Point-Step-Decreasing Diagrams*

Bertram Felgenhauer[1]

University of Innsbruck, Austria
bertram.felgenhauer@uibk.ac.at

Abstract

Inspired by Bognar's point-decreasing diagrams, we present a generalisation of decreasing diagrams in which both the objects and the steps of conversions are labeled. We argue that this extension is more powerful than decreasing diagrams. However, it remains to be seen whether this power can be exploited in practice.

1 Introduction

There are two different approaches to proving confluence of abstract rewrite systems (ARSs) by decreasing diagrams. The first approach, by van Oostrom [4], labels the rewrite steps of an abstract rewrite system (ARS). In contrast, Bognar [1] proposed a variant that labels the points (i.e., objects) of an ARS instead. In this note we use the monotonic proof order from [2] to derive a point-step version of decreasing diagrams, which combines these two ideas.

This note is based on [3, Section 3.6].

2 Preliminaries

We use standard notation for abstract rewriting. An ARS $\langle \mathcal{A}, \to \rangle$ consists of a set of objects $\mathcal{A}$ and a (rewrite) relation $\to$ on $\mathcal{A}$. We denote the inverse, reflexive closure, symmetric closure and reflexive transitive closure of $\to$ by $\leftarrow$, $\to^=$, $\leftrightarrow$ and $\to^*$, respectively. An ARS $\to$ is Church-Rosser (equivalently, confluent) if $\leftrightarrow^* \subseteq \to^* \cdot {}^*\!\leftarrow$. Let L be an alphabet equipped with a well-founded order $\succ$. If $(\to_\kappa)_{\kappa \in L}$ is a family of relations on $\mathcal{A}$ then $\langle \mathcal{A}, (\to_\kappa)_{\kappa\in L} \rangle$ is a labeled ARS. For $M \subseteq L$ we let $\to_M = \bigcup_{\kappa \in M} \to_\kappa$. We define $\vee\kappa = \{\mu \mid \kappa \succ \mu\}$ and $\vee\kappa\mu = \vee\kappa \cup \vee\mu$.

We recall Greek strings [2], which we will use to represent conversions. For each $\kappa \in L$, there are three Greek letters, accented by acute ($\acute{\kappa}$), grave ($\grave{\kappa}$) or macron ($\bar{\kappa}$) accents. The notation $\hat{\kappa}$ represents a Greek letter with arbitrary accent. Greek strings are strings over Greek letters. We use the following notation for certain regular languages on Greek letters: $\kappa\!\succ$ represents any Greek letter whose label is less than κ, $[s]$ denotes an optional string, and $\{s\}$ denotes the Kleene star of s.

Definition 1. The order $\ggg_\bullet$ on Greek strings over L is defined inductively as follows:

$$s \ggg_\bullet t \qquad \text{iff} \qquad \langle s \rangle^g \; ((\succ, \ggg_\bullet)_{lex})_{mul} \; \langle t \rangle^g$$

where $>$ compares Greek letters by forgetting the accents, $(\cdot,\cdot)_{lex}$ and $(\cdot)_{mul}$ denote the lexicographic product and the multiset extension of relations, and the map

$$\langle s \rangle^g = [(\acute{\kappa}, q) \mid s = p\acute{\kappa}q] \cup [(\grave{\kappa}, p) \mid s = p\grave{\kappa}q] \cup [(\bar{\kappa}, \epsilon) \mid s = p\bar{\kappa}q]$$

collects acute letters together with their suffixes, grave letters together with their prefixes, and macron letters together with empty strings into a multiset. We also define $\ggg_\bullet^\Lambda = ((\succ, \ggg_\bullet)_{lex})_{mul}$.

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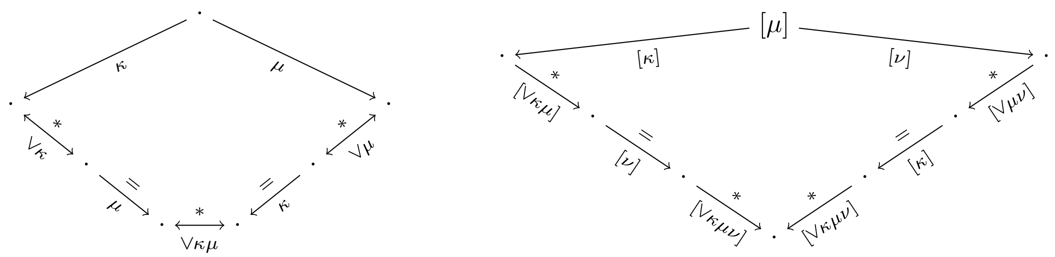

(a) Locally decreasing diagram. (b) Locally point-decreasing diagram.

Figure 1: Decreasing diagrams.

Example 2. Let $L = \{1,2,3\}$ equipped with the order $3 \succ 2 \succ 1$. We claim that $\bar{2}\grave{3}\bar{1} \ggg_\bullet \grave{3}\bar{2}\acute{2}\bar{1}$. In the resulting multiset comparison, it's easy to see that $(\grave{3},\bar{2})$ is larger than every element of the right-hand side multiset:

$$\langle \bar{2}\grave{3}\bar{1}\rangle^g = [(\bar{2},\epsilon),(\grave{3},\bar{2}),(\bar{1},\epsilon)] \ggg_\bullet^\Lambda [(\grave{3},\epsilon),(\bar{2},\epsilon),(\acute{2},\bar{1}),(\bar{1},\epsilon)] = \langle \grave{3}\bar{2}\acute{2}\bar{1}\rangle^g$$

In [2] it is shown that $\ggg_\bullet$ is a well-founded order on Greek strings that is monotonic, i.e., $q \ggg_\bullet r$ implies $pq \ggg_\bullet pr$ and $qp \ggg_\bullet rp$ for all Greek strings p, q and r.

We recall van Oostrom's and Bognar's decreasing diagrams results. (In [3, Section 3.5] it is shown that Bognar's result follows from van Oostrom's, but this is not immediately obvious.)

Theorem 3 (decreasing diagrams [4])**.** *Let $\langle \mathcal{A}, \to_\kappa\rangle_{\kappa\in L}$ be a labeled ARS. If for all $\kappa,\mu \in L$*

$$\underset{\kappa}{\leftarrow}\cdot\underset{\mu}{\rightarrow} \subseteq \underset{\vee\kappa}{\overset{*}{\longleftrightarrow}}\cdot \underset{\mu}{\overset{=}{\rightarrow}}\cdot \underset{\vee\kappa\mu}{\overset{*}{\longleftrightarrow}}\cdot\underset{\kappa}{\overset{=}{\leftarrow}}\cdot\underset{\vee\mu}{\overset{*}{\longleftrightarrow}}$$

then $\to_L$ is confluent. (See also Figure 1(a).)

Theorem 4 (point-decreasing diagrams [1])**.** *Let $\langle \mathcal{A}, \to\rangle$ be an abstract rewrite system and $\ell : \mathcal{A} \to L$ be a function labeling the objects. We annotate steps by the labels of their targets in square brackets, that is, we write $s \to_{[\ell(t)]} t$. For $M \subset L$ we define $\to_{[M]} = \bigcup_{\kappa\in M} \to_{[\kappa]}$. If every local peak $t \; {}_{[\kappa]}\!\leftarrow s \to_{[\nu]} u$ with $\mu = \ell(s)$ has a joining valley*

$$t \xrightarrow[{[\vee\kappa\mu]}]{*} \cdot \xrightarrow[{[\nu]}]{=} \cdot \xrightarrow[{[\vee\kappa\mu\nu]}]{*} \cdot \xleftarrow[{[\vee\kappa\mu\nu]}]{*} \cdot \xleftarrow[{[\kappa]}]{=} \cdot \xleftarrow[{[\vee\mu\nu]}]{*} u$$

then $\to$ is confluent. (See also Figure 1(b).)

3 Point-Step Decreasing Diagrams

In this section we present a unified result that encompasses both standard [4] and point-decreasing diagrams [1] results, and is, to our knowledge, strictly more general than either one. The key idea is to use a representation of conversions as Greek strings that alternates between steps and objects, mapping objects to macron letters and steps to accented letters.

Definition 5. Let $\langle \mathcal{A}, (\to_\kappa)_{\kappa\in L}\rangle$ be a labeled ARS. Furthermore let $\ell : \mathcal{A} \to L$ be a function labeling the objects. Then each conversion

$$s_0 \underset{\kappa_1}{\longleftrightarrow} \cdots \underset{\kappa_n}{\longleftrightarrow} s_n$$

S	$\hat{\kappa} \ggg_\bullet \{\kappa{\succ}\}$
M1	$\acute{\kappa}\bar{\mu} \ggg_\bullet \{\kappa\mu{\succ}\}[\acute{\kappa}]\{\mu{\succ}\}$
M2	$\acute{\kappa}\bar{\mu} \ggg_\bullet \{\kappa{\succ}\}\bar{\mu}\{\kappa{\succ}\}[\acute{\kappa}](\{\kappa{\succ}\} \cap \{\mu{\succ}\})$
P1	$\acute{\kappa}\bar{\mu}\grave{\nu} \ggg_\bullet \{\kappa\mu{\succ}\}[\acute{\kappa}]\{\mu{\succ}\}[\grave{\nu}]\{\mu\nu{\succ}\}$
P2	$\acute{\kappa}\bar{\mu}\grave{\nu} \ggg_\bullet \{\kappa\mu{\succ}\}[\grave{\nu}]\{\kappa\mu\nu{\succ}\}[\acute{\kappa}]\{\mu\nu{\succ}\}$
P3	$\acute{\kappa}\bar{\mu}\grave{\nu} \ggg_\bullet \{\kappa{\succ}\}\bar{\mu}\{\kappa{\succ}\}[\acute{\kappa}](\{\kappa{\succ}\} \cap \{\mu{\succ}\})[\grave{\nu}]\{\nu{\succ}\}$
P4	$\acute{\kappa}\bar{\mu}\grave{\nu} \ggg_\bullet \{\kappa{\succ}\}\bar{\mu}\{\kappa{\succ}\}[\grave{\nu}]\{\kappa\nu{\succ}\}[\acute{\kappa}]\{\nu{\succ}\}$
P5	$\acute{\kappa}\bar{\mu}\grave{\nu} \ggg_\bullet \{\kappa{\succ}\}[\grave{\nu}]\{\kappa\nu{\succ}\}\bar{\mu}\{\kappa\nu{\succ}\}[\acute{\kappa}]\{\nu{\succ}\}$

Table 1: Comparing short Greek strings.

has a *point-step interpretation* $\hat{\kappa}_1\bar{\ell}(s_1)\dots\bar{\ell}(s_{n-1})\hat{\kappa}_n$, where

$$\hat{\kappa}_i = \begin{cases} \acute{\kappa}_i & \text{if } s_{i-1} \; {}_{\kappa_i}\!\leftarrow s_i \\ \grave{\kappa}_i & \text{if } s_{i-1} \to_{\kappa_i} s_i \end{cases}$$

Note that the initial and final objects are omitted from the interpretation.

In order to obtain a point-step decreasing diagrams result, we map each conversion to its point-step interpretation, and then compare the resulting interpretations before and after pasting a local diagram by $\ggg_\bullet$. Pasting a local diagram corresponds to replacing a substring $\acute{\kappa}\bar{\mu}\grave{\nu}$ (i.e., the interpretation of a local peak without its endpoints) by some other Greek string corresponding to the joining conversion (again, without endpoints). We omit the endpoints because their labels do not change.

Lemma 6. *All comparisons from Table 1 are true.*

Proof. Cases S, M1 and M2 are covered by [2, Lemma 31]. Consider P1. Let $s = \acute{\kappa}\bar{\mu}\grave{\nu}$ and $t \in \{\kappa\mu{>}\}[\acute{\kappa}]\{\mu{>}\}[\grave{\nu}]\{\mu\nu{>}\}$. Then

$$\langle s\rangle^g = [(\acute{\kappa}, \bar{\mu}\grave{\nu}), (\bar{\mu}, \epsilon), (\grave{\nu}, \acute{\kappa}\bar{\mu})]$$

We claim that for all elements of $\langle t\rangle^g$ there is a larger element in $\langle s\rangle^g$, which establishes $\langle s\rangle^g \ggg_\bullet^\Lambda \langle t\rangle^g$. Indeed for pairs corresponding to $\{\kappa\mu{>}\}$, $\{\mu{>}\}$ or $\{\mu\nu{>}\}$, the first component is smaller than κ, μ or ν, and therefore the claim follows. The element corresponding to $[\grave{\mu}]$ (if any) is in $(\grave{\mu}, \{\kappa\mu{>}\}[\acute{\kappa}]\{\mu{>}\})$, and that is smaller than $(\grave{\nu}, \acute{\kappa}\bar{\mu})$ by property M1. For the element corresponding to $[\acute{\kappa}]$, a symmetric argument applies. Therefore, P1 holds.

For P2…P5 the same basic approach works: Apply $\langle\cdot\rangle^g$ to both sides, and then establish the resulting multiset comparison by $\ggg_\bullet^\Lambda$ using $>$ on the first component and properties M1 and M2 on the second component of the resulting pairs. □

Example 7. Let (M) denote an object whose label is in $M \subseteq L$, and let $\leftrightarrow_M^*$ stand for a conversion with all steps and intermediate objects (not including the initial and final objects of the conversion) having labels in M. Property P2 corresponds to the following conversion joining a local peak $t \;{}_\kappa\!\leftarrow s \to_\nu u$, where $\mu = \ell(s)$:

$$t \xleftrightarrow[\vee\kappa\mu]{*} (\vee\kappa\mu) \xrightarrow[\nu]{=} (\vee\kappa\mu\nu) \xleftrightarrow[\vee\kappa\mu\nu]{*} (\vee\kappa\mu\nu) \xleftarrow[\kappa]{=} (\vee\mu\nu) \xleftrightarrow[\vee\mu\nu]{} u$$

Theorem 8. *Let $\langle \mathcal{A}, (\to_\kappa)_{\kappa\in L}\rangle$ be a labeled ARS, and $\ell : \mathcal{A} \to L$ be a labeling function. Assume that every local peak $t \;{}_\kappa\!\leftarrow s \to_\nu u$, with $\mu = \ell(s)$ has a joining conversion $s \leftrightarrow^* t$ whose interpretation matches a right-hand side of P1, P2, P3, P4 or P5 from Table 1. Then $\to$ is confluent.*

Proof. Recall that $\overline{\ggg}_\bullet$ is well-founded and monotonic. We show that every conversion $t \leftrightarrow^* u$ has an equivalent valley proof $t \to^* \cdot {}^*\!\leftarrow u$ by well-founded induction on $t \leftrightarrow^* u$, measured by the interpretation of the conversion according to Definition 5 and ordered according to $\overline{\ggg}_\bullet$. If $t \leftrightarrow^* u$ is a valley proof then we are done. Otherwise, there must be a local peak, say

$$t \overset{*}{\leftrightarrow} t' \underset{\kappa}{\leftarrow} s' \underset{\nu}{\to} u' \overset{*}{\leftrightarrow} u \tag{P}$$

Let $\mu = \ell(s')$. Then the interpretation of (P) can be written as $p\acute{\kappa}\bar{\mu}\grave{\nu}r$, where p is the interpretation of $t \leftrightarrow^* t'$ followed by $\bar{\ell}(t')$ (if the conversion is non-empty) and r is the interpretation of $u' \leftrightarrow^* u$ preceded by $\bar{\ell}(u')$ (if the conversion is non-empty) By assumption, the local peak $t' \;{}_\kappa\!\leftarrow s' \to_\nu u'$ has a joining conversion $t' \leftrightarrow^* u'$ whose interpretation q satisfies $\acute{\kappa}\bar{\mu}\grave{\nu} \;\overline{\ggg}_\bullet\; q$ by Lemma 6. By monotonicity, this implies $p\acute{\kappa}\bar{\mu}\grave{\nu}r \;\overline{\ggg}_\bullet\; pqr$. Consequently, we can apply the induction hypothesis to the resulting conversion $t \leftrightarrow^* t' \leftrightarrow^* u' \leftrightarrow^* u$, whose interpretation is pqr (or smaller if $t' \leftrightarrow^* u'$ is an empty conversion), to conclude. □

Remark 9. Theorem 8 entails both decreasing diagrams and point-decreasing diagrams. To obtain decreasing diagrams, simply label all objects by a fresh label $\bot$ that is minimal with respect to $\succ$. Then case D of Table 1 (which encodes a decreasing diagram) corresponds to case P2, noting that each of the sets $\{\kappa\mu\succ\}$, $\{\kappa\mu\nu\succ\}$ and $\{\mu\nu\succ\}$ contains $\bar{\bot}$.

In order to obtain point-decreasing diagrams, let ℓ_p be the labeling function for establishing point-decreasingness. We label steps $s \to t$ by $(\ell_p(t), \top)$ and objects s by $(\ell_p(s), \bot)$, with the tuples ordered lexicographically by $\succ$ on the original L and $\top \succ \bot$. Then if we consider the joining conversion from Theorem 4, we easily see that it corresponds to case P2 of Table 6.

Corollary 10. *Let $\langle \mathcal{A}, \to \rangle$ be an ARS, and $\ell : \mathcal{A} \to L$ a labeling function. If for every peak $t \leftarrow s \to u$, either*

$$t \to v \leftarrow u \qquad \text{or} \qquad t \leftrightarrow v_1 \cdots v_n \leftrightarrow u$$

such that $\ell(s) = \ell(v)$ or $\ell(s) \succ \ell(v_i)$ for $1 \leq i \leq n$, then $\to$ is confluent.

Proof. Let $L_\bot = L \cup \{\bot\}$, where $\bot$ is a fresh label. We extend $\succ$ to $L_\bot$ by letting $\alpha \succ \bot$ for all $\alpha \in L$. Let $(\to_\alpha)_{\alpha \in L_\bot}$ be the labeled ARS given by $\to_\bot = \to$ and $\to_\alpha = \varnothing$. Then since both

$$\acute{\bot}\bar{\ell}(s)\grave{\bot} \;\overline{\ggg}_\bullet\; \grave{\bot}\bar{\ell}(v)\acute{\bot}$$

by P5 from Table 1 and

$$\acute{\bot}\bar{\ell}(s)\grave{\bot} \;\overline{\ggg}_\bullet\; \hat{\bot}\bar{\ell}(v_1)\ldots\bar{\ell}(v_n)\hat{\bot}$$

by P2, we conclude that the local diagrams are point-step decreasing and therefore $\to$ is confluent. □

Remark 11. Corollary 10 is interesting because to our knowledge, neither decreasing diagrams nor point-decreasing diagrams can prove it directly, without changing the joining conversions for the local peaks. (In particular, while step-decreasing diagrams are complete for confluence of countable ARSs, the proof picks particular joining valleys for local peaks.) This indicates that point-step decreasing diagrams strictly generalize decreasing diagrams and point-decreasing diagrams for practical purposes.

For the point-decreasing diagrams, the main obstacle presents itself as follows: In order to obtain a point-decreasing diagram (with labeling function ℓ') for joining $t \leftarrow s \to u$ as $t \leftrightarrow v_1 \ldots v_n \leftrightarrow u$ for $n > 1$, since we cannot assume anything about the labels of t and u, we will have to ensure that $\ell'(s) > \ell'(v_i)$ whenever $\ell(s) > \ell(v_i)$. This suggests using $\ell' = \ell$.

But then the case of joining $t \to v \leftarrow u$ with $\ell(s) = \ell(v)$ does not result in a point-decreasing diagram unless $\ell(t) \succ \ell(s)$ or $\ell(u) \succ \ell(s)$ holds. So the proof attempt fails.

For decreasing diagrams, the picture is less clear. Let us assume that the order $\succ$ on labels is total. The simplest labeling that makes the joining conversions $t \leftrightarrow v_1 \dots v_n \leftrightarrow u$ decreasing labels each step $s \to t$ by $\max(\ell(s), \ell(t))$. Then for the $t \to v \leftarrow u$ join, we have to join the peak $t \ {}_{\kappa}\!\leftarrow s \to_{\mu} u$ (with $\kappa = \max(\ell(s), \ell(t))$ and $\mu = \max(\ell(s), \ell(u))$) by $t \to_{\kappa} v \ {}_{\mu}\!\leftarrow u$. However, this is only a decreasing diagram if $\kappa = \mu$, and we cannot ensure this in general.

Remark 12. The results presented here extend to Church-Rosser modulo. Recall that an ARS $\to$ is Church-Rosser modulo $\vdash\!\dashv$ if $\leftrightarrow^* \subseteq \to^* \cdot \vdash\!\dashv^* \cdot {}^*\!\leftarrow$, where $\vdash\!\dashv$ is a symmetric relation. The idea is that equality steps $\vdash\!\dashv_\kappa$ are mapped to macron letters $\bar{\kappa}$ when interpreting conversions. In addition to interpretations of peaks $\acute{\kappa}\bar{\mu}\grave{\nu}$ one also needs to consider interpretations of cliffs $\acute{\kappa}\bar{\mu}\bar{\nu}$ in Lemma 6. Details can be found in [3, Chapter 3.6].

4 Conclusion

We have presented an extension of decreasing diagrams in which both steps and objects of an ARS are labeled. As argued in Remark 11, the extension appears to be more powerful than standard decreasing diagrams. Note that while decreasing diagrams are complete for confluence of countable ARSs, this fact does not often help us with finding concrete labelings that establish confluence, since cofinal derivations are hard to describe. On the other hand, Table 1 is quite intimidating, so it remains to be seen whether this result will be useful in practice. Perhaps the labeling framework from [5] could be extended to label terms as well.

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