AC-KBO Revisited*

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Abstract. We consider various definitions of AC-compatible Knuth-Bendix orders. The orders of Steinbach and of Korovin and Voronkov are revisited. The former is enhanced to a more powerful AC-compatible order and we modify the latter to amend its lack of monotonicity on non-ground terms. We further present new complexity results. An extension reflecting the recent proposal of subterm coefficients in standard Knuth-Bendix orders is also given. The various orders are compared on problems in termination and completion.

1 Introduction

Associative and commutative (AC) operators appear in many applications, e.g. in automated reasoning with respect to algebraic structures such as commutative groups or rings. We are interested in proving termination of term rewrite systems with AC symbols. AC termination is important when deciding validity in equational theories with AC operators by means of completion.

Several termination methods for plain rewriting have been extended to deal with AC symbols. Ben Cherifa and Lescanne [4] presented a characterization of polynomial interpretations that ensures compatibility with the AC axioms. There have been numerous papers on extending the recursive path order (RPO) of Dershowitz [6] to deal with AC symbols, starting with the associative path order of Bachmair and Plaisted [3] and culminating in the fully syntactic AC-RPO of Rubio [16]. Several authors [1, 7, 12, 15] adapted the influential dependency pair method of Arts and Giesl [2] to AC rewriting.

We are aware of only two papers on AC extensions of the order of Knuth and Bendix (KBO) [8]. In this paper we revisit these orders and present yet another AC-compatible KBO. Steinbach [17] presented a first version, which comes with the restriction that AC symbols are minimal in the precedence. By incorporating ideas of [16], Korovin and Voronkov [9] presented a version without this restriction. Actually, they present two versions. One is defined on ground terms and another one on arbitrary terms. For (automatically) proving AC termination of

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rewrite systems, an AC-compatible order on arbitrary terms is required.[4] We show that the second order of [9] lacks the monotonicity property which is required by the definition of simplification orders. Nevertheless we prove that the order is sound for proving termination by extending it to an AC-compatible simplification order. We furthermore present a simpler variant of this latter order which properly extends the order of [17]. In particular, Steinbach's order is a correct AC-compatible simplification order, contrary to what is claimed in [9]. We also present new complexity results which confirm that AC rewriting is much more involved than plain rewriting. Apart from these theoretical contributions, we implemented the various AC-compatible KBOs to compare them also experimentally.

The remainder of this paper is organized as follows. After recalling basic concepts of rewriting modulo AC and orders, we revisit Steinbach's order in Section 3. Section 4 is devoted to the two orders of Korovin and Voronkov. We present a first version of our AC-compatible KBO in Section 5, where we also give the non-trivial proof that it has the required properties. (The proofs in [9] are limited to the order on ground terms.) Then we prove in Section 6 that the problem of orienting a ground rewrite system with the order of Korovin and Voronkov as well as our new order is NP-hard. In Section 7 our order is strengthened with subterm coefficients. In order to show effectiveness of these orders experimental data is provided in Section 8. The paper is concluded in Section 9. Due to lack of space, some of the proofs can be found in the full version of this paper [21].

2 Preliminaries

We assume familiarity with rewriting and termination. Throughout this paper we deal with rewrite systems over a set $\mathcal{V}$ of variables and a *finite* signature $\mathcal{F}$ together with a designated subset $\mathcal{F}_{\mathsf{AC}}$ of binary AC symbols. The congruence relation induced by the equations $f(x,y) \approx f(y,x)$ and $f(f(x,y),z) \approx f(x,f(y,z))$ for all $f \in \mathcal{F}_{\mathsf{AC}}$ is denoted by $=_{\mathsf{AC}}$. A term rewrite system (TRS for short) $\mathcal{R}$ is AC terminating if the relation $=_{\mathsf{AC}} \cdot \to_{\mathcal{R}} \cdot =_{\mathsf{AC}}$ is well-founded. In this paper AC termination is established by *AC-compatible simplification orders* $\succ$, which are strict orders (i.e., irreflexive and transitive relations) closed under contexts and substitutions that have the subterm property $f(t_1,\ldots,t_n) \succ t_i$ for all $1 \leqslant i \leqslant n$ and satisfy $=_{\mathsf{AC}} \cdot \succ \cdot =_{\mathsf{AC}} \subseteq \succ$. A strict order $\succ$ is *AC-total* if $s \succ t$, $t \succ s$ or $s =_{\mathsf{AC}} t$, for all ground terms s and t. A pair $(\succsim, \succ)$ consisting of a preorder $\succsim$ and a strict order $\succ$ is said to be an *order pair* if the *compatibility* condition $\succsim \cdot \succ \cdot \succsim \subseteq \succ$ holds.

Definition 1. *Let $\succ$ be a strict order and $\succsim$ be a preorder on a set A. The* lexicographic extensions *$\succ^{\mathsf{lex}}$ and $\succsim^{\mathsf{lex}}$ are defined as follows:*

[4] Any AC-compatible reduction order $\succ_{\mathrm{g}}$ on ground terms can trivially be extended to arbitrary terms by defining $s \succ t$ if and only if $s\sigma \succ_{\mathrm{g}} t\sigma$ for all grounding substitutions σ. This is, however, only of (mild) theoretical interest.

– $\boldsymbol{x} \succsim^{\mathsf{lex}} \boldsymbol{y}$ *if* $\boldsymbol{x} \sqsupset_k^{\mathsf{lex}} \boldsymbol{y}$ *for some* $1 \leqslant k \leqslant n$,
– $\boldsymbol{x} \succ^{\mathsf{lex}} \boldsymbol{y}$ *if* $\boldsymbol{x} \sqsupset_k^{\mathsf{lex}} \boldsymbol{y}$ *for some* $1 \leqslant k < n$.

Here $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$, *and* $\boldsymbol{x} \sqsupset_k^{\mathsf{lex}} \boldsymbol{y}$ *denotes the following condition:* $x_i \succsim y_i$ *for all* $i \leqslant k$ *and either* $k < n$ *and* $x_{k+1} \succ y_{k+1}$ *or* $k = n$. *The* multiset extensions $\succ^{\mathsf{mul}}$ *and* $\succsim^{\mathsf{mul}}$ *are defined as follows:*

– $M \succsim^{\mathsf{mul}} N$ *if* $M \sqsupset_k^{\mathsf{mul}} N$ *for some* $0 \leqslant k \leqslant \min(m, n)$,
– $M \succ^{\mathsf{mul}} N$ *if* $M \sqsupset_k^{\mathsf{mul}} N$ *for some* $0 \leqslant k \leqslant \min(m-1, n)$.

Here $M \sqsupset_k^{\mathsf{mul}} N$ *if* M *and* N *consist of* $x_1, \ldots, x_m$ *and* $y_1, \ldots, y_n$ *respectively such that* $x_j \succsim y_j$ *for all* $j \leqslant k$, *and for every* $k < j \leqslant n$ *there is some* $k < i \leqslant m$ *with* $x_i \succ y_j$.

Note that these extended relations depend on both $\succsim$ and $\succ$. The following result is folklore; a recent formalization of multiset extensions in Isabelle/HOL is presented in [18].

Theorem 2. *If* $(\succsim, \succ)$ *is an order pair then* $(\succsim^{\mathsf{lex}}, \succ^{\mathsf{lex}})$ *and* $(\succsim^{\mathsf{mul}}, \succ^{\mathsf{mul}})$ *are order pairs.* □

3 Steinbach's Order

In this section we recall the AC-compatible KBO $>_{\mathsf{S}}$ of Steinbach [17], which reduces to the standard KBO if AC symbols are absent.[5] The order $>_{\mathsf{S}}$ depends on a precedence and an admissible weight function. A *precedence* $>$ is a strict order on $\mathcal{F}$. A *weight function* (w, w_0) for a signature $\mathcal{F}$ consists of a mapping $w \colon \mathcal{F} \to \mathbb{N}$ and a constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for every constant $c \in \mathcal{F}$. The *weight* of a term t is recursively computed as follows: $w(t) = w_0$ if $t \in \mathcal{V}$ and $w(f(t_1, \ldots, t_n)) = w(f) + w(t_1) + \cdots + w(t_n)$. A weight function (w, w_0) is *admissible* for $>$ if every unary f with $w(f) = 0$ satisfies $f > g$ for all function symbols g different from f. Throughout this paper we assume admissibility.

The *top-flattening* [16] of a term t with respect to an AC symbol f is the multiset $\nabla_f(t)$ defined inductively as follows: $\nabla_f(t) = \{t\}$ if $\mathsf{root}(t) \neq f$ and $\nabla_f(f(t_1, t_2)) = \nabla_f(t_1) \uplus \nabla_f(t_2)$.

Definition 3. *Let* $>$ *be a precedence and* (w, w_0) *a weight function. The order* $>_{\mathsf{S}}$ *is inductively defined as follows:* $s >_{\mathsf{S}} t$ *if* $|s|_x \geqslant |t|_x$ *for all* $x \in \mathcal{V}$ *and either* $w(s) > w(t)$, *or* $w(s) = w(t)$ *and one of the following alternatives holds:*

(0) $s = f^k(t)$ *and* $t \in \mathcal{V}$ *for some* $k > 0$,
(1) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, *and* $f > g$,
(2) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >_{\mathsf{S}}^{\mathsf{lex}} (t_1, \ldots, t_n)$,
(3) $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, *and* $\nabla_f(s) >_{\mathsf{S}}^{\mathsf{mul}} \nabla_f(t)$.

[5] The version in [17] is slightly more general, since non-AC function symbols can have arbitrary status. To simplify the discussion, we do not consider status in this paper.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>_{\mathsf{S}}^{\mathsf{lex}}$ and $>_{\mathsf{S}}^{\mathsf{mul}}$.

Cases (0)–(2) are the same as in the classical Knuth-Bendix order. In case (3) terms rooted by the same AC symbol f are treated by comparing their top-flattenings in the multiset extension of $>_{\mathsf{S}}$.

Example 4. Consider the signature $\mathcal{F} = \{\mathsf{a}, \mathsf{f}, \mathsf{g}\}$ with $\mathsf{f} \in \mathcal{F}_{\mathsf{AC}}$, precedence $\mathsf{g} > \mathsf{a} > \mathsf{f}$ and admissible weight function (w, w_0) with $w(\mathsf{f}) = w(\mathsf{g}) = 0$ and $w_0 = w(\mathsf{a}) = 1$. Let $\mathcal{R}_1$ be the following ground TRS:

$$\mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) \quad (1) \qquad\qquad \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a}))) \to \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) \quad (2)$$

For $1 \leqslant i \leqslant 2$, let ℓ_i and r_i be the left- and right-hand side of rule (i), $S_i = \nabla_{\mathsf{f}}(\ell_i)$ and $T_i = \nabla_{\mathsf{f}}(r_i)$. Both rules vacuously satisfy the variable condition. We have $w(\ell_1) = 2 = w(r_1)$ and $\mathsf{g} > \mathsf{f}$, so $\ell_1 >_{\mathsf{S}} r_1$ holds by case (1). We have $w(\ell_2) = 2 = w(r_2)$, $S_2 = \{\mathsf{a}, \mathsf{g}(\mathsf{g}(\mathsf{a}))\}$, and $T_2 = \{\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{a})\}$. Since $\mathsf{g}(\mathsf{a}) >_{\mathsf{S}} \mathsf{a}$ holds by case (1), $\mathsf{g}(\mathsf{g}(\mathsf{a})) >_{\mathsf{S}} \mathsf{g}(\mathsf{a})$ holds by case (2), and therefore $\ell_2 >_{\mathsf{S}} r_2$ by case (3).

Theorem 5 ([17]). *If every symbol in $\mathcal{F}_{\mathsf{AC}}$ is minimal with respect to $>$ then $>_{\mathsf{S}}$ is an AC-compatible simplification order.*[6] □

In Section 5 we reprove[7] Theorem 5 by showing that $>_{\mathsf{S}}$ is a special case of our new AC-compatible Knuth-Bendix order.

4 Korovin and Voronkov's Orders

In this section we recall the orders of [9]. The first one is defined on ground terms. The difference with $>_{\mathsf{S}}$ is that in case (3) of the definition a further case analysis is performed based on terms in S and T whose root symbols are not smaller than f in the precedence. Rather than recursively comparing these terms with the order being defined, a lighter non-recursive version is used in which the weights and root symbols are considered. This is formally defined below.

Given a multiset T of terms, a function symbol f, and a binary relation R on function symbols, we define the following submultisets of T:

$$T{\restriction}_{\mathcal{V}} = \{x \in T \mid x \in \mathcal{V}\} \qquad\qquad T{\restriction}_f^R = \{t \in T \setminus \mathcal{V} \mid \mathsf{root}(t)\ R\ f\}$$

Definition 6. *Let $>$ be a precedence and (w, w_0) a weight function.*[8] *First we define the auxiliary relations $=_{\mathsf{kv}}$ and $>_{\mathsf{kv}}$ as follows:*

- *$s =_{\mathsf{kv}} t$ if $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$,*
- *$s >_{\mathsf{kv}} t$ if either $w(s) > w(t)$ or both $w(s) = w(t)$ and $\mathsf{root}(s) > \mathsf{root}(t)$.*

[6] In [17] AC symbols are further required to have weight 0 because terms are flattened. Our version of $>_{\mathsf{S}}$ does not impose this restriction due to the use of top-flattening.

[7] The counterexample in [9] against the monotonicity of $>_{\mathsf{S}}$ is invalid as the condition that AC symbols are *minimal* in the precedence is not satisfied.

[8] Here we do not impose totality on precedences, cf. [9]. See also Example 25.

The order $>_{\mathsf{KV}}$ is inductively defined on ground terms as follows: $s >_{\mathsf{KV}} t$ if either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

(1) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, *and* $f > g$,

(2) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >^{\mathsf{lex}}_{\mathsf{KV}} (t_1, \ldots, t_n)$,

(3) $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, *and for* $S = \nabla_f(s)$ *and* $T = \nabla_f(t)$

- *(a)* $S{\restriction}^{\not<}_{f} >^{\mathsf{mul}}_{\mathsf{kv}} T{\restriction}^{\not<}_{f}$, *or*
- *(b)* $S{\restriction}^{\not<}_{f} =^{\mathsf{mul}}_{\mathsf{kv}} T{\restriction}^{\not<}_{f}$ *and* $|S| > |T|$, *or*
- *(c)* $S{\restriction}^{\not<}_{f} =^{\mathsf{mul}}_{\mathsf{kv}} T{\restriction}^{\not<}_{f}$, $|S| = |T|$, *and* $S >^{\mathsf{mul}}_{\mathsf{KV}} T$.

Here $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{KV}}$ and $>^{\mathsf{mul}}_{\mathsf{KV}}$ whereas $=_{\mathsf{kv}}$ is used in $>^{\mathsf{mul}}_{\mathsf{kv}}$.

Only in cases (2) and (3c) the order $>_{\mathsf{KV}}$ is used recursively. In case (3) terms rooted by the same AC symbol f are compared by extracting from the top-flattenings S and T the multisets $S{\restriction}^{\not<}_{f}$ and $T{\restriction}^{\not<}_{f}$ consisting of all terms rooted by a function symbol not smaller than f in the precedence. If $S{\restriction}^{\not<}_{f}$ is larger than $T{\restriction}^{\not<}_{f}$ in the multiset extension of $>_{\mathsf{kv}}$, we conclude in case (3a). Otherwise the multisets must be equal (with respect to $=^{\mathsf{mul}}_{\mathsf{kv}}$). If S has more terms than T, we conclude in case (3b). In the final case (3c) S and T have the same number of terms and we compare S and T in the multiset extension of $>_{\mathsf{KV}}$.

Theorem 7 ([9])**.** *The order $>_{\mathsf{KV}}$ is an AC-compatible simplification order on ground terms. If $>$ is total then $>_{\mathsf{KV}}$ is AC-total on ground terms.* □

The two orders $>_{\mathsf{KV}}$ and $>_{\mathsf{S}}$ are incomparable on ground TRSs.

Example 8. Consider again the ground TRS $\mathcal{R}_1$ of Example 4. To orient rule (1) with $>_{\mathsf{KV}}$, the weight of the unary function symbol g must be 0 and admissibility demands $\mathsf{g} > \mathsf{a}$ and $\mathsf{g} > \mathsf{f}$. Hence rule (1) is handled by case (1) of the definition. For rule (2), the multisets $S = \{\mathsf{a}, \mathsf{g}(\mathsf{g}(\mathsf{a}))\}$ and $T = \{\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{a})\}$ are compared in case (3). We have $S{\restriction}^{\not<}_{\mathsf{f}} = \{\mathsf{g}(\mathsf{g}(\mathsf{a}))\}$ if $\mathsf{f} > \mathsf{a}$ and $S{\restriction}^{\not<}_{\mathsf{f}} = S$ otherwise. In both cases we have $T{\restriction}^{\not<}_{\mathsf{f}} = T$. Note that neither $\mathsf{a} >_{\mathsf{kv}} \mathsf{g}(\mathsf{a})$ nor $\mathsf{g}(\mathsf{g}(\mathsf{a})) >_{\mathsf{kv}} \mathsf{g}(\mathsf{a})$ holds. Hence case (3a) does not apply. But also cases (3b) and (3c) are not applicable as $\mathsf{g}(\mathsf{g}(\mathsf{a})) =_{\mathsf{kv}} \mathsf{g}(\mathsf{a})$ and $\mathsf{a} \neq_{\mathsf{kv}} \mathsf{g}(\mathsf{a})$. Hence, independent of the choice of $>$, $\mathcal{R}_1$ cannot be proved terminating by $>_{\mathsf{KV}}$. Conversely, the TRS $\mathcal{R}_2$ resulting from reversing rule (2) in $\mathcal{R}_1$ can be proved terminating by $>_{\mathsf{KV}}$ but not by $>_{\mathsf{S}}$.

Next we present the second order of [9], the extension of $>_{\mathsf{KV}}$ to non-ground terms. Since it coincides with $>_{\mathsf{KV}}$ on ground terms, we use the same notation for the order.

In case (3) of the following definition, also variables appearing in the top-flattenings S and T are taken into account in the first multiset comparison. Given a relation $\sqsupset$ on terms, we write $S \sqsupset^{f} T$ for $S{\restriction}^{\not<}_{f} \sqsupset^{\mathsf{mul}} T{\restriction}^{\not<}_{f} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$. Note that $\sqsupset^{f}$ depends on a precedence $>$. Whenever we use $\sqsupset^{f}$, $>$ is defined.

Definition 9. *Let $>$ be a precedence and (w, w_0) a weight function. First we extend the orders $=_{\mathsf{kv}}$ and $>_{\mathsf{kv}}$ as follows:*

– *$s =_{\mathsf{kv}} t$ if $|s|_x = |t|_x$ for all $x \in \mathcal{V}$, $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$,*
– *$s >_{\mathsf{kv}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$ or both $w(s) = w(t)$ and $\mathsf{root}(s) > \mathsf{root}(t)$.*

The order $>_{\mathsf{KV}}$ is now inductively defined as follows: $s >_{\mathsf{KV}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

(0) $s = f^k(t)$ and $t \in \mathcal{V}$ for some $k > 0$,
(1) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f > g$,
(2) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >^{\mathsf{lex}}_{\mathsf{KV}} (t_1, \ldots, t_n)$,
(3) $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$
   (a) $S >^f_{\mathsf{kv}} T$, or
   (b) $S =^f_{\mathsf{kv}} T$ and $|S| > |T|$, or
   (c) $S =^f_{\mathsf{kv}} T$, $|S| = |T|$, and $S >^{\mathsf{mul}}_{\mathsf{KV}} T$.

Here $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{KV}}$ and $>^{\mathsf{mul}}_{\mathsf{KV}}$ whereas $=_{\mathsf{kv}}$ is used in $>^{\mathsf{mul}}_{\mathsf{kv}}$.

Contrary to what is claimed in [9], the order $>_{\mathsf{KV}}$ of Definition 9 is not a simplification order because it lacks the monotonicity property (i.e., $>_{\mathsf{KV}}$ is not closed under contexts), as shown in the following example.

Example 10. Let f be an AC symbol and g a unary function symbol with $w(\mathsf{g}) = 0$ and $\mathsf{g} > \mathsf{f}$.[9] We obviously have $\mathsf{g}(x) >_{\mathsf{KV}} x$. However, $\mathsf{f}(\mathsf{g}(x), y) >_{\mathsf{KV}} \mathsf{f}(x, y)$ does not hold. Let $S = \nabla_{\mathsf{f}}(s) = \{\mathsf{g}(x), y\}$ and $T = \nabla_{\mathsf{f}}(t) = \{x, y\}$. We have $S{\restriction}^{\not<}_{\mathsf{f}} = \{\mathsf{g}(x)\}$, $S{\restriction}_{\mathcal{V}} = \{y\}$, $T{\restriction}^{\not<}_{\mathsf{f}} = \varnothing$, and $T{\restriction}_{\mathcal{V}} = \{x, y\}$. Note that $\mathsf{g}(x) >_{\mathsf{kv}} x$ does not hold since $\mathsf{g} \not> x$. Hence case (3a) in Definition 9 does not apply. But also $\mathsf{g}(x) =_{\mathsf{kv}} x$ does not hold, excluding cases (3b) and (3c).

The example does not refute the soundness of $>_{\mathsf{KV}}$ for proving AC termination; note that also $\mathsf{f}(x, y) >_{\mathsf{KV}} \mathsf{f}(\mathsf{g}(x), y)$ does not hold. We prove soundness by extending $>_{\mathsf{KV}}$ to $>_{\mathsf{KV}'}$ which has all desired properties.

Definition 11. *The order $>_{\mathsf{KV}'}$ is obtained as in Definition 9 after replacing $=^f_{\mathsf{kv}}$ by $\geqslant^f_{\mathsf{kv}'}$ in cases (3b) and (3c), and using $\geqslant_{\mathsf{kv}'}$ as preorder in $>^{\mathsf{mul}}_{\mathsf{kv}}$ in case (3a). Here the relation $\geqslant_{\mathsf{kv}'}$ is defined as follows:*

– *$s \geqslant_{\mathsf{kv}'} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and either $\mathsf{root}(s) \geqslant \mathsf{root}(t)$ or $t \in \mathcal{V}$.*

Note that $\geqslant_{\mathsf{kv}'}$ is a preorder that contains $=_{\mathsf{AC}}$.

Example 12. Consider again Example 10. We have $\mathsf{f}(\mathsf{g}(x), y) >_{\mathsf{KV}'} \mathsf{f}(x, y)$ because now case (3c) applies: $S{\restriction}^{\not<}_{\mathsf{f}} = \{\mathsf{g}(x)\} \geqslant^{\mathsf{mul}}_{\mathsf{kv}'} \{x\} = T{\restriction}^{\not<}_{\mathsf{f}} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$, $|S| = 2 = |T|$, and $S = \{\mathsf{g}(x), y\} >^{\mathsf{mul}}_{\mathsf{KV}'} \{x, y\} = T$ because $\mathsf{g}(x) >_{\mathsf{KV}'} x$.

[9] The use of a unary function of weight 0 is not crucial, see [21].

The order $>_{\mathsf{KV}'}$ is an AC-compatible simplification order. Since the inclusion $>_{\mathsf{KV}} \subseteq >_{\mathsf{KV}'}$ obviously holds, it follows that $>_{\mathsf{KV}}$ is a sound method for establishing AC termination, despite the lack of monotonicity.

Theorem 13. *The order $>_{\mathsf{KV}'}$ is an AC-compatible simplification order.*

Proof. See [21]. $\square$

The order $>_{\mathsf{KV}'}$ lacks one important feature: a polynomial-time algorithm to decide $s >_{\mathsf{KV}'} t$ when the precedence and weight function are given. By using the reduction technique of [18, Theorem 4.2], NP-hardness of this problem can be shown. Note that for KBO the problem is known to be linear [13].

Theorem 14. *The decision problem for $>_{\mathsf{KV}'}$ is NP-hard.*

Proof. See [21]. $\square$

5 AC-KBO

In this section we present another AC-compatible simplification order. In contrast to $>_{\mathsf{KV}'}$, our new order $>_{\mathsf{ACKBO}}$ contains $>_{\mathsf{S}}$. Moreover, its definition is simpler than $>_{\mathsf{KV}'}$ since we avoid the use of an auxiliary order in case (3). Finally, $>_{\mathsf{ACKBO}}$ is decidable in polynomial-time. Hence it will be used as the basis for the extension discussed in Section 7.

Definition 15. *Let $>$ be a precedence and (w, w_0) a weight function. We define $>_{\mathsf{ACKBO}}$ inductively as follows: $s >_{\mathsf{ACKBO}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:*

(0) $s = f^k(t)$ *and* $t \in \mathcal{V}$ *for some* $k > 0$,

(1) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, *and* $f > g$,

(2) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathcal{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >^{\mathsf{lex}}_{\mathsf{ACKBO}} (t_1, \ldots, t_n)$,

(3) $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathcal{F}_{\mathsf{AC}}$, *and for* $S = \nabla_f(s)$ *and* $T = \nabla_f(t)$

  (a) $S >^{f}_{\mathsf{ACKBO}} T$, *or*

  (b) $S =^{f}_{\mathsf{AC}} T$, *and* $|S| > |T|$, *or*

  (c) $S =^{f}_{\mathsf{AC}} T$, $|S| = |T|$, *and* $S{\restriction}^{<}_{f} >^{\mathsf{mul}}_{\mathsf{ACKBO}} T{\restriction}^{<}_{f}$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{ACKBO}}$ and $>^{\mathsf{mul}}_{\mathsf{ACKBO}}$.

Note that in case (3c) we compare the multisets $S{\restriction}^{<}_{f}$ and $T{\restriction}^{<}_{f}$ rather than S and T in the multiset extension of $>_{\mathsf{ACKBO}}$.

Steinbach's order is a special case of the order defined above.

Theorem 16. *If every AC symbol has minimal precedence then $>_{\mathsf{S}} = >_{\mathsf{ACKBO}}$.*

Proof. Suppose that every function symbol in $\mathcal{F}_{\mathsf{AC}}$ is minimal with respect to $>$. We show that $s >_{\mathsf{S}} t$ if and only if $s >_{\mathsf{ACKBO}} t$ by induction on s. It is clearly sufficient to consider case (3) in Definition 3 and cases (3a)–(3c) in Definition 15. So let $s = f(s_1, s_2)$ and $t = f(t_1, t_2)$ such that $w(s) = w(t)$ and $f \in \mathcal{F}_{\mathsf{AC}}$. Let $S = \nabla_f(s)$ and $T = \nabla_f(t)$.

- Let $s >_{\mathsf{S}} t$ by case (3). We have $S >_{\mathsf{S}}^{\mathsf{mul}} T$. Since $S >_{\mathsf{S}}^{\mathsf{mul}} T$ involves only comparisons $s' >_{\mathsf{S}} t'$ for subterms s' of s, the induction hypothesis yields $S >_{\mathsf{ACKBO}}^{\mathsf{mul}} T$. Because f is minimal in $>$, $S = S{\restriction}_f^{\not<} \uplus S{\restriction}_{\mathcal{V}}$ and $T = T{\restriction}_f^{\not<} \uplus T{\restriction}_{\mathcal{V}}$. For no elements $u \in S{\restriction}_{\mathcal{V}}$ and $v \in T{\restriction}_f^{\not<}$, $u >_{\mathsf{ACKBO}} v$ or $u =_{\mathsf{AC}} v$ holds. Hence $S >_{\mathsf{ACKBO}}^{\mathsf{mul}} T$ implies $S >_{\mathsf{ACKBO}}^{f} T$ or both $S =_{\mathsf{AC}}^{f} T$ and $S{\restriction}_{\mathcal{V}} \supsetneq T{\restriction}_{\mathcal{V}}$. In the former case $s >_{\mathsf{ACKBO}} t$ is due to case (3a) in Definition 15. In the latter case we have $|S| > |T|$ and $s >_{\mathsf{ACKBO}} t$ follows by case (3b).
- Let $s >_{\mathsf{ACKBO}} t$ by applying one of the cases (3a)–(3c) in Definition 15.
  - Suppose (3a) applies. Then we have $S >_{\mathsf{ACKBO}}^{f} T$. Since f is minimal in $>$, $S{\restriction}_f^{\not<} = S - S{\restriction}_{\mathcal{V}}$ and $T{\restriction}_f^{\not<} \uplus T{\restriction}_{\mathcal{V}} = T$. Hence $S >_{\mathsf{ACKBO}}^{\mathsf{mul}} (T - S{\restriction}_{\mathcal{V}}) \uplus S{\restriction}_{\mathcal{V}} \supseteq T$. We obtain $S >_{\mathsf{S}}^{\mathsf{mul}} T$ from the induction hypothesis and thus case (3) in Definition 3 applies.
  - Suppose (3b) applies. Analogous to the previous case, the inclusion $S =_{\mathsf{AC}}^{\mathsf{mul}} (T - S{\restriction}_{\mathcal{V}}) \uplus S{\restriction}_{\mathcal{V}} \supseteq T$ holds. Since $|S| > |T|$, $S =_{\mathsf{AC}}^{\mathsf{mul}} T$ is not possible. Thus $(T - S{\restriction}_{\mathcal{V}}) \uplus S{\restriction}_{\mathcal{V}} \supsetneq T$ and hence $S >_{\mathsf{S}}^{\mathsf{mul}} T$.
  - If case (3c) applies then $S{\restriction}_f^{<} >_{\mathsf{ACKBO}}^{\mathsf{mul}} T{\restriction}_f^{<}$. This is impossible since both sides are empty as f is minimal in $>$. ☐

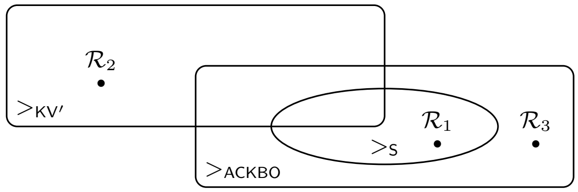
The following example shows that $>_{\mathsf{ACKBO}}$ is a proper extension of $>_{\mathsf{S}}$ and incomparable with $>_{\mathsf{KV'}}$.

Example 17. Consider the TRS $\mathcal{R}_3$ consisting of the rules

$$\begin{array}{lll} \mathsf{f}(x+y) \to \mathsf{f}(x)+y & \mathsf{h}(\mathsf{a},\mathsf{b}) \to \mathsf{h}(\mathsf{b},\mathsf{a}) & \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{a}) \to \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{b})) \\ \mathsf{g}(x)+y \to \mathsf{g}(x+y) & \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a}))) \to \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{f}(\mathsf{a})) & \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{b}) \to \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{a})) \\ \mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a}) & & \end{array}$$

over the signature $\{+, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{a}, \mathsf{b}\}$ with $+ \in \mathcal{F}_{\mathsf{AC}}$. Consider the precedence $\mathsf{f} > + > \mathsf{g} > \mathsf{a} > \mathsf{b} > \mathsf{h}$ together with the admissible weight function (w, w_0) with $w(+) = w(\mathsf{h}) = 0$, $w(\mathsf{f}) = w(\mathsf{a}) = w(\mathsf{b}) = w_0 = 1$ and $w(\mathsf{g}) = 2$. The interesting rule is $\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a})$. For $S = \nabla_+(\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}))$ and $T = \nabla_+(\mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a}))$ the multisets $S' = S{\restriction}_+^{\not<} = \{\mathsf{f}(\mathsf{a})\}$ and $T' = T{\restriction}_+^{\not<} \uplus T{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}} = \{\mathsf{f}(\mathsf{b})\}$ satisfy $S' >_{\mathsf{ACKBO}}^{\mathsf{mul}} T'$ as $\mathsf{f}(\mathsf{a}) >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{b})$, so that case (3a) of Definition 15 applies. All other rules are oriented from left to right by both $>_{\mathsf{KV'}}$ and $>_{\mathsf{ACKBO}}$, and they enforce a precedence and weight function which are identical (or very similar) to the one given above. Since $>_{\mathsf{KV'}}$ orients the rule $\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a})$ from right to left, $\mathcal{R}_3$ cannot be compatible with $>_{\mathsf{KV'}}$. It is easy to see that the rule $\mathsf{g}(x)+y \to \mathsf{g}(x+y)$ requires $+ > \mathsf{g}$, and hence $>_{\mathsf{S}}$ cannot be applied.

Fig. 1 summarizes the relationships between the orders introduced so far.



$\mathcal{R}_1$ Example 4 (and 8)
$\mathcal{R}_2$ Example 8
$\mathcal{R}_3$ Example 17

Fig. 1. Comparison.

In the following, we show that $>_{\mathsf{ACKBO}}$ is an AC-compatible simplification order. As a consequence, correctness of $>_{\mathsf{S}}$ (i.e., Theorem 5) is concluded by Theorem 16.

Lemma 18. *The pair* $(=_{\mathsf{AC}}, >_{\mathsf{ACKBO}})$ *is an order pair.*

Proof. See [21]. $\square$

The subterm property is an easy consequence of transitivity and admissibility.

Lemma 19. *The order* $>_{\mathsf{ACKBO}}$ *has the subterm property.* $\square$

Next we prove that $>_{\mathsf{ACKBO}}$ is closed under contexts. The following lemma is an auxiliary result needed for its proof. In order to reuse this lemma for the correctness proof of $>_{\mathsf{KV}'}$ in the appendix of [21], we prove it in an abstract setting.

Lemma 20. *Let* $(\succsim, \succ)$ *be an order pair and* $f \in \mathcal{F}_{\mathsf{AC}}$ *with* $f(u, v) \succ u, v$ *for all terms* u *and* v*. If* $s \succsim t$ *then* $\{s\} \succsim^{\mathsf{mul}} \nabla_f(t)$ *or* $\{s\} \succ^{\mathsf{mul}} \nabla_f(t)$*. If* $s \succ t$ *then* $\{s\} \succ^{\mathsf{mul}} \nabla_f(t)$*.*

Proof. Let $\nabla_f(t) = \{t_1, \dots, t_m\}$. If $m = 1$ then $\nabla_f(t) = \{t\}$ and the lemma holds trivially. Otherwise we get $t \succ t_j$ for all $1 \leqslant j \leqslant m$ by recursively applying the assumption. Hence $s \succ t_j$ by the transitivity of $\succ$ or the compatibility of $\succ$ and $\succsim$. We conclude that $\{s\} \succ^{\mathsf{mul}} \nabla_f(t)$. $\square$

In the following proof of closure under contexts, admissibility is essential. This is in contrast to the corresponding result for standard KBO.

Lemma 21. *If* (w, w_0) *is admissible for* $>$ *then* $>_{\mathsf{ACKBO}}$ *is closed under contexts.*

Proof. Suppose $s >_{\mathsf{ACKBO}} t$. We consider the context $h(\square, u)$ with $h \in \mathcal{F}_{\mathsf{AC}}$ and u an arbitrary term, and prove that $s' = h(s, u) >_{\mathsf{ACKBO}} h(t, u) = t'$. Closure under contexts of $>_{\mathsf{ACKBO}}$ follows then by induction; contexts rooted by a non-AC symbol are handled as in the proof for standard KBO.

If $w(s) > w(t)$ then obviously $w(s') > w(t')$. So we assume $w(s) = w(t)$. Let $S = \nabla_h(s)$, $T = \nabla_h(t)$, and $U = \nabla_h(u)$. Note that $\nabla_h(s') = S \uplus U$ and $\nabla_h(t') = T \uplus U$. Because $>_{\mathsf{ACKBO}}^{\mathsf{mul}}$ is closed under multiset sum, it suffices to show that one of the cases (3a)–(3c) of Definition 15 holds for S and T. Let $f = \mathsf{root}(s)$ and $g = \mathsf{root}(t)$. We distinguish the following cases.

- Suppose $f \nleqslant h$. We have $S = S{\upharpoonleft}_h^{\not<} = \{s\}$, and from Lemmata 19 and 20 we obtain $S >_{\mathsf{ACKBO}}^{\mathsf{mul}} T$. Since T is a superset of $T{\upharpoonleft}_h^{\not<} \uplus T{\upharpoonleft}_{\mathcal{V}} - S{\upharpoonleft}_{\mathcal{V}}$, (3a) applies.
- Suppose $f = h > g$. We have $T{\upharpoonleft}_h^{\not<} \uplus T{\upharpoonleft}_{\mathcal{V}} = \varnothing$. If $S{\upharpoonleft}_h^{\not<} \neq \varnothing$, then (3a) applies. Otherwise, since AC symbols are binary and $T = \{t\}$, $|S| \geqslant 2 > 1 = |T|$. Hence (3b) applies.
- If $f = g = h$ then $s >_{\mathsf{ACKBO}} t$ must be derived by one of the cases (3a)–(3c) for S and T.
- Suppose $f, g < h$. We have $S{\upharpoonleft}_h^{\not<} = T{\upharpoonleft}_h^{\not<} \uplus T{\upharpoonleft}_{\mathcal{V}} = \varnothing$, $|S| = |T| = 1$, and $S{\upharpoonleft}_h^{<} = \{s\} >_{\mathsf{ACKBO}}^{\mathsf{mul}} \{t\} = T{\upharpoonleft}_h^{<}$. Hence (3c) holds.

Note that $f \geqslant g$ since $w(s) = w(t)$ and $s >_{\mathsf{ACKBO}} t$. Moreover, if $t \in \mathcal{V}$ then $s = f^k(t)$ for some $k > 0$ with $w(f) = 0$, which entails $f > h$ due to admissibility. $\square$

Closure under substitutions is the trickiest part since by substituting AC-rooted terms for variables that appear in the top-flattening of a term, the structure of the term changes. In the proof, the multisets $\{t \in T \mid t \notin \mathcal{V}\}$, $\{t\sigma \mid t \in T\}$, and $\{\nabla_f(t) \mid t \in T\}$ are denoted by $T{\upharpoonleft}_{\mathcal{F}}$, $T\sigma$, and $\nabla_f(T)$, respectively.

Lemma 22. *Let $>$ be a precedence, $f \in \mathcal{F}_{\mathsf{AC}}$, and $(\succsim, \succ)$ an order pair on terms such that $\succsim$ and $\succ$ are closed under substitutions and $f(x,y) \succ x, y$. Consider terms s and t such that $S = \nabla_f(s)$, $T = \nabla_f(t)$, $S' = \nabla_f(s\sigma)$, and $T' = \nabla_f(t\sigma)$.*

(1) If $S \succ^f T$ then $S' \succ^f T'$.
(2) If $S \succsim^f T$ then $S' \succ^f T'$ or $S' \succsim^f T'$. In the latter case $|S| - |T| \leqslant |S'| - |T'|$ and $S'{\upharpoonleft}_f^{<} \succ^{\mathsf{mul}} T'{\upharpoonleft}_f^{<}$ whenever $S{\upharpoonleft}_f^{<} \succ^{\mathsf{mul}} T{\upharpoonleft}_f^{<}$.

Proof. Let v be an arbitrary term. By the assumption on $\succ$ we have either $\{v\} = \nabla_f(v)$ or both $\{v\} \succ^{\mathsf{mul}} \nabla_f(v)$ and $1 < |\nabla_f(v)|$. Hence, for any set V of terms, either $V = \nabla_f(V)$ or both $V \succ^{\mathsf{mul}} \nabla_f(V)$ and $|V| < |\nabla_f(V)|$. Moreover, for $V = \nabla_f(v)$, the following equalities hold:

$$\nabla_f(v\sigma){\upharpoonleft}_f^{\not<} = V{\upharpoonleft}_f^{\not<}\sigma \uplus \nabla_f(V{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_f^{\not<} \qquad\qquad \nabla_f(v\sigma){\upharpoonleft}_{\mathcal{V}} = \nabla_f(V{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_{\mathcal{V}}$$

To prove the lemma, assume $S \sqsupset^f T$ for $\sqsupset \in \{\succsim, \succ\}$. We have $S{\upharpoonleft}_f^{\not<} \sqsupset^{\mathsf{mul}} T{\upharpoonleft}_f^{\not<} \uplus U$ where $U = (T - S){\upharpoonleft}_{\mathcal{V}}$. Since multiset extensions preserve closure under substitutions, $S{\upharpoonleft}_f^{\not<}\sigma \sqsupset^{\mathsf{mul}} T{\upharpoonleft}_f^{\not<}\sigma \uplus U\sigma$ follows. Using the above (in)equalities, we obtain

$$\begin{array}{rll}
S'{\upharpoonleft}_f^{\not<} & = & S{\upharpoonleft}_f^{\not<}\sigma \uplus \nabla_f(S{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_f^{\not<} \\
& \sqsupset^{\mathsf{mul}} & T{\upharpoonleft}_f^{\not<}\sigma \uplus \nabla_f(S{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_f^{\not<} \uplus U\sigma \\
& O & T{\upharpoonleft}_f^{\not<}\sigma \uplus \nabla_f(S{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_f^{\not<} \uplus \nabla_f(U\sigma) \\
& = & T{\upharpoonleft}_f^{\not<}\sigma \uplus \nabla_f(S{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_f^{\not<} \uplus \nabla_f(U\sigma){\upharpoonleft}_{\mathcal{V}} \uplus \nabla_f(U\sigma){\upharpoonleft}_f^{\not<} \uplus \nabla_f(U\sigma){\upharpoonleft}_f^{<} \\
& P & T{\upharpoonleft}_f^{\not<}\sigma \uplus \nabla_f(T{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_f^{\not<} \uplus \nabla_f(U\sigma){\upharpoonleft}_{\mathcal{V}} \\
& = & T{\upharpoonleft}_f^{\not<}\sigma \uplus \nabla_f(T{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_f^{\not<} \uplus \nabla_f(T{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_{\mathcal{V}} - \nabla_f(S{\upharpoonleft}_{\mathcal{V}}\sigma){\upharpoonleft}_{\mathcal{V}} \\
& = & T'{\upharpoonleft}_f^{\not<} \uplus T'{\upharpoonleft}_{\mathcal{V}} - S'{\upharpoonleft}_{\mathcal{V}}
\end{array}$$

Here O denotes $=$ if $U\sigma = \nabla_f(U\sigma)$ and $\succ^{\mathsf{mul}}$ if $|U\sigma| < |\nabla_f(U\sigma)|$, while P denotes $=$ if $U\sigma{\restriction}_f^{<} = \varnothing$ and $\supsetneq$ otherwise. Since $(\succsim^{\mathsf{mul}}, \succ^{\mathsf{mul}})$ is an order pair with $\supseteq \subseteq \succsim^{\mathsf{mul}}$ and $\supsetneq \subseteq \succ^{\mathsf{mul}}$, we obtain $S' \sqsupset^f T'$.

It remains to show (2). If $S' \not\succ^f T'$ then O and P are both $=$ and thus $U\sigma = \nabla_f(U\sigma)$ and $U\sigma{\restriction}_f^{<} = \varnothing$. Let $X = S{\restriction}_{\mathcal{V}} \cap T{\restriction}_{\mathcal{V}}$. We have $U = T{\restriction}_{\mathcal{V}} - X$.

– Since $|W{\restriction}_{\mathcal{F}}\sigma| = |W{\restriction}_{\mathcal{F}}|$ and $|W| \leqslant |\nabla_f(W)|$ for an arbitrary set W of terms, we have $|S'| \geqslant |S| - |X| + |\nabla_f(X\sigma)|$. From $|U\sigma| = |U| = |T{\restriction}_{\mathcal{V}}| - |X|$ we obtain $|T'| = |T{\restriction}_{\mathcal{F}}\sigma| + |\nabla_f(U\sigma)| + |\nabla_f(X\sigma)| = |T| - |X| + |\nabla_f(X\sigma)|$. Hence $|S| - |T| \leqslant |S'| - |T'|$ as desired.
– Suppose $S{\restriction}_f^{<} \succ^{\mathsf{mul}} T{\restriction}_f^{<}$. From $U\sigma{\restriction}_f^{<} = \varnothing$ we infer $T{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<} \subseteq S{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<}$. Because $S'{\restriction}_f^{<} = S{\restriction}_f^{<}\sigma \uplus S{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<}$ and $T'{\restriction}_f^{<} = T{\restriction}_f^{<}\sigma \uplus T{\restriction}_{\mathcal{V}}\sigma{\restriction}_f^{<}$, closure under substitutions of $\succ^{\mathsf{mul}}$ (which it inherits from $\succ$ and $\succsim$) yields the desired $S'{\restriction}_f^{<} \succ^{\mathsf{mul}} T'{\restriction}_f^{<}$. $\square$

Lemma 23. *$>_{\mathsf{ACKBO}}$ is closed under substitutions.*

Proof. If $s >_{\mathsf{ACKBO}} t$ is obtained by cases (0)–(1) in Definition 15, the proof for standard KBO goes through. If (3a) or (3b) is used to obtain $s >_{\mathsf{ACKBO}} t$, according to Lemma 22 one of these cases also applies to $s\sigma >_{\mathsf{ACKBO}} t\sigma$. The final case is (3c). So $\nabla_f(s){\restriction}_f^{<} >_{\mathsf{ACKBO}}^{\mathsf{mul}} \nabla_f(t){\restriction}_f^{<}$. Suppose $s\sigma >_{\mathsf{ACKBO}} t\sigma$ cannot be obtained by (3a) or (3b). Lemma 22(2) yields $|\nabla_f(s\sigma)| = |\nabla_f(t\sigma)|$ and $\nabla_f(s\sigma){\restriction}_f^{<} >_{\mathsf{ACKBO}}^{\mathsf{mul}} \nabla_f(t\sigma){\restriction}_f^{<}$. Hence case (3c) is applicable to obtain $s\sigma >_{\mathsf{ACKBO}} t\sigma$. $\square$

We arrive at the main theorem of this section.

Theorem 24. *The order $>_{\mathsf{ACKBO}}$ is an AC-compatible simplification order.* $\square$

Since we deal with finite non-variadic signatures, simplification orders are well-founded. The following example shows that AC-KBO is not *incremental*, i.e., orientability is not necessarily preserved when the precedence is extended. This is in contrast to the AC-RPO of Rubio [16]. However, this is not necessarily a disadvantage; actually, the example shows that by allowing partial precedences more TRSs can be proved to be AC terminating using AC-KBO.

Example 25. Consider the TRS $\mathcal{R}$ consisting of the rules

$$\mathsf{a} \circ (\mathsf{b} \bullet \mathsf{c}) \to \mathsf{b} \circ \mathsf{f}(\mathsf{a} \bullet \mathsf{c}) \qquad\qquad \mathsf{a} \bullet (\mathsf{b} \circ \mathsf{c}) \to \mathsf{b} \bullet \mathsf{f}(\mathsf{a} \circ \mathsf{c})$$

over the signature $\mathcal{F} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{f}, \circ, \bullet\}$ with $\circ, \bullet \in \mathcal{F}_{\mathsf{AC}}$. By taking the precedence $\mathsf{f} > \mathsf{a}, \mathsf{b}, \mathsf{c}, \circ, \bullet$ and admissible weight function (w, w_0) with $w(\mathsf{f}) = w(\circ) = w(\bullet) = 0$, $w_0 = w(\mathsf{a}) = w(\mathsf{c}) = 1$, and $w(\mathsf{b}) = 2$, the resulting $>_{\mathsf{ACKBO}}$ orients both rules from left to right. It is essential that $\circ$ and $\bullet$ are incomparable in the precedence: We must have $w(\mathsf{f}) = 0$, so $\mathsf{f} > \mathsf{a}, \mathsf{b}, \mathsf{c}, \circ, \bullet$ is enforced by admissibility. If $\circ > \bullet$ then the first rule can only be oriented from left to right if $\mathsf{a} >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{a} \bullet \mathsf{c})$ holds, which contradicts the subterm property. If $\bullet > \circ$ then we use the second rule to obtain the impossible $\mathsf{a} >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{a} \bullet \mathsf{c})$. Similarly, $\mathcal{R}$ is also orientable by $>_{\mathsf{KV'}}$ but we must adopt a non-total precedence.

The final theorem in this section is easily proved.

Theorem 26. *If $>$ is total then $>_{\mathsf{ACKBO}}$ is AC-total on ground terms.* $\square$

6 NP-Hardness of Orientability

It is well-known [10] that KBO orientability is decidable in polynomial time. In this section we show that $>_{\mathsf{KV}}$ orientability is NP-hard even for ground TRSs. The corresponding result for $>_{\mathsf{ACKBO}}$ is given in the full version of this paper [21]. To this end, we reduce a SAT instance to an orientability problem.

Let $\phi = \{C_1, \ldots, C_n\}$ be a CNF SAT problem over propositional variables $p_1, \ldots, p_m$. We consider the signature $\mathcal{F}_\phi$ consisting of an AC symbol $+$, constants c and $\mathsf{d}_1, \ldots, \mathsf{d}_n$, and unary function symbols $p_1, \ldots, p_m$, a, b, and e_i^j for all $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. We define a ground TRS $\mathcal{R}_\phi$ on $\mathcal{T}(\mathcal{F}_\phi)$ such that $>_{\mathsf{KV}}$ orients $\mathcal{R}_\phi$ if and only if ϕ is satisfiable. The TRS $\mathcal{R}_\phi$ will contain the following base system $\mathcal{R}_0$ that enforces certain constraints on the precedence and the weight function:

$$\begin{array}{ccc} \mathsf{a}(\mathsf{c}+\mathsf{c}) \to \mathsf{a}(\mathsf{c})+\mathsf{c} & \mathsf{b}(\mathsf{c})+\mathsf{c} \to \mathsf{b}(\mathsf{c}+\mathsf{c}) & \mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{c}))) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(\mathsf{c}))) \\ \mathsf{a}(p_1(\mathsf{c})) \to \mathsf{b}(p_2(\mathsf{c})) \quad \cdots & \mathsf{a}(p_m(\mathsf{c})) \to \mathsf{b}(\mathsf{a}(\mathsf{c})) & \mathsf{a}(\mathsf{a}(\mathsf{c})) \to \mathsf{b}(p_1(\mathsf{c})) \end{array}$$

Lemma 27. *The order $>_{\mathsf{KV}}$ is compatible with $\mathcal{R}_0$ if and only if $\mathsf{a} > + > \mathsf{b}$ and $w(\mathsf{a}) = w(\mathsf{b}) = w(p_j)$ for all $1 \leqslant j \leqslant m$.* $\square$

Consider the clause C_i of the form $\{p'_1, \ldots, p'_k, \neg p''_1, \ldots, \neg p''_l\}$. Let U, U', V, and W denote the followings multisets:

$$\begin{array}{ll} U = \{p'_1(\mathsf{b}(\mathsf{d}_i)), \ldots, p'_k(\mathsf{b}(\mathsf{d}_i))\} & V = \{p''_0(\mathsf{e}_i^{0,1}), \ldots, p''_{l-1}(\mathsf{e}_i^{l-1,l}), p''_l(\mathsf{e}_i^{l,0})\} \\ U' = \{\mathsf{b}(p'_1(\mathsf{d}_i)), \ldots, \mathsf{b}(p'_k(\mathsf{d}_i))\} & W = \{p''_0(\mathsf{e}_i^{0,0}), \ldots, p''_l(\mathsf{e}_i^{l,l})\} \end{array}$$

where we write p''_0 for a and $\mathsf{e}_i^{j,k}$ for $\mathsf{e}_i^j(\mathsf{e}_i^k(\mathsf{c}))$. The TRS $\mathcal{R}_\phi$ is defined as the union of $\mathcal{R}_0$ and $\{\ell_i \to r_i \mid 1 \leqslant i \leqslant n\}$ with

$$\ell_i = \mathsf{b}(\mathsf{b}(\mathsf{c}+\mathsf{c})) + \textstyle\sum U + \sum V \qquad r_i = \mathsf{b}(\mathsf{c}) + \mathsf{b}(\mathsf{c}) + \sum U' + \sum W$$

Note that the symbols d_i and $\mathsf{e}_i^0, \ldots, \mathsf{e}_i^l$ are specific to the rule $\ell_i \to r_i$.

Lemma 28. *Let $\mathsf{a} > + > \mathsf{b}$. Then, $\mathcal{R}_\phi \subseteq >_{\mathsf{KV}}$ for some (w, w_0) if and only if for every i there is some p such that $p \in C_i$ with $p \not< +$ or $\neg p \in C_i$ with $+ > p$.*

Proof. For the "if" direction we reason as follows. Consider a (partial) weight function w such that $w(\mathsf{a}) = w(\mathsf{b}) = w(p_j)$ for all $1 \leqslant j \leqslant m$. We obtain $\mathcal{R}_0 \subseteq >_{\mathsf{KV}}$ from Lemma 27. Furthermore, consider $C_i = \{p'_1, \ldots, p'_k, \neg p''_1, \ldots, \neg p''_l\}$ and ℓ_i, r_i, U, V and W defined above. Let $L = \nabla_+(\ell_i)$ and $R = \nabla_+(r_i)$. We clearly have $L{\restriction}^{\not<}_{+} = U{\restriction}^{\not<}_{+} \cup V{\restriction}^{\not<}_{+}$ and $R{\restriction}^{\not<}_{+} = W{\restriction}^{\not<}_{+}$. It is easy to show that $w(\ell_i) = w(r_i)$. We show $\ell_i >_{\mathsf{KV}} r_i$ by distinguishing two cases.

1. First suppose that $p'_j \not< +$ for some $1 \leqslant j \leqslant k$. We have $p'_j(\mathsf{b}(\mathsf{d}_i)) \in U{\restriction}^{\not<}_{+}$. Extend the weight function w such that $w(\mathsf{d}_i) = 1{+}2{\cdot}\max\{w(\mathsf{e}_i^0), \ldots, w(\mathsf{e}_i^l)\}$. Then $p'_j(\mathsf{b}(\mathsf{d}_i)) >_{\mathsf{kv}} t$ for all terms $t \in W$ and hence $L{\restriction}^{\not<}_{+} >^{\mathsf{mul}}_{\mathsf{kv}} R{\restriction}^{\not<}_{+}$. Therefore $\ell_i >_{\mathsf{KV}} r_i$ by case (3a).
2. Otherwise, $U{\restriction}^{\not<}_{+} = \varnothing$ holds. By assumption $+ > p''_j$ for some $1 \leqslant j \leqslant l$. Consider the smallest m such that $+ > p''_m$. Extend the weight function w such that $w(\mathsf{e}_i^m) = 1 + 2 \cdot \max\{w(\mathsf{e}_i^j) \mid j \neq m\}$. Then $w(p''_{m-1}(\mathsf{e}_i^{m-1,m})) > w(p''_j(\mathsf{e}_i^{j,j}))$ for all $j \neq m$. From $p''_{m-1} > +$ we infer $p''_{m-1}(\mathsf{e}_i^{m-1,m}) \in V{\restriction}^{\not<}_{+}$. (Note that $p''_{m-1} = \mathsf{a} > +$ if $m = 1$.) By definition of m, $p''_m(\mathsf{e}_i^{m,m}) \notin W{\restriction}^{\not<}_{+}$. It follows that $L{\restriction}^{\not<}_{+} >^{\mathsf{mul}}_{\mathsf{kv}} R{\restriction}^{\not<}_{+}$ and thus $\ell_i >_{\mathsf{KV}} r_i$ by case (3a).

Next we prove the "only if" direction. So suppose there exists a weight function w such that $\mathcal{R}_\phi \subseteq >_{\mathsf{KV}}$. We obtain $w(\mathsf{a}) = w(\mathsf{b}) = w(p_j)$ for all $1 \leqslant j \leqslant m$ from Lemma 27. It follows that $w(\ell_i) = w(r_i)$ for every $C_i \in \phi$. Suppose for a proof by contradiction that there exists $C_i \in \phi$ such that $+ > p$ for all $p \in C_i$ and $p \not< +$ whenever $\neg p \in C_i$. So $L{\restriction}^{\not<}_{+} = V$ and $R{\restriction}^{\not<}_{+} = W$. Since $|R| = |L| + 1$, we must have $\ell_i >_{\mathsf{KV}} r_i$ by case (3a) and thus $V >_{\mathsf{kv}} W$. Let s be a term in V of maximal weight. We must have $w(s) \geqslant w(t)$ for all terms $t \in W$. By construction of the terms in V and W, this is only possible if all symbols e_i^j have the same weight. It follows that all terms in V and W have the same weight. Since $|V| = |W|$ and for every term $s' \in V$ there exists a unique term $t' \in W$ with $\mathsf{root}(s') = \mathsf{root}(t')$, we conclude $V =_{\mathsf{kv}} W$, which provides the desired contradiction. ☐

After these preliminaries we are ready to prove NP-hardness.

Theorem 29. *The (ground) orientability problem for $>_{\mathsf{KV}}$ is NP-hard.*

Proof. It is sufficient to prove that a CNF formula $\phi = \{C_1, \ldots, C_n\}$ is satisfiable if and only if the corresponding $\mathcal{R}_\phi$ is orientable by $>_{\mathsf{KV}}$. Note that the size of $\mathcal{R}_\phi$ is linear in the size of ϕ. First suppose that ϕ is satisfiable. Let α be a satisfying assignment for the atoms $p_1, \ldots, p_m$. Define the precedence $>$ as follows: $\mathsf{a} > + > \mathsf{b}$ and $p_j > +$ if $\alpha(p_j)$ is true and $+ > p_j$ if $\alpha(p_j)$ is false. Then $\mathcal{R}_\phi \subseteq >_{\mathsf{KV}}$ follows from Lemma 28. Conversely, if $\mathcal{R}_\phi$ is compatible with $>_{\mathsf{KV}}$ then we define an assignment α for the atoms in ϕ as follows: $\alpha(p)$ is true if $p \not< +$ and $\alpha(p)$ is false if $+ > p$. We claim that α satisfies ϕ. Let C_i be a clause in ϕ. According to Lemma 28, $p \not< +$ for one of the atoms p in C_i or $+ > p$ for one of the negative literals $\neg p$ in C_i. Hence α satisfies C_i by definition. ☐

7 Subterm Coefficients

Subterm coefficients were introduced in [14] in order to cope with rewrite rules like $\mathsf{f}(x) \to \mathsf{g}(x, x)$ which violate the variable condition. A *subterm coefficient function* is a partial mapping $sc\colon \mathcal{F} \times \mathbb{N} \to \mathbb{N}$ such that for a function symbol f of arity n we have $sc(f, i) > 0$ for all $1 \leqslant i \leqslant n$. Given a weight function (w, w_0)

and a subterm coefficient function sc, the weight of a term is inductively defined as follows:

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \sum\limits_{1 \leqslant i \leqslant n} s(f,i) \cdot w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

The *variable coefficient* $\mathsf{vc}(x,t)$ of a variable x in a term t is inductively defined as follows: $\mathsf{vc}(x,t) = 1$ if $t = x$, $\mathsf{vc}(x,t) = 0$ if $t \in \mathcal{V} \setminus \{x\}$, and $\mathsf{vc}(x, f(t_1, \ldots, t_n)) = sc(f,1) \cdot \mathsf{vc}(x,t_1) + \cdots + sc(f,n) \cdot \mathsf{vc}(x,t_n)$.

Definition 30. *The order $>^{sc}_{\mathsf{ACKBO}}$ is obtained from Definition 15 by replacing the condition "$|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$" with "$\mathsf{vc}(x,s) \geqslant \mathsf{vc}(x,t)$ for all $x \in \mathcal{V}$" and using the modified weight function introduced above.*

In order to guarantee AC compatibility of $>^{sc}_{\mathsf{ACKBO}}$, the subterm coefficient function sc has to assign the value 1 to arguments of AC symbols. This follows by considering the terms $t \circ (u \circ v)$ and $(t \circ u) \circ v$ for an AC symbol $\circ$ with $sc(\circ, 1) = m$ and $sc(\circ, 2) = n$. We have

$$\begin{aligned} w(t \circ (u \circ v)) &= 2 \cdot w(\circ) + m \cdot w(t) + mn \cdot w(u) + n^2 \cdot w(v) \\ w((t \circ u) \circ v) &= 2 \cdot w(\circ) + m^2 \cdot w(t) + mn \cdot w(u) + n \cdot w(v) \end{aligned}$$

Since $w(t \circ (u \circ v)) = w((t \circ u) \circ v)$ must hold for all possible terms t, u, and v, it follows that $m = m^2$ and $n^2 = n$, implying $m = n = 1$.[10] The proof of the following theorem is very similar to the one of Theorem 24 and hence omitted.

Theorem 31. *If $sc(f,1) = sc(f,2) = 1$ for every function symbol $f \in \mathcal{F}_{\mathsf{AC}}$ then $>^{sc}_{\mathsf{ACKBO}}$ is an AC-compatible simplification order.* □

Example 32. Consider the following TRS $\mathcal{R}$ with $\mathsf{f} \in \mathcal{F}_{\mathsf{AC}}$:

$$\mathsf{g}(\mathsf{0}, \mathsf{f}(x,x)) \to x \quad (1) \qquad \mathsf{g}(\mathsf{s}(x), y) \to \mathsf{g}(\mathsf{f}(x,y), \mathsf{0}) \quad (3)$$

$$\mathsf{g}(x, \mathsf{s}(y)) \to \mathsf{g}(\mathsf{f}(x,y), \mathsf{0}) \quad (2) \qquad \mathsf{g}(\mathsf{f}(x,y), \mathsf{0}) \to \mathsf{f}(\mathsf{g}(x,\mathsf{0}), \mathsf{g}(y,\mathsf{0})) \quad (4)$$

Termination of $\mathcal{R}$ was shown using AC dependency pairs in [11, Example 4.2.30]. Consider a precedence $\mathsf{g} > \mathsf{f} > \mathsf{s} > \mathsf{0}$, and weights and subterm coefficients given by $w_0 = 1$ and the following interpretation $\mathcal{A}$, mapping function symbols in $\mathcal{F}$ to linear polynomials over $\mathbb{N}$:

$$\mathsf{s}_{\mathcal{A}}(x) = x + 6 \qquad \mathsf{g}_{\mathcal{A}}(x,y) = 4x + 4y + 5 \qquad \mathsf{f}_{\mathcal{A}}(x,y) = x + y + 3 \qquad \mathsf{0}_{\mathcal{A}} = 1$$

It is easy to check that the first three rules result in a weight decrease. The left- and right-hand side of rule (4) are both interpreted as $4x+4y+21$, so both terms have weight 29, but since $\mathsf{g} > \mathsf{f}$ we conclude termination of $\mathcal{R}$ from case (1) in Definition 15 (30). Note that termination of $\mathcal{R}$ cannot be shown by AC-RPO or any of the previously considered versions of AC-KBO.

[10] This condition is also obtained by restricting [4, Proposition 4] to linear polynomials.

Table 1. Experiments on 145 termination and 67 completion problems.

	orientability			AC-DP			completion		
method	yes	time	∞	yes	time	∞	yes	time	∞
AC-KBO	32	1.7	0	66	463.1	3	25	2278.6	37
Steinbach	23	1.6	0	50	463.2	2	24	2235.4	36
Korovin & Voronkov	30	2.0	0	66	474.3	4	25	2279.4	37
KV′	30	2.1	0	66	472.4	3	25	2279.6	37
subterm coefficients	37	47.1	0	68	464.7	2	28	1724.7	26
AC-RPO	63	2.8	0	79	501.5	4	28	1701.6	26
total	72			94			31		

8 Experiments

We ran experiments on a server equipped with eight dual-core AMD Opteron® processors 885 running at a clock rate of 2.6GHz with 64GB of main memory. The different versions of AC-KBO considered in this paper as well as AC-RPO [16] were implemented on top of $\mathsf{T_TT_2}$ using encodings in SAT/SMT. These encodings resemble those for standard KBO [22] and transfinite KBO [20]. The encoding of multiset extensions of order pairs are based on [5], but careful modifications were required to deal with submultisets induced by the precedence.

For termination experiments, our test set comprises all AC problems in the *Termination Problem Data Base*,[11] all examples in this paper, some further problems harvested from the literature, and constraint systems produced by the completion tool mkbtt [19] (145 TRSs in total). The timeout was set to 60 seconds. The results are summarized in Table 1, where we list for each order the number of successful termination proofs, the total time, and the number of timeouts (column ∞). The 'orientability' column directly applies the order to orient all the rules. Although AC-RPO succeeds on more input problems, termination of 9 TRSs could only be established by (variants of) AC-KBO. We found that our definition of AC-KBO is about equally powerful as Korovin and Voronkov's order, but both are considerably more useful than Steinbach's version. When it comes to proving termination, we did not observe a difference between Definitions 9 and 11. Subterm coefficients clearly increase the success rate, although efficiency is affected. In all settings partial precedences were allowed.

The 'AC-DP' column applies the order in the AC-dependency pair framework of [1], in combination with *argument filterings* and *usable rules*. Here AC symbols in dependency pairs are *unmarked*, as proposed in [15]. In this setting the variants of AC-KBO become considerably more powerful and competitive to AC-RPO, since argument filterings relax the variable condition, as pointed out in [22].

For completion experiments, we ran the normalized completion tool mkbtt with AC-RPO and the variants of AC-KBO for termination checks on 67 equational systems collected from the literature. The overall timeout was set to 60 seconds, the timeout for each termination check to 1.5 seconds. Table 1 summa-

[11] `http://termination-portal.org/wiki/TPDB`

Table 2. Complexity results (KV is the ground version of $>_{\mathsf{KV}}$).

problem	KBO	AC-KBO	KV	KV′
membership	P	P	P	NP-hard
orientability	P	NP-hard	NP-hard	NP-hard

rizes our results, listing for each order the number of successful completions, the total time, and the number of timeouts. All experimental details, source code, and $\mathsf{T_TT_2}$ binaries are available online.[12]

The following example can be completed using AC-KBO, whereas AC-RPO does not succeed.

Example 33. Consider the following TRS $\mathcal{R}$ [15] for addition of binary numbers:

$$\begin{array}{lll} \# + \mathsf{0} \to \# & x\mathsf{0} + y\mathsf{0} \to (x + y)\mathsf{0} & x\mathsf{1} + y\mathsf{1} \to (x + y + \#\mathsf{1})\mathsf{0} \\ x + \# \to x & x\mathsf{0} + y\mathsf{1} \to (x + y)\mathsf{1} & \end{array}$$

Here $+ \in \mathcal{F}_{\mathsf{AC}}$, $\mathsf{0}$ and $\mathsf{1}$ are unary operators in postfix notation, and $\#$ denotes the empty bit sequence. For example, $\#\mathsf{100}$ represents the number 4. This TRS is not compatible with AC-RPO but AC termination can easily be shown by AC-KBO, for instance with the weight function (w, w_0) with $w(+) = 0$, $w_0 = w(\mathsf{0}) = w(\#) = 1$, and $w(\mathsf{1}) = 3$. The system can be completed into an AC convergent TRS using AC-KBO but not with AC-RPO.

9 Conclusion

We revisited the two variants of AC-compatible extensions of KBO. We extended the first version $>_{\mathsf{S}}$ introduced by Steinbach [17] to a new version $>_{\mathsf{ACKBO}}$, and presented a rigorous correctness proof. By this we conclude correctness of $>_{\mathsf{S}}$, which had been put in doubt in [9]. We also modified the order $>_{\mathsf{KV}}$ by Korovin and Voronkov [9] to a new version $>_{\mathsf{KV}'}$ which is monotone on non-ground terms, in contrast to $>_{\mathsf{KV}}$. We also presented several complexity results regarding these variants (see Table 2). While a polynomial time algorithm is known for the orientability problem of standard KBO [10], the problem becomes NP-hard even for the ground version of $>_{\mathsf{KV}}$, as well as for our $>_{\mathsf{ACKBO}}$. Somewhat unexpectedly, even deciding $>_{\mathsf{KV}'}$ is NP-hard while it is linear for standard KBO [13]. In contrast, the corresponding problem is polynomial-time for our $>_{\mathsf{ACKBO}}$. Finally, we implemented these variants of AC-compatible KBO as well as the AC-dependency pair framework of Alarcón *et al.* [1]. We presented full experimental results both for termination proving and normalized completion.

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[12] `http://cl-informatik.uibk.ac.at/software/ackbo`

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