Binomial Interpretations*

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Abstract

Polynomial interpretations can be used for proving termination of term rewrite systems. In this note, we contemplate *binomial* interpretations based on binomial coefficients, and show that they form a suitable basis for obtaining (weakly) monotone algebras. The main motivation is that this representation covers examples with negative coefficients like $f(x) = 2x^2 - x + 1$, and even some polynomials with rational coefficients like $f(x) = x(x-1)/2$ that map natural numbers to natural numbers.

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1 Introduction

Using well-founded monotone algebras is a general and common method for proving termination of term rewrite systems. Many algebras have been suggested for this purpose. Here we are mainly interested in polynomial interpretations (introduced by Lankford, [6]). In [8] it is shown among other things that polynomial interpretations over the real numbers do not subsume polynomial interpretations over the natural numbers. Ultimately, the reason for this surprising result lies in the fact that there are polynomials that are non-negative for every natural number, but negative when evaluated for some real numbers. The example $f(x) = x(x-1)/2$ shows that there are polynomials with non-integer coefficients that nevertheless evaluate to integers at every integer argument. Binomial functions (see below) capture these polynomials precisely.

We are not the first to use binomial functions this way. Girard et al. [1] extend linear logic with resources bounded by *resource polynomials*, which are binomial functions with non-negative coefficients. In more recent work, Hofmann et al. [4, 3] use resource polynomials for amortized resource analysis of programs. The observation that binomial functions are closed under composition is much older. The earliest appearances that we are aware of originate in the study of nilpotent groups [2] and of recursively equivalent sets [7].

In the remainder of the paper, we exhibit some fundamental properties of binomial coefficients in Section 2, then sketch binomial interpretations in Section 3. In Section 4, we compare the power of binomial interpretations to standard polynomial interpretations.

2 Fundamentals

▶ **Definition 1.** For $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers), and x element of some ring, the *falling power*, $x^{\underline{n}}$, is defined as follows: (This notation is used by Knuth in [5])

$$x^{\underline{0}} = 1 \qquad\qquad x^{\underline{n}} = x \cdot (x-1)^{\underline{n-1}}$$

Falling powers are closely related to binomial coefficients. In fact, we can define binomial coefficients in terms of falling powers.

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▶ **Definition 2.** For $k \in \mathbb{N}$, the *binomial coefficient* $\binom{x}{k}$ is defined as $\binom{x}{k} = \frac{x^{\underline{k}}}{k!}$.

▶ Remark. Over some rings, the fraction $\frac{x^{\underline{k}}}{k!}$ may not have a value for some x. For example, in the polynomial ring $\mathbb{Z}[x]$, $\binom{x}{2}$ does not exist. The fraction may also have several values (if the ring is not torsion-free). But we will only work over $\mathbb{Z}$ or $\mathbb{R}$, where this does not happen.

It is clear from the definition that the binomial coefficient $\binom{x}{k}$ is a polynomial in x of degree k with rational coefficients. It is well known that $\binom{x}{k} \in \mathbb{Z}$ whenever $x \in \mathbb{Z}$.

▶ **Lemma 3.** *Binomial coefficients satisfy a tremendous number of identities. We exhibit two of them. (Only (1) is used later, but the second one gives some insight into why binomial functions are closed under composition.)*

$$\binom{x+1}{k+1} - \binom{x}{k+1} = \binom{x}{k} \tag{1}$$

$$\binom{x+y}{k} = \sum_{i=0}^{k} \binom{x}{i}\binom{y}{k-i} \tag{2}$$

3 Binomial Interpretations

▶ **Definition 4.** A *monomial* over the variables V is a finite product $\prod_{v \in V'} \binom{v}{k_v}$ such that $V' \subseteq V$ and $0 < k_v \in \mathbb{N}$. We write 1 if the product is empty and v for $\binom{v}{1}$. A *binomial function* f over a domain D is a linear combination of monomials, $f(v_1, \ldots, v_n) = \sum_{m \in M} a_m \cdot m$ where $a_m \in D$ and M is a finite set of monomials over the variables $V = \{v_1, \ldots, v_n\}$. We can evaluate binomial functions in the obvious way, substituting values for the formal variables.

We define a difference operator on binomial functions, justified by the identity (1): We let $\Delta_v \left(\sum_{m \in M} a_m \cdot m\right) = \sum_{m \in M} a_m \cdot \Delta_v m$, where on monomials, $\Delta_v \prod_{w \in V} \binom{w}{k_w} = 0$ if $v \notin V$ or $k_v = 0$. Otherwise, $\Delta_v \prod_{w \in V} \binom{w}{k_w} = \binom{w}{k'_w}$ where $k'_v = k_v - 1$ and $k'_w = k_w$ for $w \neq v$. It is easy to see that $f(v_1, \ldots, v_i + 1, \ldots, v_n) - f(v_1, \ldots, v_i, \ldots, v_n) = (\Delta_{v_i} f)(v_1, \ldots, v_i, \ldots, v_n)$ for all binomial functions f and variables v_i.

It is known that binomial functions over $\mathbb{N}$ are closed under addition, multiplication and composition [1]. In practice, the best way to compute the results of these operations appears to be to use the identity $(\Delta_v^n \binom{v}{k})(0) = \delta_{n,k}$, where $\delta_{n,k} = 1$ if $n = k$ and $\delta_{n,k} = 0$ otherwise. Once the degree d of a unary binomial function $f(x)$ is known, one can compute its coefficients from $f(0), f(1), \ldots, f(d)$. This can be extended to multiple variables by treating a binomial function in V with $v \in V$ as a unary function in v with coefficients that are binomial functions over $V \setminus \{v\}$. The difference operator also plays a crucial role in showing that all integer-valued (over the integers) polynomials can be expressed by binomial functions, as follows. Let $f(v)$ be an integer-valued polynomial of degree $d > 0$. Then $(\Delta_v f)(v) = f(v+1) - f(v)$ is an integer-valued polynomial of degree $d - 1$. The function f can be reconstructed from $\Delta_v f$ and $f(0)$, and ultimately from the values $f_i = (\Delta_v^i f)(0)$ for $0 \leqslant i \leqslant d$, and we have just seen that these define a binomial function. In fact, $f(v) = \sum_{i=0}^{d} f_i \binom{v}{i}$.

▶ **Definition 5.** Let $\mathcal{F}$ be a signature where each $f \in \mathcal{F}$ has an arity $\mathsf{ari}(f)$. Furthermore let $V = \{v_1, v_2, \ldots\}$ be a countable set of variables. A *binomial $\mathcal{F}$-algebra $\mathcal{A}$ over a domain D* assigns to each $f \in \mathcal{F}$ an interpretation $f_{\mathcal{A}}$ that is a binomial function over D with variables $\{v_1, \ldots v_{\mathsf{ari}(f)}\}$. A binomial $\mathcal{F}$-algebra induces an $\mathcal{F}$-algebra with carrier D by evaluating the binomial functions.

To use a binomial $\mathcal{F}$-algebra for proving termination of a TRS $\mathcal{R}$, it has to induce a well-founded monotone algebra that is compatible with $\mathcal{R}$.

▶ **Theorem 6.** *Let $\mathcal{A}$ be a $\mathcal{F}$-algebra over $\mathbb{N}$. Then $\mathcal{A}$ is a well-founded monotone algebra, provided that for all $f \in \mathcal{F}$, $f_{\mathcal{A}}(v_1, \ldots, v_n) = \sum_{m \in M} a_m \cdot m$ implies $a_{v_i} \geqslant 1$ for $1 \leqslant i \leqslant n$. Furthermore, for any binomial function $f(v_1, \ldots, v_n) = \sum_{m \in M} a_m \cdot m$ over $\mathbb{N}$, we have $f(v_1, \ldots, v_n) > 0$ for all possible values of $v_i \in \mathbb{N}$, if, and only if, $a_1 \geqslant 0$.*

Proof. Note that over $\mathbb{N}$, all binomial functions are weakly monotone and nowhere negative. The theorem follows easily from that observation. ◀

4 Comparison to Polynomial Interpretations

We will show below that neither polynomial interpretations over $\mathbb{R}$ nor over $\mathbb{N}$ subsume binomial interpretations. Note that *linear* binomial interpretations are identical to linear polynomial interpretations over the integers—the increased power requires higher degree polynomials. Using the method by Neurauter and Middeldorp [8], which can force weakly compatible polynomial interpretations to be linear with non-integer coefficients, it is clear that binomial interpretations do not subsume polynomial interpretations over $\mathbb{Q}$. On the other hand, if negative coefficients are allowed, binomial interpretations subsume polynomial interpretations over $\mathbb{N}$ with integer coefficients, by way of the identity

$$x^k = \sum_{i=0}^{k} i! \left\{ {n \atop i} \right\} \binom{x}{i}$$

where $\left\{ {n \atop i} \right\}$ denotes Stirling numbers of the second kind, which are non-negative integers. The same relation allows us to transform polynomial interpretations with non-negative coefficients to binomial interpretations with non-negative coefficients.

We adapt an example from [8] to show that binomial interpretations are not subsumed by polynomial interpretations over $\mathbb{R}$ or $\mathbb{N}$. Let $\mathcal{R}$ be the following TRS.

$$\begin{aligned}
\mathsf{s}(\mathsf{0}) &\to \mathsf{f}(\mathsf{0}) && (1) & \mathsf{s}(\mathsf{f}(\mathsf{s}(x))) &\to \mathsf{h}(\mathsf{f}(x), \mathsf{g}(x)) && (6)\\
\mathsf{s}^2(\mathsf{0}) &\to \mathsf{f}(\mathsf{s}(\mathsf{0})) && (2) & \mathsf{f}(\mathsf{g}(\mathsf{s}(x))) &\to \mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{s}(x)))) && (7)\\
\mathsf{g}(x) &\to \mathsf{h}(x, \mathsf{h}(x, x)) && (3) & \mathsf{h}(\mathsf{s}^2(x), \mathsf{h}(x, x)) &\to \mathsf{g}(x) && (8)\\
\mathsf{s}(x) &\to \mathsf{h}(\mathsf{0}, x) && (4) & \mathsf{s}(x) &\to \mathsf{h}(x, \mathsf{0}) && (9)\\
\mathsf{g}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{s}(\mathsf{g}(x))) && (5) & \mathsf{h}(\mathsf{f}(x), \mathsf{s}(\mathsf{g}(x))) &\to \mathsf{f}(\mathsf{s}(x)) && (10)
\end{aligned}$$

▶ **Theorem 7.** *Termination of the TRS $\mathcal{R}$ can be shown by a binomial interpretation.*

Proof. We let $[\mathsf{0}] = 0$, $[\mathsf{s}](x) = x+1$, $[\mathsf{f}](x) = 3\binom{x}{2}+x$, $[\mathsf{g}](x) = 3x+1$, $[\mathsf{h}](x, y) = x+y$. These are strictly monotone functions on $\mathbb{N}$. For compatibility with $\mathcal{R}$, we obtain the following constraints.

$$\begin{aligned}
1 &> 0 && (1) & 3\tbinom{x}{2} + 4x + 2 &> 3\tbinom{x}{2} + 4x + 1 && (6)\\
2 &> 1 && (2) & 27\tbinom{x}{2} + 48x + 22 &> 27\tbinom{x}{2} + 36x + 13 && (7)\\
3x + 1 &> 3x && (3) & 3x + 2 &> 3x + 1 && (8)\\
x + 1 &> x && (4) & x + 1 &> x && (9)\\
3x + 4 &> 3x + 3 && (5) & 3\tbinom{x}{2} + 4x + 2 &> \tbinom{x}{2} + 4x + 1 && (10)
\end{aligned}$$

Since these constraints are all satisfied, we conclude that $\mathcal{R}$ is terminating. ◀

▶ **Theorem 8.** *Termination of $\mathcal{R}$ cannot be shown using polynomial interpretations over $\mathbb{N}$ or $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$.*

Proof. The argument follows that from [8]. We will first show that regardless of the domain $\mathbb{N}$ or $\mathbb{R}^+$, a polynomial interpretation that is compatible with $\mathcal{R}$ must assign $[\mathsf{f}]$ a quadratic polynomial with leading coefficient $\frac{3}{2s_0}$ for some $s_0 \in \mathbb{N}$, ruling out $\mathbb{N}$ as a domain. The second part of the proof is devoted to showing that $[\mathsf{s}](x) = x + \delta$, where δ is the parameter defining the well-founded order $<_\delta$ on $\mathbb{R}^+$. Using this fact we will conclude that $[\mathsf{f}](x) < 0$ for some $x \in \mathbb{R}$, establishing the claim for the domain $\mathbb{R}^+$.

For the first part, we can treat both domains $\mathbb{N}$ and $\mathbb{R}^+$ simultaneously, as follows. When working over $\mathbb{R}^+$, we use $>_\delta$ as well-founded order and $\geqslant$ as compatible quasi-order to obtain a well-founded monotone algebra, where $a >_\delta b$ iff $a \geqslant b+\delta$ and $\delta > 0$ is a fixed real numgber. Over $\mathbb{N}$, the well-founded order and quasi-order are $>$ and $\geqslant$, respectively. If we let $\delta = 1$ over $\mathbb{N}$, then $>_\delta \, = \, >$, and the two definitions of the orders coincide.

Assume that we are given polynomials $[\mathsf{0}] = z$, $[\mathsf{s}] = s$, $[\mathsf{f}] = f$, $[\mathsf{g}] = g$ and $[\mathsf{h}] = h$ with coefficients in $\mathbb{R}$ ($\mathbb{Z}$) for domain $\mathbb{R}^+$ ($\mathbb{N}$). Furthermore let these polynomials be strictly monotone with respect to $>_\delta$ over the domain and compatible with $\mathcal{R}$. To establish compatibility with the rules, we evaluate both sides of all rules and compare the resulting polynomials. First consider rules (7) and (5), and compare the degrees of both sides: We have $\deg(f)\deg(g) \geqslant \deg(g)^2 \deg(f)$ and $\deg(g)\deg(s) \geqslant \deg(s)^2\deg(g)$, from which we conclude that $\deg(g) = \deg(s) = 1$ (note that because of strict monotonicity, none of the polynomials can be constant). So $g(x) = g_1 x + g_0$ and $s(x) = s_1 x + s_0$ for some $g_1, s_1 \geqslant 1$ and $g_0, s_0 \geqslant 0$. Furthermore by comparing the leading coefficients of (5), namely $g_1 s_1 \geqslant s_1^2 g_1$ we see that $s_1 = 1$. Next we find constraints on h. To that end, consider (3). Since the left-hand side evaluates to a linear polynomial, so must the right-hand side. Therefore, we may assume that $h(x,y) = h_x x + h_y y + h_0$ where $h_x, h_y \geqslant 1$ and $h_0 \geqslant 0$. By comparing leading coefficients of (4) and (9) we find that $s_1 \geqslant h_x$ and $s_1 \geqslant h_y$, i.e., $h_x = h_y = 1$. Using these values, we can find a lower bound on s_0 from the compatibility of (9), namely $s_0 \geqslant z_0 + h_0 + \delta$, which implies $s_0 \geqslant \delta > 0$. Finally we find a bound on the degree of f. Using (10) we conclude that $x + 2s_0 + h_0 \geqslant f(x+s_0) - f(x)$ Because $s_0 \geqslant \delta > 0$, the degree of $f(x+s_0) - f(x)$ is one less than that of $f(x)$, and since the left-hand side is linear, f can at most be quadratic. To summarize, we can express z, s, f, g and h as follows.

$$z = z_0 \qquad s(x) = x + s_0 \qquad f(x) = f_2 x^2 + f_1 x + f_0$$
$$g(x) = g_1 x + g_0 \qquad h(x,y) = x + y + h_0$$

We also know that $z_0, f_0, g_0, h_0 \geqslant 0$ and $s_0 \geqslant \delta$. Next we compare the leading coefficients in (3,8). For (3), we get $g_1 \geqslant 3$, while for (8), $3 \geqslant g_1$. Therefore, $g_1 = 3$.

Now let us determine f_2. From compatibility of (6) we find that $f_2 x^2 + (2f_2 s_0 + f_1)x + O(1) >_\delta f_2 x^2 + (f_1 + g_1)x + O(1)$, where $O(1)$ stands for a constant term not containing x. From this we conclude that $2f_2 s_0 \geqslant g_1$. Similarly from compatibility (10) we have $f_2x^2 + (f_1+g_1)x + O(1) >_\delta f_2 x^2 + (2f_2 s_0 + f_1)x + O(1)$, which implies $g_1 \geqslant 2 f_2 s_0$. Therefore, $f_2 = \frac{g_1}{2s_0} = \frac{3}{2s_0}$. In particular, no polynomial interpretation over $\mathbb{N}$ can exist, because s_0 and f_2 cannot both be integers.

Therefore, from now on, we assume that we are given a polynomial interpretation over $\mathbb{R}$. Our next step will be to determine s_0. We already know that $s_0 \geqslant \delta$. By strict monotonicity, we must have $f(\delta) - f(0) \geqslant \delta$, which is equivalent to $f_2\delta + f_1 \geqslant 1$. Now consider (2). We have $z_0 + 2s_0 - \delta \geqslant f_2(z_0+s_0)^2 + f_1(z_0+s_0) + f_0 \geqslant f_2 s_0(z_0+s_0) + (1 - f_2\delta)(z_0+s_0)$. Therefore, $s_0 - \delta \geqslant f_2(z_0+s_0)(s_0-\delta) \geqslant f_2 s_0(s_0 - \delta) = \frac{3}{2}(s_0-\delta)$, which implies $\delta \geqslant s_0$, from which we conclude that $s_0 = \delta$. Using (4), this implies $z_0 + h_0 \leqslant 0$, i.e., $z_0 = h_0 = 0$. Then, from (1), we conclude that $f_0 = 0$. Finally, we consider (2) once more. Compatibility now implies $2\delta - \delta \geqslant \frac{3}{2}\delta + f_1\delta$, or $-\frac{1}{2} \geqslant f_1$. This, however, leads to a contradiction, since

$f(\frac{\delta}{6}) = \frac{3}{2\delta} \cdot \frac{\delta^2}{6^2} + f_1 \frac{\delta}{6} \leqslant \frac{1}{24}\delta - \frac{1}{12}\delta < 0$ lies outside the domain $\mathbb{R}^+$. ◀

5 Conclusion

We have described an extension of polynomial interpretations with integer coefficients using binomial coefficients. These binomial interpretations arise naturally as a characterization of integer-valued polynomials with integer arguments and rational coefficients. We have also shown that binomial interpretations are not subsumed by polynomial interpretations over the real numbers.

As future work, we plan to incorporate binomial interpretations into $\mathsf{T_TT_2}$.

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