Point-Decreasing Diagrams Revisited*

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Abstract

In this note we revisit Bognar's point version of decreasing diagrams. We show that it is an instance of van Oostrom's decreasing diagrams. Furthermore we demonstrate that the point version of decreasing diagrams is complete for confluence of *finite* abstract rewrite systems, contradicting a counterexample by Bognar.

1 Introduction

The decreasing diagrams technique [4] is a powerful confluence criterion for abstract rewrite systems (ARSs), based on labeling the rewrite steps of the system. The criterion is complete for confluence of countable ARSs. In [1], Bognar introduced a point version of decreasing diagrams, where labels are assigned to the objects instead of the rewrite steps. In order to prove this result, Bognar modified van Oostrom's proof [4], which is based on lexicographic path measures.

In this note, we revisit Bognar's point-decreasing diagrams. We give a new proof of the result based on van Oostrom's decreasing diagrams. We also show that point-decreasing diagrams are complete for confluence of *finite* ARSs.

This note is based on [3, Section 3.5].

2 Preliminaries

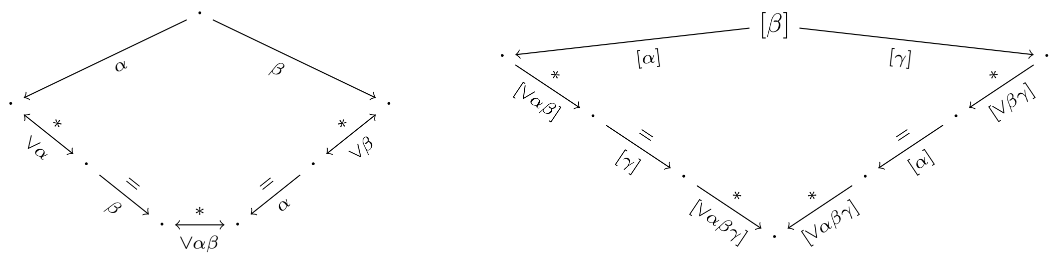
We use standard notation for abstract rewriting. An ARS $\langle \mathcal{A}, \to \rangle$ consists of a set of objects $\mathcal{A}$ and a (rewrite) relation $\to$ on $\mathcal{A}$. Let $\to^0 = \equiv$ and $\to^{n+1} = \to^n \cdot \to$. We denote the inverse, reflexive closure, symmetric closure and reflexive transitive closure of $\to$ by $\leftarrow$, $\to^=$, $\leftrightarrow$ and $\to^*$, respectively. We also consider labeled ARSs $\langle \mathcal{A}, (\to_\alpha)_{\alpha \in L} \rangle$, where L is a set of labels equipped with a well-founded order $>$ and $(\to_\alpha)_{\alpha \in L}$ is a family of relations on $\mathcal{A}$. For $M \subseteq L$ we let $\to_M = \bigcup_{\alpha \in M} \to_\alpha$. We define $\curlyvee\alpha = \{\beta \mid \alpha > \beta\}$ and $\curlyvee\alpha\beta = \curlyvee\alpha \cup \curlyvee\beta$. Recall van Oostrom's decreasing diagrams result:

Theorem 1. *Let $\langle \mathcal{A}, \to_\alpha \rangle_{\alpha \in L}$ be a labeled ARS. If for all $\alpha, \beta \in L$*

$$\underset{\alpha}{\leftarrow} \cdot \underset{\beta}{\to} \subseteq \underset{\curlyvee\alpha}{\overset{*}{\leftrightarrow}} \cdot \underset{\beta}{\overset{=}{\to}} \cdot \underset{\curlyvee\alpha\beta}{\overset{*}{\leftrightarrow}} \cdot \underset{\alpha}{\overset{=}{\leftarrow}} \cdot \underset{\curlyvee\beta}{\overset{*}{\leftrightarrow}}$$

then $\to_L$ is confluent. (See also Figure 1(a).)

We will refer to van Oostrom's decreasing diagrams as the step version of decreasing diagrams, to distinguish it from Bognar's point version.



(a) Locally step-decreasing diagram.

(b) Locally point-decreasing diagram.

Figure 1: Decreasing diagrams.

3 Point-Decreasing Diagrams

In this section we consider the point version of decreasing diagrams proposed by Bognar in [1], and show how it follows from step-decreasing diagrams. For point-decreasing diagrams, the objects (i.e., the points) rather than the steps of an abstract rewrite system are labeled.

Formally, we consider *point-labeled ARSs* $\langle \mathcal{A}, \to, \ell \rangle$ which consist of an ARSs $\langle \mathcal{A}, \to \rangle$ together with a function labeling the objects $\ell : \mathcal{A} \to W$, where W is a set of labels equipped with a well-founded order $>$. We annotate steps by the labels of their targets in square brackets, that is, we write $s \to_{[\ell(t)]} t$. We also allow sets inside the square brackets, in which case the target of the step may have any label from this set: $\to_{[M]} = \bigcup_{\alpha \in M} \to_{[\alpha]}$.

Bognar's version of local decreasingness [1, Corollary 8] can be stated as follows.

Theorem 2. *Let $\langle \mathcal{A}, \to, \ell \rangle$ be a point-labeled abstract rewrite system. Then $\to$ is confluent if every local peak $t \; {}_{[\alpha]}\!\leftarrow s \to_{[\gamma]} u$ with $\beta = \ell(s)$ has a joining valley*

$$t \xrightarrow[{[\vee\alpha\beta]}]{*} \cdot \xrightarrow[{[\gamma]}]{=} \cdot \xrightarrow[{[\vee\alpha\beta\gamma]}]{*} \cdot \xleftarrow[{[\vee\alpha\beta\gamma]}]{*} \cdot \xleftarrow[{[\alpha]}]{=} \cdot \xleftarrow[{[\vee\beta\gamma]}]{*} u \tag{PD}$$

(resulting in a locally point-decreasing diagram*, Figure 1(b)).*

Proof. Because any well-founded order can be extended to a well-order, and because locally point-decreasing diagrams are preserved when the order $>$ on W is extended, we may assume w.l.o.g. that $>$ is a well-order. We label steps by pairs from $W \times \{\bot, \top\}$, ordered lexicographically, using the order $\top > \bot$ on the second component. Note that this gives a well-order on $W \times \{\bot, \top\}$. Each step $s \to t$ is labeled by $\max(\langle \ell(s), \bot \rangle, \langle \ell(t), \top \rangle)$. In particular, the peak $t \; {}_{[\alpha]}\!\leftarrow s \to_{[\gamma]} u$ with $\beta = \ell(s)$ is labeled by $A = \max(\langle \beta, \bot \rangle, \langle \alpha, \top \rangle)$ to the left and $B = \max(\langle \beta, \bot \rangle, \langle \gamma, \top \rangle)$ to the right. We claim that using this labeling, the point-decreasing diagrams (PD) become decreasing diagrams. Consider a step $v \to w$ of the valley, with label $C = \max(\langle \ell(v), \bot \rangle, \langle \ell(w), \top \rangle)$. There are three cases.

1. Let $v \to w$ be from the $t \to^*_{[\vee\alpha\beta]} \cdot$ subderivation of the valley. The source v of such a step satisfies $\alpha \geqslant \ell(v)$ (hence $\langle \alpha, \top \rangle > \langle \ell(v), \bot \rangle$) or $\beta > \ell(v)$ (hence $\langle \beta, \bot \rangle > \langle \ell(v), \bot \rangle$), while the target w satisfies $\alpha > \ell(w)$ (hence $\langle \alpha, \top \rangle > \langle \ell(w), \top \rangle$)) or $\beta > \ell(w)$ (hence $\langle \beta, \bot \rangle > \langle \ell(w), \top \rangle$). Therefore, $A > C$.

*This research was supported by the Austrian Science Fund (FWF) projects P22467 and P27528.

2. Assume that $v \to w$ corresponds to the optional step $\cdot \to_{[\gamma]}^{=} \cdot$ of the valley. Then $\alpha \geqslant \ell(v)$ or $\beta > \ell(v)$, and $\ell(w) = \gamma$. We have $\langle \gamma, \top \rangle = \langle \ell(w), \top \rangle$, $\langle \alpha, \top \rangle > \langle \ell(v), \bot \rangle$ and $\langle \beta, \bot \rangle > \langle \ell(v), \bot \rangle$. Consequently, $B = C$ or $A > C$.
3. Let $v \to w$ be from the $\cdot \to_{[\vee\alpha\beta\gamma]}^{*} \cdot$ part of the valley. Then $\alpha \geqslant \ell(v)$, $\beta \geqslant \ell(v)$ or $\gamma > \ell(v)$, and $\alpha > \ell(w)$, $\beta > \ell(w)$ or $\gamma > \ell(w)$. Consequently, $\langle \alpha, \top \rangle > \langle \ell(v), \bot \rangle$, $\langle \beta, \top \rangle > \langle \ell(v), \bot \rangle$ or $\langle \gamma, \bot \rangle > \langle \ell(v), \bot \rangle$, and $\langle \alpha, \top \rangle > \langle \ell(w), \top \rangle$, $\langle \beta, \top \rangle > \langle \ell(w), \top \rangle$ or $\langle \gamma, \bot \rangle > \langle \ell(w), \top \rangle$. Consequently, $A > C$ or $B > C$ follows.

A symmetric argument applies to steps $w \leftarrow v$ on the left side of the valley. Therefore,

$$t \xrightarrow[\vee A]{*} \cdot \xrightarrow[B]{=} \cdot \xrightarrow[\vee AB]{*} \cdot \xleftarrow[\vee AB]{*} \cdot \xleftarrow[A]{=} \cdot \xleftarrow[\vee B]{*} u$$

This is a step-decreasing diagram. Since we started from an arbitrary peak, and because the order on the set of labels $L \times \{\bot, \top\}$ is well-founded, we conclude that the ARS $\to$ is decreasing by Theorem 1. □

Remark 3. In fact the proof of Theorem 2 shows that any point-decreasingly confluent system is also step-decreasing using the same joining sequences for local peaks. This detail is important for applications of decreasing diagrams, where one usually attempts to find suitable labelings for a given set of joining sequences. Our proof also provides some insight into how van Oostrom's original proof for step-decreasing diagrams relates to Bognar's adaptation of that proof for point-decreasing diagrams.

Remark 4. Condition (PD) only considers local peaks, which differs from Bognar's definition [1], which defines decreasing diagrams for peaks and valleys of arbitrary size, based on van Oostrom's *lexicographic path measure* [4]. Furthermore [1] assumes that $>$ is a well-order, whereas we allow an arbitrary well-founded order; note however, that this extra degree of freedom does not any power because any well-founded order can be extended to a well-order. If one assumes $>$ to be a well-order, then condition (PD) is equivalent to Bognar's decreasing diagrams for *local* peaks. This can be seen as follows.

We use notation from [1]. Any locally decreasing diagram can be written as

$$j \xleftarrow[{[j]}]{} i \xrightarrow[{[k]}]{} k \qquad j \xrightarrow[{[\tau']}]{*} l \xleftarrow[{[\sigma']}]{*} k$$

where σ' and τ' are strings of labels satisfying

$$|i; j; \tau'| \preceq_{\#} [i] \cup_{\#} {}_{i\leqslant}|j| \cup_{\#} {}_{i\leqslant}|k| \tag{DCR1}$$

and the symmetric property (DCR2) which is obtained by swapping the roles of j, τ' and k, σ'. Condition (DCR1) can be simplified as follows.

$$[i] \cup_{\#} {}_{i\leqslant}|j| \cup_{\#} {}_{\max(i,j)\leqslant}|\tau'| \preceq_{\#} [i] \cup_{\#} {}_{i\leqslant}|j| \cup_{\#} {}_{i\leqslant}|k|$$
$${}_{\max(i,j)\leqslant}|\tau'| \preceq_{\#} {}_{i\leqslant}|k|$$

The right-hand side is an empty multiset if $i > k$, in which case τ' must consist of labels all smaller than $\max(i, j)$ (including labels smaller than or equal to k). If $k \geqslant i$, then the right-hand side is the singleton multiset $[k]$, and τ' must consist of some labels smaller than $\max(i, j)$, optionally followed by k, followed by further labels smaller than $\max(i, j, k)$. Condition (PD) arises from these observations and the fact that a comparison by $\max(i, j)$ (or $\max(i, j, k)$) can be performed by comparing to each of i, j (or i, j, k) and taking the disjunction of the comparison results.

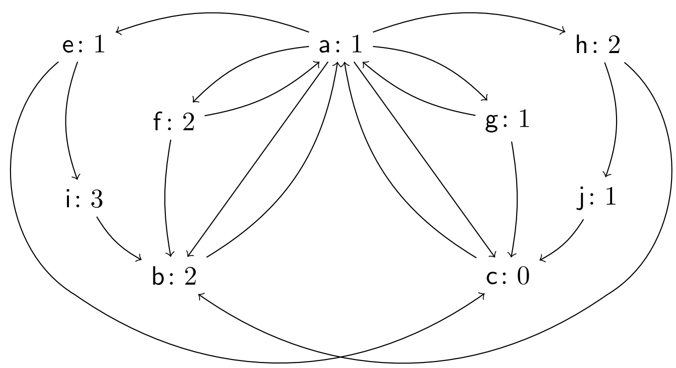


Figure 2: Labeling the "Maja the Bee"-example from [2].

Finally, we consider the question of completeness of the point version of decreasing diagrams.

Theorem 5. *Point-decreasing diagrams are complete for confluence of finite ARSs.*

Proof. Let $\langle \mathcal{A}, \to \rangle$ be a confluent, finite ARS. The relation $\leftrightarrow^*$ is an equivalence relation that partitions $\mathcal{A}$ into equivalence classes, the *components* of $\mathcal{A}$. Because $\mathcal{A}$ is finite, there are only finitely many components of $\mathcal{A}$ and each component contains finitely many objects from $\mathcal{A}$. Let $C \subseteq A$ be a component of $\mathcal{A}$. For all $s, t \in C$ we have $s \leftrightarrow^* t$. Therefore, by finiteness of C and confluence, we can choose an object $f_C \in A$ that is reachable from all elements of C. Furthermore, because $\to^* \subseteq \leftrightarrow^*$, $f_C \in C$. Let

$$F = \{f_C \mid C \text{ is a component of } \mathcal{A}\}$$

By this construction, every object $s \in \mathcal{A}$ reaches exactly one element of F, namely f_{C_s}, where C_s denotes the component which contains s. Let

$$\ell(a) = \min\{n \in \mathbb{N} \mid a \xrightarrow{n} f_{C_a}\}$$

Note that for any $s \in \mathcal{A}$ with $n = \ell(s)$, we have

$$s \xrightarrow[{[\vee n]}]{n} f_{C_s}$$

We claim that the labeling function ℓ makes $\langle \mathcal{A}, \to \rangle$ point-decreasing. To see why, it suffices to consider a local peak $t \;{}_{[n]}{\leftarrow}\; s \to_{[m]} u$, and note that

$$t \xrightarrow[{[\vee n]}]{*} f_{C_t} = f_{C_u} \xleftarrow[{[\vee m]}]{*} u \qquad \square$$

Remark 6. In the proof of Theorem 5, we use natural numbers as labels, ordered by the usual order, which is a well-order. Therefore, it applies to Bognar's original definition of point-decreasing diagrams as well.

Example 7. We consider the "Maja the Bee" example by Bognar and Klop [2], which has been presented as a counterexample to the completeness of the point-version of decreasing diagrams. The example is reproduced in Figure 2. There is only a single component $C = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{e}, \mathsf{f}, \mathsf{g}, \mathsf{h}\}$, and we pick $f_C = \mathsf{c}$, which is reachable from all objects in C. (In fact, C is strongly connected, and we could pick any element of C.) The resulting labels are displayed in Figure 2. Consider the local peak $\mathsf{e} \;{}_{[1]}\!\leftarrow \mathsf{a} \rightarrow_{[2]} \mathsf{f}$. We obtain the joining valley $\mathsf{e} \rightarrow_{[0]} \mathsf{c} \;{}_{[0]}\!\leftarrow \mathsf{a} \;{}_{[1]}\!\leftarrow \mathsf{f}$, which passes through a.

This particular peak is of interest because in [2], it is argued that any conversion between e and f that passes through a cannot result in a point-decreasing diagram. Evidently, that is not the case with our labeling, due to the fact that $\ell(\mathsf{f}) > \ell(\mathsf{a})$.

4 Conclusion

We have presented a new proof of Bognar's point version of decreasing diagrams based on van Oostrom's decreasing diagrams. Furthermore, we showed that point-decreasing diagrams are complete for confluence of finite ARSs.

The question whether the point version of decreasing diagrams is complete for countable ARSs remains open.

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