Transforming Termination by Self-Labelling

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Abstract. We introduce a new technique for proving termination of term rewriting systems. The technique, a specialization of Zantema's semantic labelling technique, is especially useful for establishing the correctness of transformation methods that attempt to prove termination by transforming term rewriting systems into systems whose termination is easier to prove. We apply the technique to modularity, distribution elimination, and currying, resulting in new results, shorter correctness proofs, and a positive solution to an open problem.

1   Introduction

Termination is an undecidable property of term rewriting systems. In the literature (Dershowitz [4] contains an early survey of termination techniques) several methods for proving termination are described that are quite successful in practice. We can distinguish roughly two kinds of termination methods:

1. basic methods like recursive path order and polynomial interpretations that apply directly to a given term rewriting system, and
2. methods that attempt to prove termination by transforming a given term rewriting system into a term rewriting system whose termination is easier to prove, e.g. by a method of the first kind, and implies termination of the given system.

Transformation orders (Bellegarde and Lescanne [1]), distribution elimination (Zantema [19]), and semantic labelling (Zantema [18]) are examples of methods of the second kind. The starting point of the present paper is the observation that semantic labelling is in a sense too powerful. We show that any terminating term rewriting system can be transformed by semantic labelling into a system whose termination can be shown by the recursive path order. The proof of this result gives rise to a new termination method which we name *self-labelling*. We show that self-labelling is especially useful for proving the correctness of termination methods of the second kind:

1. Using self-labelling we prove a new modularity result: the extension of any terminating term rewriting system with a terminating recursive program scheme that defines new functions is again terminating.

2. Using self-labelling we give a positive solution to an open problem in [19] concerning distribution elimination: right-linearity is not necessary for the correctness of distribution elimination in the absence of distribution rules. The proof reveals how to improve distribution elimination in the absence of distribution rules.

3. Using self-labelling we give a short proof of the main result of Kennaway, Klop, Sleep, and de Vries [10], the correctness of currying, which for the purpose of this paper we view as a termination method of the second kind.

The proofs of the above results are remarkably similar.

The remainder of this paper is organized as follows. In the next section we recapitulate semantic labelling. In Sect. 3 we show that every terminating term rewriting system can be transformed by semantic labelling into a term rewriting system whose termination is very easy to prove. This completeness result gives rise to the self-labelling technique. In Sect. 4 we obtain a new modularity result with self-labelling. In Sect. 5 we use self-labelling to solve the conjecture concerning distribution elimination. The self-labelling proof gives rise to a stronger result, which we explain in Sect. 6. Our final illustration of the strength of self-labelling can be found in Sect. 7 where we present a short proof of the preservation of termination under currying.

2 Preliminaries

We assume the reader is familiar with the basics of term rewriting (as expounded in [6, 11]). This paper deals with the termination property. A term rewriting system (TRS for short) $(\mathcal{F}, \mathcal{R})$ is said to be terminating if it doesn't admit infinite rewrite sequences. It is well-known that a TRS $(\mathcal{F}, \mathcal{R})$ is terminating if and only if there exists a reduction order—a well-founded order that is closed under contexts and substitutions—on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that orients the rewrite rules of $\mathcal{R}$ from left to right. Another well-known fact states that $(\to_{\mathcal{R}} \cup \rhd)^+$ is a well-founded order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ for any terminating TRS $(\mathcal{F}, \mathcal{R})$. Here $s \rhd t$ if and only if t is a proper subterm of s. Observe that $(\to_{\mathcal{R}} \cup \rhd)^+$ is in general not a reduction order as it lacks closure under contexts. In this paper we make use of the fact that termination (confluence) is preserved under signature extension, which follows from modularity considerations ([14, 16]).

In this preliminary section we briefly recall the ingredients of semantic labelling (Zantema [18]). Actually we present a special case which is sufficient for our purposes. Let $(\mathcal{F}, \mathcal{R})$ be a TRS and $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ an $\mathcal{F}$-algebra with non-empty carrier A. Let $\succ$ be a well-founded order on A, write $\succcurlyeq$ for the union of $\succ$ and equality. We say that the pair $(\mathcal{A}, \succ)$ is a *quasi-model* for $(\mathcal{F}, \mathcal{R})$ if

1. the interpretation $f_{\mathcal{A}}$ of every n-ary function symbol $f \in \mathcal{F}$ is weakly monotone (with respect to $\succ$) in all its n coordinates, i.e., $f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \succcurlyeq f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$ for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i \succ b$, and

2. $(\mathcal{A}, \succ)$ and $(\mathcal{F}, \mathcal{R})$ are *compatible*, i.e., $[\alpha](l) \succsim [\alpha](r)$ for every rewrite rule $l \to r \in \mathcal{R}$ and assignment $\alpha: \mathcal{V} \to A$. Here $[\alpha]$ denotes the unique homomorphism from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{A}$ that extends α, i.e.,

$$[\alpha](t) = \begin{cases} \alpha(t) & \text{if } t \in \mathcal{V}, \\ f_{\mathcal{A}}([\alpha](t_1), \ldots, [\alpha](t_n)) & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

The above takes care of the semantical content of semantic labelling. We now describe the labelling part. We label function symbols from $\mathcal{F}$ with elements of A. Formally, we consider the labelled signature $\mathcal{F}_{\mathrm{lab}} = \{f_a \mid f \in \mathcal{F} \text{ and } a \in A\}$ where each f_a has the same arity as f. For every assignment α we inductively define a labelling function lab_α from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}_{\mathrm{lab}}, \mathcal{V})$ as follows:

$$\mathrm{lab}_\alpha(t) = \begin{cases} t & \text{if } t \in \mathcal{V}, \\ f_{[\alpha](t)}(\mathrm{lab}_\alpha(t_1), \ldots, \mathrm{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

So function symbols in t are simply labelled by the value (under the assignment α) of the corresponding subterms. We define the TRSs $\mathcal{R}_{\mathrm{lab}}$ and $\mathrm{dec}(\mathcal{F}, \succ)$ over the signature $\mathcal{F}_{\mathrm{lab}}$ as follows:

$$\mathcal{R}_{\mathrm{lab}} = \{\mathrm{lab}_\alpha(l) \to \mathrm{lab}_\alpha(r) \mid l \to r \in \mathcal{R} \text{ and } \alpha: \mathcal{V} \to A\},$$
$$\mathrm{dec}(\mathcal{F}, \succ) = \{f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid f \in \mathcal{F} \text{ and } a, b \in A \text{ with } a \succ b\}.$$

The following theorem is a special case of the main result of Zantema [18].

Theorem 1. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS, $\mathcal{A}$ an $\mathcal{F}$-algebra, and $\succ$ a well-founded order on the carrier of $\mathcal{A}$. If $(\mathcal{A}, \succ)$ is a quasi-model then termination of $(\mathcal{F}, \mathcal{R})$ is equivalent to termination of $(\mathcal{F}_{\mathrm{lab}}, \mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ))$.* $\square$

Observe that in the above approach the labelling part of semantic labelling is completely determined by the semantics. This is not the case for semantic labelling as defined in [18], but for our purpose it suffices.

If termination of $(\mathcal{F}_{\mathrm{lab}}, \mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ))$ is proved by means of a recursive path order, as will be the case with self-labelling, then a corresponding termination ordering for $(\mathcal{F}, \mathcal{R})$ can be described as a semantic path order as defined in [9].

3 Self-Labelling

In this section we show that every terminating TRS can be transformed by semantic labelling into a TRS whose termination is very easily established. The proof of this result forms the basis of a powerful technique for proving the correctness of transformation techniques for establishing termination.

Definition 2. A TRS $(\mathcal{F}, \mathcal{R})$ is called *precedence terminating* if there exists a well-founded order $\sqsupset$ on $\mathcal{F}$ such that $\mathrm{root}(l) \sqsupset f$ for every rewrite rule $l \to r \in \mathcal{R}$ and every function symbol $f \in \mathcal{F}\mathrm{un}(r)$.

Lemma 3. *Every precedence terminating TRS is terminating.*

Proof. Let $(\mathcal{F}, \mathcal{R})$ be a precedence terminating TRS. So there exists a well-founded order $\sqsupset$ on $\mathcal{F}$ that satisfies the condition of Definition 2. An easy induction argument on the structure of r reveals that $l \sqsupset_{\mathrm{rpo}} r$ for every $l \to r \in \mathcal{R}$. Here $\sqsupset_{\mathrm{rpo}}$ denotes the recursive path order (Dershowitz [3]) induced by the precedence $\sqsupset$. Since $\sqsupset_{\mathrm{rpo}}$ is a reduction order, termination of $(\mathcal{F}, \mathcal{R})$ follows. $\square$

The next result states that any terminating TRS can be transformed by semantic labelling into a precedence terminating TRS.

Theorem 4. *For every terminating TRS $(\mathcal{F}, \mathcal{R})$ there exists a quasi-model $(\mathcal{A}, \succ)$ such that $(\mathcal{F}_{\mathrm{lab}}, \mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ))$ is precedence terminating.*

Proof. As $\mathcal{F}$-algebra $\mathcal{A}$ we take the term algebra $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We equip $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with the well-founded order $\succ = \to_{\mathcal{R}}^+$. (Well-foundedness is an immediate consequence of the termination of $\mathcal{R}$.) Because rewriting is closed under contexts, all algebra operations are (strictly) monotone in all their coordinates. Because assignments in the term algebra $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are substitutions and rewriting is closed under substitutions, $(\mathcal{A}, \succ)$ is a quasi-model for $(\mathcal{F}, \mathcal{R})$. It remains to show that $(\mathcal{F}_{\mathrm{lab}}, \mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ))$ is precedence terminating. To this end we define a well-founded order $\sqsupset$ on $\mathcal{F}_{\mathrm{lab}}$ as follows: $f_s \sqsupset g_t$ if and only if $s \ (\to_{\mathcal{R}} \cup \rhd)^+ \ t$. Let $l \to r$ be a rewrite rule of $\mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ)$.

1. If $l \to r \in \mathcal{R}_{\mathrm{lab}}$ then there exist an assignment $\alpha \colon \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a rewrite rule $l' \to r' \in \mathcal{R}$ such that $l = \mathrm{lab}_\alpha(l')$ and $r = \mathrm{lab}_\alpha(r')$. The label of $\mathrm{root}(l)$ is $[\alpha](l') = l'\alpha$. Let ℓ be the label of a function symbol in r. By construction $\ell = [\alpha](t) = t\alpha$ for some subterm t of r'. Hence $l'\alpha \to_{\mathcal{R}} r'\alpha \trianglerighteq \ell$. So $\mathrm{root}(l) \sqsupset f$ for every $f \in \mathcal{F}\mathrm{un}(r)$.
2. If $l \to r \in \mathrm{dec}(\mathcal{F}, \succ)$ then $l = f_s(x_1, \ldots, x_n)$ and $r = f_t(x_1, \ldots, x_n)$ with $s \to_{\mathcal{R}}^+ t$. Clearly $\mathrm{root}(l) = f_s \sqsupset f_t$.

$\square$

The particular use of semantic labelling in the above proof (i.e., choosing the term algebra as semantics and thus labelling function symbols with terms) is what we will call *self-labelling*. One may argue that Theorem 4 is completely useless, since the construction of the quasi-model in the proof relies on the fact that $(\mathcal{F}, \mathcal{R})$ is terminating. Nevertheless, in the following sections we will see how self-labelling gives rise to many new results and significant simplifications of existing results on the correctness of transformation techniques for establishing termination. Below we sketch the general framework.

Let Φ be a transformation on TRSs, designed to make the task of proving termination easier. In two of the three applications we give, the TRS $\Phi(\mathcal{F}, \mathcal{R})$ is a subsystem of $(\mathcal{F}, \mathcal{R})$. The crucial point is proving correctness of the transformation, i.e., proving that termination of $\Phi(\mathcal{F}, \mathcal{R})$ implies termination of the original TRS $(\mathcal{F}, \mathcal{R})$. Write $\Phi(\mathcal{F}, \mathcal{R}) = (\mathcal{F}', \mathcal{R}')$. The basic idea is to label the TRS $(\mathcal{F}, \mathcal{R})$ with terms of $(\mathcal{F}', \mathcal{R}')$. This is achieved by executing the following steps:

1. Turn the term algebra $\mathcal{T}(\mathcal{F}', \mathcal{V})$ into an $\mathcal{F}$-algebra $\mathcal{A}$ by choosing *suitable* interpretations for the function symbols in $\mathcal{F} \setminus \mathcal{F}'$ and taking term construction as interpretation of the function symbols in $\mathcal{F} \cap \mathcal{F}'$.
2. Equip the $\mathcal{F}$-algebra $\mathcal{A}$ with the well-founded order $\succ = \rightarrow_{\mathcal{R}'}^{+}$.
3. Show that $(\mathcal{A}, \succ)$ is a quasi-model for $(\mathcal{F}, \mathcal{R})$.
4. Define $f_s \sqsupset g_t$ if and only if $s \, (\rightarrow_{\mathcal{R}'} \cup \rhd)^{+} \, t$, for $f, g \in \mathcal{F} \cap \mathcal{F}'$ and extend this to a well-founded order $\sqsupset$ on $\mathcal{F}_{\mathrm{lab}}$ such that the TRS $(\mathcal{F}_{\mathrm{lab}}, \mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ))$ is precedence terminating with respect to $\sqsupset$.

At this point termination of $(\mathcal{F}, \mathcal{R})$ and thus the correctness of the transformation Φ is a consequence of Theorem 1.

We would like to stress that the only creative step in this scheme is the choice of the interpretations for the function symbols that disappear during the transformation Φ; the choice of $\sqsupset$ will then be implied from the requirement of precedence termination.

4 Modularity

Our first application of self-labelling is a new modularity result. Modularity is concerned with the preservation of properties under combinations of TRSs. Recently the focus in modularity research (Ohlebusch [15] contains a recent overview) has shifted to so-called *hierarchical* combinations ([5, 12, 13]). We will prove the following result: the combination of an arbitrary terminating TRS and a terminating recursive program scheme that defines new functions is terminating. A *recursive program scheme* (RPS for short) is a TRS $(\mathcal{F}, \mathcal{R})$ whose rewrite rules have the form

$$f(x_1, \ldots, x_n) \rightarrow t$$

with $x_1, \ldots, x_n$ pairwise distinct variables such that for every function symbol $f \in \mathcal{F}$ there is at most one such rule. The subset of $\mathcal{F}$ consisting of all f such that there is a corresponding rule in $\mathcal{R}$ is denoted by $\mathcal{F}_{\mathcal{D}}$. In the literature RPSs are assumed to be finite, but we don't need that restriction here. From a rewriting point of view RPSs are quite simple: every RPS is confluent and termination of RPSs is decidable. Moreover, the normals forms of an RPS $(\mathcal{F}, \mathcal{R})$ constitute the set $\mathcal{T}(\mathcal{F} \setminus \mathcal{F}_{\mathcal{D}}, \mathcal{V})$ of terms that do not contain function symbols in $\mathcal{F}_{\mathcal{D}}$. Below we make use of the following fact.

Lemma 5. *An RPS is terminating if and only if it is precedence terminating.*

Proof. The "if" direction is trivial. Let $(\mathcal{F}, \mathcal{R})$ be a terminating RPS. Define a binary relation $\rightarrowtail$ on $\mathcal{F}$ as follows: $f \rightarrowtail g$ if and only if there exists a rewrite rule $l \rightarrow r$ in $\mathcal{R}$ such that $\mathrm{root}(l) = f$ and $g \in \mathcal{F}\mathrm{un}(r)$. Termination of $(\mathcal{F}, \mathcal{R})$ implies that $\rightarrowtail^{+}$ is a well-founded order on $\mathcal{F}$. Hence $(\mathcal{F}, \mathcal{R})$ is precedence terminating with respect to $\rightarrowtail^{+}$. $\square$

Theorem 6. *Let $(\mathcal{F}, \mathcal{R})$ be a terminating TRS and $(\mathcal{G}, \mathcal{S})$ a terminating RPS satisfying $\mathcal{F} \cap \mathcal{G}_{\mathcal{D}} = \varnothing$. Then $(\mathcal{F} \cup \mathcal{G}, \mathcal{R} \cup \mathcal{S})$ is terminating.*

Proof. Let $\mathcal{F}' = \mathcal{F} \cup (\mathcal{G} \setminus \mathcal{G}_\mathcal{D})$. Using the technique of self-labelling, we show how termination of $(\mathcal{F} \cup \mathcal{G}, \mathcal{R} \cup \mathcal{S})$ follows from termination of $(\mathcal{F}', \mathcal{R})$. We turn $\mathcal{T}(\mathcal{F}', \mathcal{V})$ into an $\mathcal{F} \cup \mathcal{G}$-algebra $\mathcal{A}$ by defining $f_\mathcal{A}$ for every $f \in \mathcal{G}_\mathcal{D}$ as follows:

$$f_\mathcal{A}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)\downarrow_\mathcal{S}$$

for all terms $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}', \mathcal{V})$. Here $f(t_1, \ldots, t_n)\downarrow_\mathcal{S}$ denotes the (unique) normal form of $f(t_1, \ldots, t_n)$ with respect to the complete, i.e. confluent and terminating, RPS $(\mathcal{F} \cup \mathcal{G}, \mathcal{S})$. Note that $f(t_1, \ldots, t_n)\downarrow_\mathcal{S} \in \mathcal{T}(\mathcal{F}', \mathcal{V})$. As well-founded order $\succ$ on $\mathcal{T}(\mathcal{F}', \mathcal{V})$ we take $\rightarrow_\mathcal{R}^+$. We claim that $(\mathcal{A}, \succ)$ is a quasi-model for $(\mathcal{F} \cup \mathcal{G}, \mathcal{R} \cup \mathcal{S})$.

First we show by induction on $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{G}, \mathcal{V})$ that $[\alpha](t) = t\alpha\downarrow_\mathcal{S}$ for all assignments $\alpha\colon \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}', \mathcal{V})$. If $t \in \mathcal{V}$ then $[\alpha](t) = \alpha(t) = t\alpha = t\alpha\downarrow_\mathcal{S}$ because $t\alpha \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ is a normal form with respect to $\mathcal{S}$. Let $t = f(t_1, \ldots, t_n)$. We have $[\alpha](t) = f_\mathcal{A}([\alpha](t_1), \ldots, [\alpha](t_n))$. From the induction hypothesis we obtain $[\alpha](t_i) = t_i\alpha\downarrow_\mathcal{S}$ for all $i \in \{1, \ldots, n\}$ and thus $[\alpha](t) = f_\mathcal{A}(t_1\alpha\downarrow_\mathcal{S}, \ldots, t_n\alpha\downarrow_\mathcal{S})$. If $f \notin \mathcal{G}_\mathcal{D}$ then $[\alpha](t) = f(t_1\alpha\downarrow_\mathcal{S}, \ldots, t_n\alpha\downarrow_\mathcal{S})$ and if $f \in \mathcal{G}_\mathcal{D}$ then $[\alpha](t) = f(t_1\alpha\downarrow_\mathcal{S}, \ldots, t_n\alpha\downarrow_\mathcal{S})\downarrow_\mathcal{S}$. In both cases we have $[\alpha](t) = f(t_1\alpha, \ldots, t_n\alpha)\downarrow_\mathcal{S} = t\alpha\downarrow_\mathcal{S}$. The above property enables us to prove compatibility of $(\mathcal{A}, \succ)$ and $(\mathcal{F} \cup \mathcal{G}, \mathcal{R} \cup \mathcal{S})$. Let $l \rightarrow r \in \mathcal{R} \cup \mathcal{S}$ and $\alpha\colon \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}', \mathcal{V})$. We have to show that $[\alpha](l) \succcurlyeq [\alpha](r)$. If $l \rightarrow r \in \mathcal{R}$ then $[\alpha](l) = l\alpha \rightarrow_\mathcal{R} r\alpha = [\alpha](r)$. If $l \rightarrow r \in \mathcal{S}$ then $[\alpha](l) = l\alpha\downarrow_\mathcal{S}$ and $[\alpha](r) = r\alpha\downarrow_\mathcal{S}$. Because $l\alpha \rightarrow_\mathcal{S} r\alpha$, confluence of $\mathcal{S}$ yields $[\alpha](l) = [\alpha](r)$.

We now show that every algebra operation is weakly monotone in all its coordinates. For $f_\mathcal{A}$ with $f \in \mathcal{F}'$ this is a consequence of closure under contexts of the rewrite relation $\rightarrow_\mathcal{R}$. Let f be an n-ary function symbol in $\mathcal{G}_\mathcal{D}$ and $s_1, \ldots, s_n, t \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ such that $s_i \succ t$. Here i is an arbitrary element of $\{1, \ldots, n\}$. We show that $f_\mathcal{A}(s_1, \ldots, s_i, \ldots, s_n) \succcurlyeq f_\mathcal{A}(s_1, \ldots, t, \ldots, s_n)$. To this end we make use of the fact that $t\alpha\downarrow_\mathcal{S} = t\downarrow_\mathcal{S}\alpha$ for all terms $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{G}, \mathcal{V})$ and assignments $\alpha\colon \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}', \mathcal{V})$. This property is an easy consequence of the special structure of the left-hand sides of the rewrite rules of the RPS $\mathcal{S}$. Let z be a fresh variable and define $s = f(s_1, \ldots, z, \ldots, s_n)$. We have $f_\mathcal{A}(s_1, \ldots, s_i, \ldots, s_n) = s\alpha\downarrow_\mathcal{S} = s\downarrow_\mathcal{S}\alpha$ and $f_\mathcal{A}(s_1, \ldots, t, \ldots, s_n) = s\beta\downarrow_\mathcal{S} = s\downarrow_\mathcal{S}\beta$. Here the substitutions (assignments) α and β are defined by $\alpha = \{z \mapsto s_i\}$ and $\beta = \{z \mapsto t\}$. Because $\alpha(x) \succcurlyeq \beta(x)$ for every variable x, the desired $s\downarrow_\mathcal{S}\alpha \succcurlyeq s\downarrow_\mathcal{S}\beta$ is a consequence of closure under contexts of the rewrite relation $\rightarrow_\mathcal{R}$.

It remains to show that $\mathcal{R}_{\mathrm{lab}} \cup \mathcal{S}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F} \cup \mathcal{G}, \succ)$ is precedence terminating. To this end we equip the labelled signature $\mathcal{F}_{\mathrm{lab}} \cup \mathcal{G}_{\mathrm{lab}}$ with a proper order $\sqsupset$ defined as follows: $f_s \sqsupset g_t$ if and only if

1. $s \, (\rightarrow_\mathcal{R} \cup \rhd)^+ \, t$ and either $f, g \in \mathcal{F}'$ or $f, g \in \mathcal{G}_\mathcal{D}$, or
2. $f \in \mathcal{G}_\mathcal{D}$, $g \in \mathcal{G}$, and $f \blacktriangleright g$.

Here $\blacktriangleright$ is any well-founded order on $\mathcal{G}$ such that $\mathcal{S}$ is precedence terminating with respect to $\blacktriangleright$. The existence of $\blacktriangleright$ is guaranteed by Lemma 5. From well-foundedness of $(\rightarrow_\mathcal{R} \cup \rhd)^+$ and $\blacktriangleright$ it follows that $\sqsupset$ is a well-founded order on $\mathcal{F}_{\mathrm{lab}} \cup \mathcal{G}_{\mathrm{lab}}$. The rewrite rules in $\mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F} \cup \mathcal{G}, \succ)$ are taken care of by the

first clause of the definition of $\sqsupseteq$, just as in the proof of Theorem 4. For the rules in $\mathcal{S}_{\mathrm{lab}}$ we use the second clause. □

We would like to remark that neither the results of Krishna Rao [12, 13] nor the colorful theorems of Dershowitz [5] apply, because we don't put any restrictions on the base system $\mathcal{R}$. One easily shows that $\mathcal{S}$ quasi-commutes ([2]) over right-linear $\mathcal{R}$, but this doesn't hold for arbitrary TRSs $\mathcal{R}$.

As a very special case of Theorem 6 we mention that the disjoint union of any terminating TRS $\mathcal{R}$ and the TRS $\mathcal{S}$ consisting of the single projection rule $g(x, y) \to x$ is terminating. This is to be contrasted with the celebrated counterexample of Toyama [17] against the preservation of termination under disjoint unions in which one of the TRSs consists of both projection rules $g(x, y) \to x$ and $g(x, y) \to y$.

5 Distribution Elimination

Our second application of self-labelling is the proof of a conjecture of Zantema [19] concerning distribution elimination.

Let $(\mathcal{F}, \mathcal{R})$ be a TRS and let $e \in \mathcal{F}$ be a designated function symbol whose arity is at least one. A rewrite rule $l \to r \in \mathcal{R}$ is called a *distribution rule* for e if $l = C[e(x_1, \ldots, x_n)]$ and $r = e(C[x_1], \ldots, C[x_n])$ for some non-empty context C in which e doesn't occur and pairwise different variables $x_1, \ldots, x_n$. Distribution elimination is a technique that transforms $(\mathcal{F}, \mathcal{R})$ by eliminating all distribution rules for e and removing the symbol e from the right-hand sides of the other rules. First we inductively define a mapping E_{distr} that assigns to every term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ a non-empty subset of $\mathcal{T}(\mathcal{F} \setminus \{e\}, \mathcal{V})$, as follows:

$$
E_{\mathrm{distr}}(t) = \begin{cases}
\{t\} & \text{if } t \in \mathcal{V}, \\
\displaystyle\bigcup_{i=1}^{n} E_{\mathrm{distr}}(t_i) & \text{if } t = e(t_1, \ldots, t_n), \\
\{f(s_1, \ldots, s_n) \mid s_i \in E_{\mathrm{distr}}(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \text{ with } f \neq e.
\end{cases}
$$

The mapping E_{distr} is illustrated in Fig. 1, where we assume that the numbered contexts do not contain any occurrences of e. It is extended to rewrite systems as follows: $E_{\mathrm{distr}}(\mathcal{R}) = \{l \to r' \mid l \to r \in \mathcal{R} \text{ is not a distribution rule for } e \text{ and } r' \in E_{\mathrm{distr}}(r)\}$. Observe that e does not occur in $E_{\mathrm{distr}}(\mathcal{R})$ if and only if e does not occur in the left-hand sides of rewrite rules of $\mathcal{R}$ that are not distribution rules for e.

One of the main results of Zantema [19] is stated below.

Theorem 7. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS and let $e \in \mathcal{F}$ be a non-constant symbol which does not occur in the left-hand sides of rewrite rules of $\mathcal{R}$ that are not distribution rules for e.*

  1. If $E_{\mathrm{distr}}(\mathcal{R})$ is terminating and right-linear then $\mathcal{R}$ is terminating.
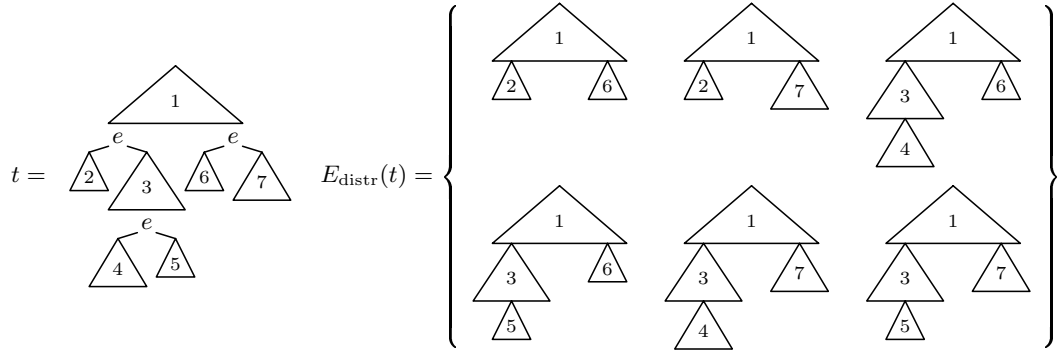
Fig. 1. The mapping E_{distr}.

2. If $E_{\mathrm{distr}}(\mathcal{R})$ is simply terminating and right-linear then $\mathcal{R}$ is simply terminating.
3. If $E_{\mathrm{distr}}(\mathcal{R})$ is totally terminating then $\mathcal{R}$ is totally terminating.

$\square$

The following example from [19] shows that right-linearity is essential in parts 1 and 2.

Example 1. Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} f(a,b,x) \to f(x,x,e(a,b)) \\ f(e(x,y),z,w) \to e(f(x,z,w),f(y,z,w)) \\ f(x,e(y,z),w) \to e(f(x,y,w),f(x,z,w)) \end{array} \right\}.$$

The last two rules are distribution rules for e and e does not occur in the left-hand side of the first rule. The TRS $E_{\mathrm{distr}}(\mathcal{R}) = \{f(a,b,x) \to f(x,x,a), f(a,b,x) \to f(x,x,b)\}$ can be shown to be simply terminating, while the term $f(a,b,e(a,b))$ admits an infinite reduction in $\mathcal{R}$.

In [19] it is conjectured that in the absence of distribution rules for e the right-linearity assumption in part 1 of Theorem 7 can be omitted. Before proving this conjecture with the technique of self-labelling, we show that a similar statement for simple termination doesn't hold, i.e., right-linearity is essential in part 2 of Theorem 7 even in the absence of distribution rules for e.

Example 2. Let $\mathcal{R}'$ consist of the first rule of the TRS $\mathcal{R}$ of Example 1. Simple termination of $E_{\mathrm{distr}}(\mathcal{R}') = E_{\mathrm{distr}}(\mathcal{R})$ was established in Example 1, but $\mathcal{R}'$ fails to be simply terminating as $s = f(a,b,e(a,b)) \to_{\mathcal{R}'} f(e(a,b),e(a,b),e(a,b)) = t$ with s embedded in t. However, termination of $\mathcal{R}'$ follows from Theorem 8 below.

Theorem 8. *Let $(\mathcal{F},\mathcal{R})$ be a TRS and let $e \in \mathcal{F}$ be a non-constant symbol which does not occur in the left-hand sides of rewrite rules of $\mathcal{R}$. If $E_{\mathrm{distr}}(\mathcal{R})$ is terminating then $\mathcal{R}$ is terminating.*

Proof. We turn the term algebra $\mathcal{T}(\mathcal{F} \setminus \{e\}, \mathcal{V})$ into an $\mathcal{F}$-algebra $\mathcal{A}$ by defining

$$e_{\mathcal{A}}(t_1, \ldots, t_n) = t_{\pi}$$

for all terms $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F} \setminus \{e\}, \mathcal{V})$. Here π is an *arbitrary* but fixed element of $\{1, \ldots, n\}$. So $e_{\mathcal{A}}$ is simply projection onto the π-th coordinate. We equip $\mathcal{A}$ with the well-founded order $\succ = \to^{+}_{E_{\mathrm{distr}}(\mathcal{R})}$. We show that $(\mathcal{A}, \succ)$ is a quasi-model for $(\mathcal{F}, \mathcal{R})$. It is very easy to see that $e_{\mathcal{A}}$ is weakly monotone in all its coordinates. All other operations are strictly monotone in all their coordinates (as $\to^{+}_{E_{\mathrm{distr}}(\mathcal{R})}$ is closed under contexts). Let ε be the identity assignment from $\mathcal{V}$ to $\mathcal{V}$. We denote $[\varepsilon](t)$ by $\langle t \rangle$. An easy induction proof shows that $[\alpha](t) = \langle t \rangle \alpha$ for all terms $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and assignments $\alpha \colon \mathcal{V} \to \mathcal{T}(\mathcal{F} \setminus \{e\}, \mathcal{V})$. Also the following two properties are easily shown by induction on the structure of $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$: 1. $\langle t \rangle \in E_{\mathrm{distr}}(t)$ and 2. if $s \trianglelefteq t$ then there exists a term $t' \in E_{\mathrm{distr}}(t)$ such that $\langle s \rangle \trianglelefteq t'$.

1. If $t \in \mathcal{V}$ then $\langle t \rangle = t$ and $E_{\mathrm{distr}}(t) = \{t\}$. For the induction step we distinguish two cases. If $t = e(t_1, \ldots, t_n)$ then $\langle t \rangle = \langle t_{\pi} \rangle$ and $E_{\mathrm{distr}}(t) = \bigcup_{i=1}^{n} E_{\mathrm{distr}}(t_i)$. We have $\langle t_{\pi} \rangle \in E_{\mathrm{distr}}(t_{\pi})$ according to the induction hypothesis. Hence $\langle t \rangle \in E_{\mathrm{distr}}(t)$. If $t = f(t_1, \ldots, t_n)$ with $f \neq e$ then $\langle t \rangle = f(\langle t_1 \rangle, \ldots, \langle t_n \rangle)$ and $E_{\mathrm{distr}}(t) = \{f(s_1, \ldots, s_n) \mid s_i \in E_{\mathrm{distr}}(t_i)\}$. The induction hypothesis yields $\langle t_i \rangle \in E_{\mathrm{distr}}(t_i)$ for all $i = 1, \ldots, n$. Hence also in this case we obtain the desired $\langle t \rangle \in E_{\mathrm{distr}}(t)$.
2. Observe that for $s = t$ the statement follows from property 1 because we can take $t' = \langle t \rangle$. This observation also takes care of the base of the induction. Suppose $t = f(t_1, \ldots, t_n)$ and let s be a proper subterm of t, so s is a subterm of t_k for some $k \in \{1, \ldots, n\}$. From the induction hypothesis we obtain a term $t'_k \in E_{\mathrm{distr}}(t_k)$ such that $\langle s \rangle \trianglelefteq t'_k$. Again we distinguish two cases. If $f = e$ then $E_{\mathrm{distr}}(t) = \bigcup_{i=1}^{n} E_{\mathrm{distr}}(t_i)$ and thus we can take $t' = t'_k$. If $f \neq e$ then $E_{\mathrm{distr}}(t) = \{f(s_1, \ldots, s_n) \mid s_i \in E_{\mathrm{distr}}(t_i)\}$. Let $t' = f(\langle t_1 \rangle, \ldots, t'_k, \ldots, \langle t_n \rangle)$. Using property 1 we infer that $t' \in E_{\mathrm{distr}}(t)$. Clearly $\langle s \rangle \trianglelefteq t'$.

Now let $l \to r$ be an arbitrary rewrite rule of $\mathcal{R}$ and $\alpha \colon \mathcal{V} \to \mathcal{T}(\mathcal{F} \setminus \{e\}, \mathcal{V})$ an arbitrary assignment. We have $[\alpha](l) = \langle l \rangle \alpha$ and $[\alpha](r) = \langle r \rangle \alpha$. Since e doesn't occur in l, $\langle l \rangle = l$ and hence $[\alpha](l) = l\alpha$. Because $\langle r \rangle \in E_{\mathrm{distr}}(r)$, the rule $l \to \langle r \rangle$ belongs to $E_{\mathrm{distr}}(\mathcal{R})$. Therefore $l\alpha \to_{E_{\mathrm{distr}}(\mathcal{R})} \langle r \rangle \alpha$ and thus also $[\alpha](l) \succcurlyeq [\alpha](r)$.

Define a well-founded order $\sqsupset$ on $\mathcal{F}_{\mathrm{lab}}$ as follows: $f_s \sqsupset g_t$ if and only if $s \, (\to_{E_{\mathrm{distr}}(\mathcal{R})} \cup \vartriangleright)^{+} t$. We will show that $(\mathcal{F}_{\mathrm{lab}}, \mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ))$ is precedence terminating with respect to $\sqsupset$. Let $l \to r$ be a rewrite rule in $\mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}, \succ)$. We distinguish two cases. If $l \to r \in \mathcal{R}_{\mathrm{lab}}$ then there exist an assignment $\alpha \colon \mathcal{V} \to \mathcal{T}(\mathcal{F} \setminus \{e\}, \mathcal{V})$ and a rewrite rule $l' \to r' \in \mathcal{R}$ such that $l = \mathrm{lab}_{\alpha}(l')$ and $r = \mathrm{lab}_{\alpha}(r')$. The label of $\mathrm{root}(l)$ is $[\alpha](l') = \langle l' \rangle \alpha = l'\alpha$. Let ℓ be the label of a function symbol in r. By construction $\ell = [\alpha](t) = \langle t \rangle \alpha$ for some subterm t of r'. According to property 2 above, $\langle t \rangle$ is a subterm of some $r'' \in E_{\mathrm{distr}}(r)$. By definition $l' \to r'' \in E_{\mathrm{distr}}(\mathcal{R})$. Hence $l'\alpha \to_{E_{\mathrm{distr}}(\mathcal{R})} r''\alpha \trianglerighteq \ell$. So $\mathrm{root}(l) \sqsupset f$ for every $f \in \mathcal{F}\mathrm{un}(r)$. If $l \to r \in \mathrm{dec}(\mathcal{F}, \succ)$ then $l = f_s(x_1, \ldots, x_n)$ and $r = f_t(x_1, \ldots, x_n)$ with $f \in \mathcal{F}$ and $s \succ t$. In this case we clearly have $\mathrm{root}(l) = f_s \sqsupset f_t$. $\qquad\square$

The only creative element in the above proof is the choice of $e_{\mathcal{A}}$. The rest is a routine verification of the two proof obligations of self-labelling.

6 Distribution Elimination Revisited

In the proof of Theorem 8 we saw that we can take any projection function as semantics for e. This freedom makes it possible to improve distribution elimination (in the absence of distribution rules) by reducing the size of $E_{\mathrm{distr}}(\mathcal{R})$ while preserving correctness of the transformation.

What are the essential properties of E_{distr} that make the proof of Theorem 8 work? A careful inspection reveals, apart from the obvious termination requirement for $E_{\mathrm{distr}}(\mathcal{R})$, the following two properties:

1. $\langle t \rangle \in E_{\mathrm{distr}}(t)$, and
2. if $s \trianglelefteq t$ then there exists a term $t' \in E_{\mathrm{distr}}(t)$ such that $\langle s \rangle \trianglelefteq t'$,

for every $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Below we define a new transformation E^{π}_{distr} that satisfies these two properties. The transformation is parameterized by the argument positions π of the function symbol e. The definition relies on the $\mathcal{F}$-algebra defined in the proof of Theorem 8 in that we use $\langle t \rangle$.

Definition 9. Let $(\mathcal{F}, \mathcal{R})$ be a TRS and let $e \in \mathcal{F}$ be a function symbol whose arity is at least one. Fix $\pi \in \{1, \ldots, \mathrm{arity}(e)\}$. We inductively define mappings ϕ and E^{π}_{distr} that assigns to every term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ a subset of $\mathcal{T}(\mathcal{F} \setminus \{e\}, \mathcal{V})$, as follows:

$$
\phi(t) = \begin{cases}
\varnothing & \text{if } t \in \mathcal{V}, \\
\phi(t_{\pi}) \cup \bigcup_{i \neq \pi} E^{\pi}_{\mathrm{distr}}(t_i) & \text{if } t = e(t_1, \ldots, t_n), \\
\bigcup_{i=1}^{n} \phi(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ with } f \neq e,
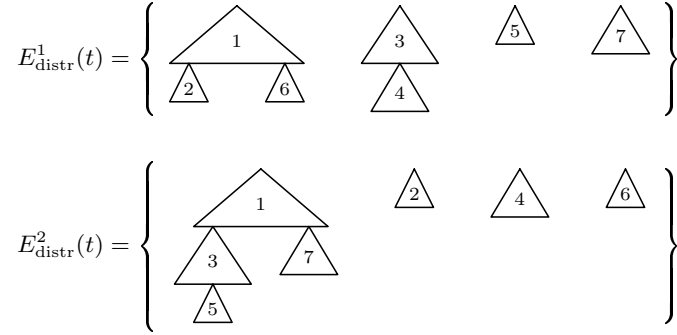\end{cases}
$$

and

$$
E^{\pi}_{\mathrm{distr}}(t) = \phi(t) \cup \{\langle t \rangle\}.
$$

We extend the mapping E^{π}_{distr} to $\mathcal{R}$ as follows: $E^{\pi}_{\mathrm{distr}}(\mathcal{R}) = \{l \to r' \mid l \to r \in \mathcal{R}$ is not a distribution rule for e and $r' \in E^{\pi}_{\mathrm{distr}}(r)\}$.

Figure 2 shows the effect of E^1_{distr} and E^2_{distr} on the term t of Fig. 1. Observe that each numbered context occurs exactly once in each set. The following lemma states that E^{π}_{distr} has the two required properties.

Lemma 10. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS and let e and π be as above. For every $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ we have*

1. $\langle t \rangle \in E^{\pi}_{\mathrm{distr}}(t)$, *and*
2. *if $s \trianglelefteq t$ then there exists a term $t' \in E^{\pi}_{\mathrm{distr}}(t)$ such that $\langle s \rangle \trianglelefteq t'$.*

$$E_{\text{distr}}^1(t) = \left\{ \vphantom{\begin{array}{c}1\\2\\6\\4\end{array}} \right.$$

$$E_{\text{distr}}^2(t) = \left\{ \vphantom{\begin{array}{c}1\\3\\7\\5\end{array}} \right.$$

Fig. 2. The mapping E_{distr}^π.

Proof. The first statement holds by definition. The second statement we prove by induction on the structure of $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. If $s = t$ then the result follows from the first statement. Hence we may assume that $s \lhd t$. This is only possible if t is a non-variable term $f(t_1, \ldots, t_n)$. There exists a $k \in \{1, \ldots, n\}$ such that $s \trianglelefteq t_k$. The induction hypothesis yields a term $t_k' \in E_{\text{distr}}^\pi(t_k) = \phi(t_k) \cup \{\langle t_k \rangle\}$ such that $\langle s \rangle \trianglelefteq t_k'$. We distinguish two cases. Suppose $f = e$. In this case we have

$$E_{\text{distr}}^\pi(t) = \phi(t) \cup \{\langle t \rangle\} = \phi(t_\pi) \cup \{\langle t \rangle\} \cup \bigcup_{i \neq \pi} E_{\text{distr}}^\pi(t_i).$$

If $k = \pi$ then $t_k' \in \phi(t_\pi) \cup \{\langle t_k \rangle\} = \phi(t_\pi) \cup \{\langle t \rangle\} \subseteq E_{\text{distr}}^\pi(t)$. If $k \neq \pi$ then $t_k' \in E_{\text{distr}}^\pi(t_k) \subseteq E_{\text{distr}}^\pi(t)$. Hence in both cases we can take $t' = t_k'$. Suppose $f \neq e$. We have

$$E_{\text{distr}}^\pi(t) = \phi(t) \cup \{\langle t \rangle\} = \bigcup_{i=1}^n \phi(t_i) \cup \{f(\langle t_1 \rangle, \ldots, \langle t_n \rangle)\}.$$

If $t_k' \in \phi(t_k)$ then clearly $t_k' \in E_{\text{distr}}^\pi(t)$ and hence we can take $t' = t_k'$. If $t_k' = \langle t_k \rangle$ then we take $t' = f(\langle t_1 \rangle, \ldots, \langle t_n \rangle)$ which satisfies $\langle s \rangle \trianglelefteq t_k' \trianglelefteq t'$. □

Hence we obtain the following result along the lines of the proof of Theorem 8.

Theorem 11. *Let $(\mathcal{F}, \mathcal{R})$ be a TRS and let $e \in \mathcal{F}$ be a non-constant symbol which does not occur in the left-hand sides of rewrite rules of $\mathcal{R}$. If $E_{\text{distr}}^\pi(\mathcal{R})$ is terminating for some $\pi \in \{1, \ldots, \text{arity}(e)\}$ then $\mathcal{R}$ is terminating.* □

Example 3. Consider the TRS $\mathcal{R} = \{f(a) \to f(e(a, b))\}$. Distribution elimination results in the non-terminating TRS $E_{\text{distr}}(\mathcal{R}) = \{f(a) \to f(a), f(a) \to f(b)\}$. The termination of the TRS

$$E_{\text{distr}}^2(\mathcal{R}) = \left\{ \begin{array}{l} f(a) \to f(b) \\ f(a) \to a \end{array} \right\}$$

can be verified by recursive path order. Hence termination of $\mathcal{R}$ follows from Theorem 11. Observe that $E_{\text{distr}}^1(\mathcal{R})$ fails to be terminating.

An obvious question is whether E_{distr}^{π} works in combination with distribution rules, i.e., does Theorem 7 hold for E_{distr}^{π}? The following example shows that the answer is negative.

Example 4. Consider the non-terminating TRS

$$\mathcal{R} = \left\{ \begin{array}{r} f(a,b) \to f(e(a,b), e(a,b)) \\ f(e(x,y),z) \to e(f(x,z), f(y,z)) \\ f(x, e(y,z)) \to e(f(x,y), f(x,z)) \end{array} \right\}.$$

The TRS $E_{\text{distr}}^{\pi}(\mathcal{R})$ is right-linear and (simply and totally) terminating for both choices of π. For instance,

$$E_{\text{distr}}^{1}(\mathcal{R}) = \left\{ \begin{array}{l} f(a,b) \to f(a,a) \\ f(a,b) \to b \end{array} \right\}.$$

A natural question to ask is whether we need the assumption in Theorems 8 and 11 that e does not occur in the left-hand sides of the rewrite rules in $\mathcal{R}$. In the proof of Theorem 8 this assumption is only used to conclude that $\langle l \rangle = l$ (where l is the left-hand side of a rewrite rule in $\mathcal{R}$). We need this identity because the left-hand sides of rewrite rules in $E_{\text{distr}}(\mathcal{R})$ and $E_{\text{distr}}^{\pi}(\mathcal{R})$ are of the form l rather than $\langle l \rangle$. This implies that we can completely remove the restriction that e does not occur in the left-hand sides of rules in $\mathcal{R}$, provided we change $E_{\text{distr}}^{\pi}(\mathcal{R})$ accordingly: $E_{\text{distr}}^{\pi}(\mathcal{R}) = \{\langle l \rangle \to r' \mid l \to r \in \mathcal{R} \text{ and } r' \in E_{\text{distr}}^{\pi}(r)\}$. This extension is useful since it enables us to conclude the termination of a non-simply terminating TRS like $\mathcal{R} = \{f(e(a,b),a) \to f(e(a,b), e(a,b))\}$ by transforming it into the TRS

$$E_{\text{distr}}^{2}(\mathcal{R}) = \left\{ \begin{array}{l} f(b,a) \to f(b,b) \\ f(b,a) \to a \end{array} \right\}$$

whose termination can be verified by recursive path order.

The transformation E_{distr}^{π} is similar in spirit to the *dummy elimination* technique of Ferreira and Zantema [8]: a function symbol is eliminated from (the right-hand sides of) the rewrite rules without duplicating the other parts. In dummy elimination this is achieved by introducing a fresh constant $\diamond$ to separate those parts, rather than gluing different parts together. The effect of dummy elimination—E_{dummy}—on the term t of Fig. 1 is shown in Fig. 3. Observe that $E_{\text{dummy}}(t)$ shares with $E_{\text{distr}}^{\pi}(t)$ the characteristic that each numbered contexts occurs exactly once. An application of E_{dummy} to the TRS $\mathcal{R}$ of Example 3 results in the (terminating) TRS

$$E_{\text{dummy}}(\mathcal{R}) = \left\{ \begin{array}{l} f(a) \to f(\diamond) \\ f(a) \to a \\ f(a) \to b \end{array} \right\}.$$

A correctness proof of dummy elimination can be given along the lines of the proof of Theorem 8, because the two key properties identified earlier hold for

Fig. 3. The mapping E_{dummy}.

E_{dummy} as well, provided we change the interpretation of e to $e_{\mathcal{A}}(t_1, \ldots, t_n) = \diamond$ for all $t_1, \ldots, t_n \in \mathcal{T}((\mathcal{F} \setminus \{e\}) \cup \{\diamond\}, \mathcal{V})$.

A thorough investigation of the relative strength of (variants of) distribution elimination and dummy elimination can be found in Ferreira [7].

7 Currying

In this final section we show that the main result of Kennaway, Klop, Sleep, and de Vries [10]—the preservation of termination under *currying*—is easily proved by self-labelling. Currying is the transformation on TRSs defined below.

Definition 12. With every TRS $(\mathcal{F}, \mathcal{R})$ we associate a TRS $(\mathcal{F}_{@}, \mathcal{R}_{@})$ as follows: the signature $\mathcal{F}_{@}$ contains all function symbols of $\mathcal{F}$ together with

1. function symbols f_i of arity i for every $f \in \mathcal{F}$ of arity n with $0 \leqslant i < n$,
2. a binary function symbol @, called *application*,

and $\mathcal{R}_{@}$ is the extension of $\mathcal{R}$ with all rewrite rules

$$@(f_i(x_1, \ldots, x_i), y) \to f_{i+1}(x_1, \ldots, x_i, y)$$

with $f \in \mathcal{F}$ of arity $n \geqslant 1$ and $0 \leqslant i < n$. Here $x_1, \ldots, x_i, y$ are pairwise different variables and f_{i+1} denotes f if $i + 1 = n$.

Clearly termination of $\mathcal{R}_{@}$ implies termination of $\mathcal{R}$.

Theorem 13 Kennaway *et al.* [10]. *If $\mathcal{R}$ is a terminating TRS then $\mathcal{R}_{@}$ is terminating.* $\qquad \square$

The proof in [10] is rather involved. We present a self-labelling proof.

Proof. Let $\mathcal{F}' = \mathcal{F}_{@} \setminus \{@\}$. The question is how termination of $(\mathcal{F}_{@}, \mathcal{R}_{@})$ follows from termination of $(\mathcal{F}', \mathcal{R})$. We turn $\mathcal{T}(\mathcal{F}', \mathcal{V})$ into an $\mathcal{F}_{@}$-algebra $\mathcal{A}$ by defining $@_{\mathcal{A}}(s, t)$ by induction on the structure of s, as follows:

$$@_{\mathcal{A}}(s, t) = \begin{cases} t & \text{if } s \in \mathcal{V}, \\ f_{i+1}(s_1, \ldots, s_i, t) & \text{if } s = f_i(s_1, \ldots, s_i) \text{ with } i < \mathrm{arity}(f), \\ f(@_{\mathcal{A}}(s_1, t), \ldots, @_{\mathcal{A}}(s_n, t)) & \text{if } s = f(s_1, \ldots, s_n). \end{cases}$$

As well-founded order $\succ$ on $\mathcal{T}(\mathcal{F}', \mathcal{V})$ we take $\to_{\mathcal{R}}^+$.

We show that $(\mathcal{A}, \succ)$ is a quasi-model for $(\mathcal{F}_{@}, \mathcal{R}_{@})$. We claim that every algebra operation is strictly monotone in all its coordinates. Here we consider

only the first coordinate of $@_{\mathcal{A}}$, which is the most interesting case. Before proceeding we mention the following fact, which is easily proved by induction on the structure of s: if $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $t \in \mathcal{T}(\mathcal{F}', \mathcal{V})$, and $\sigma \in \Sigma(\mathcal{F}', \mathcal{V})$ then $@_{\mathcal{A}}(s\sigma, t) = s\,@_{\mathcal{A}}(\sigma, t)$. Here the substitution $@_{\mathcal{A}}(\sigma, t)$ is defined as the mapping that assigns to every variable x the term $@_{\mathcal{A}}(x\sigma, t)$. We show that $@_{\mathcal{A}}(s, t) \to_{\mathcal{R}} @_{\mathcal{A}}(u, t)$ whenever $s, t, u \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ with $s \to_{\mathcal{R}} u$ by induction on the structure of s. Strict monotonicity of $@_{\mathcal{A}}$ in its first coordinate follows from this by an obvious induction argument. Since s cannot be a variable, we have either $s = f_i(s_1, \ldots, s_i)$ with $i < \mathrm{arity}(f)$ or $s = f(s_1, \ldots, s_n)$. In the former case we have $@_{\mathcal{A}}(s, t) = f_{i+1}(s_1, \ldots, s_i, t)$. Moreover, as s is root-stable, u must be of the form $f_i(s_1, \ldots, u_j, \ldots, s_i)$ with $s_j \to_{\mathcal{R}} u_j$. Hence $@_{\mathcal{A}}(s, t) \to_{\mathcal{R}} f_{i+1}(s_1, \ldots, u_j, \ldots, s_i, t) = @_{\mathcal{A}}(u, t)$. Suppose $s = f(s_1, \ldots, s_n)$. If the rewrite step from s to u takes place in one of the arguments of s then $v = f(s_1, \ldots, u_j, \ldots, s_n)$ with $s_j \to_{\mathcal{R}} u_j$. From the induction hypothesis we obtain $@_{\mathcal{A}}(s_j, t) \to_{\mathcal{R}} @_{\mathcal{A}}(u_j, t)$ and therefore

$$
\begin{aligned}
@_{\mathcal{A}}(s, t) &= f(@_{\mathcal{A}}(s_1, t), \ldots, @_{\mathcal{A}}(s_j, t), \ldots, @_{\mathcal{A}}(s_n, t)) \\
&\to_{\mathcal{R}} f(@_{\mathcal{A}}(s_1, t), \ldots, @_{\mathcal{A}}(u_j, t), \ldots, @_{\mathcal{A}}(s_n, t)) \\
&= @_{\mathcal{A}}(u, t).
\end{aligned}
$$

If the rewrite step from s to u takes place at the root of s then $s = l\sigma$ and $u = r\sigma$ for some rewrite rule $l \to r \in \mathcal{R}$ and substitution $\sigma \in \Sigma(\mathcal{F}', \mathcal{V})$. Because l and r do not contain function symbols from $\mathcal{F}' \setminus \mathcal{F}$, we obtain $@_{\mathcal{A}}(s, t) = l\,@_{\mathcal{A}}(\sigma, t)$ and $@_{\mathcal{A}}(u, t) = r\,@_{\mathcal{A}}(\sigma, t)$ from the above fact. Therefore also in this case we have $@_{\mathcal{A}}(s, t) \to_{\mathcal{R}} @_{\mathcal{A}}(u, t)$. In order to conclude that $(\mathcal{A}, \succ)$ is a quasi-model for $(\mathcal{F}_@, \mathcal{R}_@)$, it remains to show that $[\alpha](l) \succsim [\alpha](r)$ for every rewrite rule $l \to r \in \mathcal{R}_@$ and assignment α from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}', \mathcal{V})$. If $l \to r \in \mathcal{R}$ then $[\alpha](l) = l\alpha \to_{\mathcal{R}} r\alpha = [\alpha](r)$. Otherwise $l = @(f_i(x_1, \ldots, x_i), y)$ and $r = f_{i+1}(x_1, \ldots, x_i, y)$ for some $f \in \mathcal{F}$ and $i < \mathrm{arity}(f)$, in which case we have $[\alpha](l) = f_{i+1}(x_1, \ldots, x_i, y)\alpha = [\alpha](r)$ by definition.

To conclude our proof we show that $(\mathcal{R}_@)_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}_@, \succ)$ is precedence terminating with respect to the well-founded order $\sqsupset$ defined as follows: $f_s \sqsupset g_t$ if and only if

1. $s\,(\to_{\mathcal{R}} \cup \rhd)^+\,t$ and either $f, g \neq @$ or $f, g = @$, or
2. $f = @$ and $g \neq @$.

It is easy to see that $\sqsupset$ is indeed a well-founded order. Clearly $(\mathcal{R}_@)_{\mathrm{lab}} = \mathcal{R}_{\mathrm{lab}} \cup (\mathcal{R}_@ \setminus \mathcal{R})_{\mathrm{lab}}$. The rewrite rules in $\mathcal{R}_{\mathrm{lab}} \cup \mathrm{dec}(\mathcal{F}_@, \succ)$ are taken care of by the first clause of the definition of $\sqsupset$, just as in the proof of Theorem 4. For the rules in $(\mathcal{R}_@ \setminus \mathcal{R})_{\mathrm{lab}}$ we use the second clause. $\qquad\square$

The reader is invited to compare our proof with the one of Kennaway *et al.* [10].

References

1. F. Bellegarde and P. Lescanne, *Termination by Completion*, Applicable Algebra in Engineering, Communication and Computing **1** (1990) 79–96.
2. L. Bachmair and N. Dershowitz, *Commutation, Transformation, and Termination*, Proceedings of the 8th Conference on Automated Deduction, Oxford, Lecture Notes in Computer Science **230** (1986) 5–20.
3. N. Dershowitz, *Orderings for Term-Rewriting Systems*, Theoretical Computer Science **17** (1982) 279–301.
4. N. Dershowitz, *Termination of Rewriting*, Journal of Symbolic Computation **3**(1) (1987) 69–116.
5. N. Dershowitz, *Hierarchical Termination*, Proceedings of the 4th International Workshop on Conditional (and Typed) Rewriting Systems, Jerusalem, Lecture Notes in Computer Science **968** (1995) 89–105.
6. N. Dershowitz and J.-P. Jouannaud, *Rewrite Systems*, in: Handbook of Theoretical Computer Science, Vol. B (ed. J. van Leeuwen), North-Holland (1990) 243–320.
7. M.C.F. Ferreira, *Termination of Rewriting: Well-foundedness, Totality and Transformations*, Ph.D. thesis, University of Utrecht, 1995.
8. M.C.F. Ferreira and H. Zantema, *Dummy Elimination: Making Termination Easier*, Proceedings of the 10th International Conference on Fundamentals of Computation Theory, Dresden, Lecture Notes in Computer Science **965** (1995) 243–252.
9. S. Kamin and J.J. Lévy, *Two Generalizations of the Recursive Path Ordering*, unpublished manuscript, University of Illinois, 1980.
10. J.R. Kennaway, J.W. Klop, R. Sleep, and F.-J. de Vries, *On Comparing Curried and Uncurried Rewrite Systems*, report CS-R9350, CWI, Amsterdam, 1993. To appear in the Journal of Symbolic Computation.
11. J.W. Klop, *Term Rewriting Systems*, in: Handbook of Logic in Computer Science, Vol. II (eds. S. Abramsky, D. Gabbay and T. Maibaum), Oxford University Press (1992) 1–116.
12. M.R.K. Krishna Rao, *Modular Proofs for Completeness of Hierarchical Term Rewriting Systems*, Theoretical Computer Science **151** (1995) 487–512.
13. M.R.K. Krishna Rao, *Simple Termination of Hierarchical Combinations of Term Rewriting Systems*, Proceedings of the 2nd International Symposium on Theoretical Aspects of Computer Software, Sendai, Lecture Notes in Computer Science **789** (1994) 203–223.
14. A. Middeldorp, *A Sufficient Condition for the Termination of the Direct Sum of Term Rewriting Systems*, Proceedings of the 4th IEEE Symposium on Logic in Computer Science, Pacific Grove (1989) 396–401.
15. E. Ohlebusch, *Modular Properties of Composable Term Rewriting Systems*, Ph.D. thesis, Universität Bielefeld, 1994.
16. Y. Toyama, *On the Church-Rosser Property for the Direct Sum of Term Rewriting Systems*, Journal of the ACM **34**(1) (1987) 128–143.
17. Y. Toyama, *Counterexamples to Termination for the Direct Sum of Term Rewriting Systems*, Information Processing Letters **25** (1987) 141–143.
18. H. Zantema, *Termination of Term Rewriting by Semantic Labelling*, Fundamenta Informaticae **24** (1995) 89–105.
19. H. Zantema, *Termination of Term Rewriting: Interpretation and Type Elimination*, Journal of Symbolic Computation **17** (1994) 23–50.