \mathbb{C}[D'\sigma]$. Clearly, D is a witness of $t_1 \xrightarrow{\circ}_{\mathcal{R}} u_1$. We will compose D and $B \setminus \Delta_2$ into a single proof term $C\colon t_1 \xrightarrow{\circ}_{\mathcal{R}} u$. Let $B' = B - \Delta_2$. Since (Δ_1,Δ_2) is an innermost overlap with $\Delta_1 \geqslant \Delta_2$, $\blacktriangle(B',\Delta_1) = \varnothing$. Hence $B' \setminus \Delta_1$ is well-defined. If we show that $\blacktriangle(B' \setminus \Delta_1, D) = \varnothing$ then we can define C as $(B' \setminus \Delta_1) \sqcup D$. We have $\blacktriangle(D') \subseteq \mathcal{P}\mathsf{os}_{\mathcal{F}}(t_1')$ and thus $\blacktriangle(D) \subseteq \{qp \mid p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(t_1')\}$ where q is the position of the hole in $\mathbb{C}$. In words, D affects only the t_1' part of $\mathbb{C}[t_1'\sigma]$. Let $\Delta \in B'$ be an arbitrary redex. We show that $\blacktriangle(\Delta \setminus \Delta_1, D) = \varnothing$. We split B' into two parts: $B' = B_1 \sqcup B_2$ with

$$B_1 = B' - B_2 \qquad\qquad B_2 = \bigsqcup\{\Delta \mid \Delta \in B' \text{ and } \Delta > \Delta_2\}$$

Depending on the position of Δ, we distinguish two cases:

(a) If $\Delta \in B_1$ then Δ does not overlap Δ_1. So $\blacktriangle(\Delta \setminus \Delta_1)$ consists entirely of positions in $\mathbb{C}$ and thus $\blacktriangle(\Delta \setminus \Delta_1, D) = \varnothing$.

(b) Otherwise, $\Delta \in B_2$. Since (Δ_1, Δ_2) is an innermost overlap, $\blacktriangle(\Delta, \Delta_1) = \varnothing$. Because $\Delta \neq \Delta_2$ we also have $\blacktriangle(\Delta, \Delta_2) = \varnothing$. The crucial observation is that no redex in $\Delta \setminus \Delta_1$ overlaps with D. This follows because for the originating term s' of the critical peak we have $\mathcal{P}\mathsf{os}_{\mathcal{F}}(s') = \blacktriangle(\Delta_1') \cup \blacktriangle(\Delta_2')$, due to the fact that the involved rules are left-linear. Hence all redexes in $\Delta \setminus \Delta_1$ appear in the substitution part σ of $\mathbb{C}[t_1'\sigma]$.

We conclude that $\blacktriangle(B' \setminus \Delta_1, D) = \varnothing$. It remains to show that $\blacktriangle(A, B) \supsetneq \blacktriangle(A \setminus \Delta_1, C)$. We have $\blacktriangle(B_1 \setminus \Delta_1) = \blacktriangle(B_1)$ and hence $\blacktriangle(C) = \blacktriangle(B_1) \cup \blacktriangle(B_2 \setminus \Delta_1) \cup \blacktriangle(D)$. Because (Δ_1, Δ_2) is an innermost overlap, $\blacktriangle(A, B_2) = \varnothing$ and therefore

$$\blacktriangle(A, B) = \blacktriangle(A, \Delta_2) \cup \blacktriangle(A, B_1)$$

and $\blacktriangle(A \setminus \Delta_1, B_2 \setminus \Delta_1) = \varnothing$. Because all redexes in B_1 occur above Δ_2, they do not overlap with Δ_1 and thus $\blacktriangle(A \setminus \Delta_1, B \setminus \Delta_1) = \blacktriangle(A \setminus \Delta_1, B_1 \setminus \Delta_1) = \blacktriangle(A, B_1)$. We obtain

$$\begin{aligned}\blacktriangle(A \setminus \Delta_1, C) &= \blacktriangle(A \setminus \Delta_1, B' \setminus \Delta_1) \cup \blacktriangle(A \setminus \Delta_1, D) \\ &= \blacktriangle(A, B_1) \cup \blacktriangle(A \setminus \Delta_1, D)\end{aligned}$$

and so it remains to show that $\blacktriangle(A, \Delta_2) \supsetneq \blacktriangle(A \setminus \Delta_1, D)$. We split A into three parts: $A = A_1 \sqcup \Delta_1 \sqcup A_2$ with $A_1 = A - \Delta_1 - A_2$ and $A_2 = \bigsqcup\{\Delta \mid \Delta \in A \text{ and } \Delta > \Delta_1\}$. We have $\blacktriangle(A, \Delta_2) = \blacktriangle(A_1, \Delta_2) \cup \blacktriangle(\Delta_1, \Delta_2)$ and

$$\blacktriangle(A \setminus \Delta_1, D) = \blacktriangle(A_1 \setminus \Delta_1, D) \cup \blacktriangle(\Delta_1 \setminus \Delta_1, D) \cup \blacktriangle(A_2 \setminus \Delta_1, D)$$

The second component is clearly empty but also the third component reduces to the empty set, by repeating the reasoning in case (b) above. The first component equals $\blacktriangle(A_1, D)$ since redexes in A_1 occur above Δ_1. So $\blacktriangle(A \setminus \Delta_1, D) = \blacktriangle(A_1, D)$. Since $\blacktriangle(\Delta_1, \Delta_2) \neq \varnothing$, it suffices to show the inclusion $\blacktriangle(A_1, D) \subseteq \blacktriangle(A_1, \Delta_2)$, which follows from the observation that overlaps between A_1 and D are restricted to positions in $\blacktriangle(\Delta_2) - \blacktriangle(\Delta_1)$. ☐

We are ready to show the aforementioned lemma.

Proof of Lemma 4.1. By induction on the finite set $\blacktriangle(A, B)$ with respect to the well-founded order $\supsetneq$. Let

$$t \xleftarrow[\mathcal{R}]{A}\!\!\circ\; s \xrightarrow[\mathcal{S}]{B}\!\!\circ\; u$$

If $\blacktriangle(A, B)$ is empty then (a) holds by the second part of Lemma 2.1. Otherwise, there is an overlap in $A \times B$. We distinguish two cases.

- If a non-closed overlap (Δ_1, Δ_2) exists in $A \times B$, (b) holds because

$$t \xleftarrow[\mathcal{R}]{A \setminus \Delta_1}\!\circ \cdot \xleftarrow[\mathcal{R}]{\Delta_1} s \xrightarrow[\mathcal{S}]{\Delta_2} \cdot \xrightarrow[\mathcal{S}]{B \setminus \Delta_2}\!\circ\; u$$

  and the two inner steps correspond to applications of rules from $\mathsf{CPS}'_{\mathcal{S}}(\mathcal{R})$ and $\mathsf{CPS}'_{\mathcal{R}}(\mathcal{S})$.

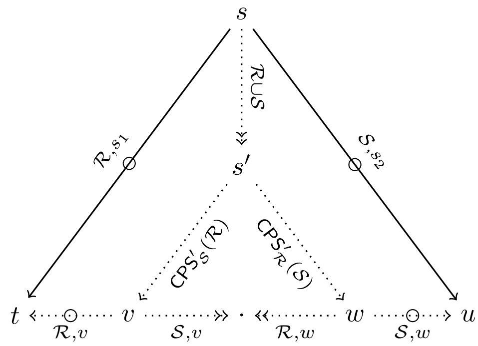
- In the other case all overlaps are closed. Let (Δ_1, Δ_2) be an innermost overlap. Without loss of generality we assume $\Delta_1 \geqslant \Delta_2$. According to Lemma 4.2 there exists some proof term C such that $\blacktriangle(A, B) \supsetneq \blacktriangle(A \setminus \Delta_1, C)$. Let s' be the term such that $s \xrightarrow{\Delta_1} s'$. (In the proof of Lemma 4.2 s' was called t_1.) Because

$$t \xleftarrow[\mathcal{R}]{A \setminus \Delta_1} s' \xrightarrow[\mathcal{S}]{C} u$$

the induction hypothesis applies. □

Theorem 4.3. *Left-linear locally commuting TRSs $\mathcal{R}$ and $\mathcal{S}$ commute if $\mathsf{CPS}'_{\mathcal{S}}(\mathcal{R}) \cup \mathsf{CPS}'_{\mathcal{R}}(\mathcal{S})$ is relatively terminating over $\mathcal{R} \cup \mathcal{S}$.*

Proof. Use Lemma 4.1 to replace Figure 1(b) by the following diagram:



□

Due to lack of space, we omit the extension that weakens the joinability requirement for overlays, but note that the result of Theorem 4.3 remains true if we strengthen the notion of $(\mathcal{R}, \mathcal{S})$-closedness by allowing $t \twoheadrightarrow^*_{\mathcal{S}} \cdot {}_{\mathcal{R}}\twoheadleftarrow u$ when $p = \epsilon$. Future work includes the extension to critical valleys [9] as well as automation. Experimental data on the confluence problem database Cops[1] for the presented theorems will be reported at the workshop.

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[1] `http://coco.nue.riec.tohoku.ac.jp/`