Cumulative Habilitation Thesis

Proof Theory at Work:
Complexity Analysis of Term Rewrite Systems

Georg Moser

August 2009
dedicated to Claudia
Contents

1 Introduction 1

2 Summary 3

2.1 Status of Research 3

2.1.1 Termination in Rewriting 3

2.1.2 Complexity of Rewrite Systems 5

2.1.3 Proof-Theoretic Analysis of Complexity Analysis 7

2.2 Research Program 9

2.2.1 Modern Termination Techniques 9

2.2.2 Low-Complexity Bounding Functions 13

2.2.3 Strategies and Higher-Order Rewriting 15

2.3 Contributions 16

2.3.1 Modern Termination Techniques 16

2.3.2 Low-Complexity Bounding Functions 19

2.4 Related Work 20

2.5 Future Research 21

2.6 Conclusion 22

3 Relating Derivation Lengths with the Slow-Growing Hierarchy Directly 25

3.1 Introduction 25

3.2 The Lexicographic Path Ordering 27

3.3 Ordinal Terms and the Lexicographic Path Ordering 28

3.4 Fundamental Sequences and Sub-recursive Hierarchies 30

3.5 The Interpretation Theorem 32

3.6 Collapsing Theorem 38

3.7 Complexity Bounds 39

3.8 Conclusion 39

4 Proofs of Termination of Rewrite Systems for Polytime Functions 41

4.1 Introduction 41

4.2 A Rewrite System for FP 42

4.3 A Path Ordering for FP 46

4.4 Predicative Recursion and POP 50

4.5 Conclusion 52
Contents

5 Derivational Complexity of Knuth-Bendix Orders revisited 53
  5.1 Introduction 53
  5.2 Preliminaries 55
  5.3 The Knuth Bendix Orders 57
  5.4 Exploiting the Order-Type of KBOs 58
  5.5 Derivation Height of Knuth-Bendix Orders 59
  5.6 Bounding the Growth of \( \text{Ah}_n \) 62
  5.7 Derivation Height of TRSs over Infinite Signatures Compatible with KBOs 65

6 Complexity Analysis by Rewriting 69
  6.1 Introduction 69
  6.2 Preliminaries 72
  6.3 Main Result 73
  6.4 Polynomial Path Order on Sequences 76
  6.5 Predicative Interpretation 78
  6.6 Experimental Data 80
  6.7 An Application: Complexity of Scheme Programs 83
  6.8 Conclusion 84

7 Proving Quadratic Derivational Complexities using Context Dependent Interpretations 87
  7.1 Introduction 87
  7.2 Context Dependent Interpretations 89
  7.3 Automated Search for Context Dependent Interpretations 91
  7.4 Derivational Complexities Induced by Polynomial Context Dependent Interpretations 95
  7.5 Experimental Results 98
  7.6 Conclusion 101

8 Automated Complexity Analysis Based on the Dependency Pair Method 103
  8.1 Introduction 103
  8.2 Preliminaries 105
  8.3 The Dependency Pair Method 107
  8.4 Usable Rules 109
  8.5 The Weight Gap Principle 112
  8.6 Reduction Pairs and Argument Filterings 114
  8.7 Experiments 117
  8.8 Conclusion 118

9 Complexity, Graphs, and the Dependency Pair Method 121
  9.1 Introduction 122
  9.2 Preliminaries 123
  9.3 Complexity Analysis Based on the Dependency Pair Method 125
  9.4 Dependency Graphs 126

iv
Contents

9.4.1 From Cycle Analysis to Path Detection 127
9.4.2 Refinement Based on Path Detection 128
9.5 Conclusion 134
Preface

This cumulative habilitation thesis is based on the following publications studying the complexity of term rewrite systems. It goes without saying that for all included papers I wrote the manuscript.


*Own Contribution:* Andreas Weiermann introduced me to the field of complexity analysis of term rewriting systems and suggested the topic of the paper. He made suggestions towards the proof methodology, but I was in charge to fill in the details. Hence I crafted the central notions and theorems and proved their correctness.


*Own Contribution:* Toshiyasu Arai suggested a close study of a term-rewriting characterisation of the polytime computable function given by Beckmann, Weiermann (Arch. Math. Log. 36(1):11–30, 1996). This study forms the basis of the paper. The established results were obtained jointly through iterated revision of the introduced concepts, theorems, and proofs.


*Own Contribution:* The theoretical contribution of this paper was entirely my work. Martin Avanzini essentially implemented the technique and was responsible to provide the experimental evidence.

*Own Contribution:* The paper provides an extension and clarification of Andreas Schnabl’s master thesis conducted under my supervision. Apart from introducing Schnabl to the topic of the paper, I refined and smoothened essentially all introduced concepts and extended the theoretical contributions of the paper.


*Own Contribution:* I suggested the topic of complexity analysis to Nao Hirokawa and extended initial ideas that were presented by him. For example the focus shift towards the dependency pair method was my idea. The finally established results were obtained jointly through iterated revision of the introduced concepts, theorems, and proofs.


*Own Contribution:* This paper extends the results presented in paper (vi) and the initial idea of the extension was the result of joint discussion. However I was in charge of filling in the details. Hence I crafted the central notions and theorems and proved their correctness. Apart from his contribution to the initial idea, Hirokawa’s main contribution was the implementation of the method and he was responsible for providing the experimental evidence.

In order to keep a uniform presentation textual changes to these papers have been necessary. Furthermore minor shortcomings have been repaired.

As an indication of the broadness of my research, below I also mention selected publications from the areas automated deduction, proof theory, and rewriting.


**Acknowledgements**

I would like to take this opportunity to thank all my colleagues that contributed to this thesis. In particular my gratitude goes to my co-authors Andreas Schabl, Andreas Weiermann, Martin Avanzini, Nao Hirokawa, and Toshiaysu Arai, who kindly allowed me to include our joint papers into this thesis.

Special thanks go to my past and present colleagues in the Computational Logic Group. I want to thank Aart Middeldorp, Andreas Schnabl, Anna-Maria Scheiring, Christian Sternagel, Christian Vogt, Clemens Ballarin, Friedrich Neurauter, Harald Zankl, Martin Avanzini, Martin Korp, Martina Ingenhaeff, Mona Kornherr, René Thiemann, Sandra Adelt, Simon Bailey, and Stefan Blom, for providing a pleasant and inspiring working environment over the years.

Last but certainly not least, I want to express my gratitude towards my parents Helga Moser and Gerhard Margreiter (†) who do and did their best in supporting me throughout my studies and research.

Innsbruck, December 30, 2009

Georg Moser
1

Introduction

This thesis is concerned with investigations into the complexity of term rewriting systems. Moreover the majority of the presented work deals with the automation of such a complexity analysis. The aim of this introduction is to present the main ideas in an easily accessible fashion to make the result presented accessible to the general public. Necessarily some technical points are stated in an over-simplified way. I kindly refer the knowledgeable reader to the technical summary as presented in Chapter 2.

Since the advent of programming languages formalisms and tools have been thought and developed to express and prove properties of programs. The general goal of the here presented work is to develop logical tools for analysing the complexity of programs (automatically whenever possible).

When reasoning about properties of programs, we need to fix the programs we aim to analyse. For a number of reasons this is a difficult choice. On one hand, we would want our analysis to be directly transferable into applications, so that we indeed can analyse existing software packages without further ado. On the other hand, we want our results to be as general as possible, hence our analysis should abstract from individual features of programming languages.

There is no decisive answer to these conflicting priorities, but for me the generality of the obtained results appears more important. Successful investigation of abstract programs can often be adapted to real-life contexts with ease, while the generalisation of tools and methods invented in a specific setting to a more abstract level, may prove to be difficult or even impossible. Still it is crucial that the applicability of the introduced general concepts is not lost.

The most abstract formalism we may consider in computer science, are so-called abstract models of computation. Such abstractions have been intensively studied by mathematicians and logicians at the beginning of the 20th century. Then the problem was to fix a suitable mathematical notion of computation. Several equivalent concepts like combinatory logic, λ-calculus, recursive functions, register machines, and Turing machines have been put forward by Curry, Church, Gödel, Kleene, Turing and others. The central computational mechanism of Church’s λ-calculus is rewriting and bluntly we could argue that the λ-calculus is simply rewriting in disguise.

Term rewriting is a conceptually simple, but powerful abstract model of computation. Let me describe this computation model in its most abstract and most simple form.
We assert a collection of widgets together with rules that govern the replacement of
one widget by another. For example, if we take the set of states of a computer as our
collection of widgets and allow the replacement of one state by another, whenever the
latter is reachable (in some sense) from the first, then this constitutes a term rewrite
system. A fundamental property of rewrite systems is termination, the non-existence
of endless rewrite steps, or replacements of one widget by another. Conclusively strong
techniques have been designed to ensure termination of term rewrite systems. In recent
years the emphasis shifted towards techniques that automatically verify termination of
a given term rewrite system.

Observe that despite its simplicity, rewriting is an abstract computation model that
is equivalent to all notions of computability mentioned above. For example any OCaml
program is easily representable as a rewrite system. If this encoding is done carefully
enough, then termination techniques for rewriting become applicable to show termination
of programs, sometimes even fully automatically. The use of a functional programming
language as an example may seem restrictive, as the representation of an OCaml pro-
gram as a term rewrite system is typically simple. However recent work aims at the
incorporation of imperative programming languages like Java or C. For example Java
Bytecode programs become applicable to this setting if a termination graph, representing
the program flow, is provided in a pre-processing step. The structural information
of these graphs can then be encoded as term rewrite systems.

Once we have verified termination of a given term rewrite system, we have (perhaps
automatically) established a very important property. But in accordance with the legacy
of rewriting as a computation model, we should strive for more. For a terminating sys-

tem, we can consider the following problem: Given some widget, how many replacement
steps can we perform till no more replacement is possible? Termination assert that this
problem is well-defined.

This naturally entails investigations into the complexity of term rewrite systems. A
term rewrite system is considered of higher complexity, if the number of possible rewrite
steps is larger. In other words the complexity of a rewrite system is measured through
the maximal possible computation steps possible with this abstract program.

The investigations into this problem are the topic of my thesis. In line with earlier
results presented in the literature the study is performed as an analysis of termination
methods. I.e., instead of directly seeking techniques to establish the complexity of a
given term rewrite system I have studied the complexity induced by termination meth-
ods. These investigations cover well-established termination orders as well as modern
(automatable) termination techniques.

Through this indirect study, a higher level of understanding is possible. Not only have
I provided new techniques to analyse the complexity of a given rewrite system, but at
the same time I have rendered insights into the expressivity of termination methods.
These results may lead to a new generation of termination provers for term rewrite
system. Currently a termination prover, if successful, will simply output yes. In the
future, termination provers can perhaps be identified with complexity analysers: given
a terminating rewrite system the laconic yes of the prover is replaced by expressive
information on the complexity of the system.
2
Summary

2.1 Status of Research

As already mentioned in the introduction, term rewriting is a conceptually simple, but powerful abstract model of computation. The foundation of rewriting is equational logic and term rewrite systems (TRSs for short) are conceivable as sets of directed equations.

To be a bit more formal, let $F$ denote a finite set of function symbols, i.e., a signature and let $V$ denote a countable set of variables. Then the set of terms over $F$ and $V$ is denoted as $T(F,V)$. A TRS $R$ is a set of rewrite rules $l \rightarrow r$, where $l$ and $r$ are terms. The rewrite relation $\rightarrow_R$ is the least binary relation on the set of terms $T(F,V)$ containing $R$ such that (i) if $s \rightarrow_R t$ and $\sigma$ a substitution, then $s\sigma \rightarrow_R t\sigma$ holds, and (ii) if $s \rightarrow_R t$, then for all $f \in F$: $f(\ldots,s,\ldots) \rightarrow_R f(\ldots,t,\ldots)$ holds. We sometimes write $\rightarrow$, instead of $\rightarrow_R$, if no confusion can arise from this.

The implicit orientation of equations in rewrite systems naturally gives rise to computations, where a term is rewritten by successively replacing subterms by equal terms until no further reduction is possible. Such a sequence of rewrite steps is also called a derivation. Term rewriting forms a Turing complete model of computation, hence fundamental questions as for example termination of a given TRS, are undecidable in general. Furthermore term rewriting underlies much of declarative programming. As a special form of equational logic it has also found many applications in automated deduction and verification.

In this thesis, I will consider termination problems in term rewriting and the complexity of term rewrite systems as measured by the maximal length of derivations.

2.1.1 Termination in Rewriting

In the area of term rewriting [137] powerful methods have been introduced to establish termination of a given TRS $R$. Earlier research mainly concentrated on inventing suitable reduction orders—for example simplification orders, see [137] Chapter 6]—capable of proving termination directly.

As an example let us consider the multiset path order (MPO for short), cf. [14]. Let $>$ denote a strict partial order on the signature $F$. We call $>$ a precedence. Let $s, t$ be
terms. For \( s, t \not\in V \), we can write \( s = f(s_1, \ldots, s_m), t = g(t_1, \ldots, t_n) \). We define \( s >_{mpo} t \) if one of the following alternatives hold:

- there exists \( i \in \{1, \ldots, m\} \) such that \( s_i >_{mpo} t \) or \( s_i = t \),
- \( f > g \) and \( s >_{mpo} t_j \) for all \( 1 \leq j \leq n \), or
- \( f = g \) and \( \{s_1, \ldots, s_m\} >_{mpo} \{t_1, \ldots, t_n\} \).

Here \( >_{mpo} \) denotes the multiset extension of \( >_{mpo} \).

The definition of MPO entails, that if a TRS \( R \) is compatible with an MPO \( >_{mpo} \), i.e., if \( R \subseteq >_{mpo} \) then termination of \( R \) follows. Clearly the reverse direction need not hold. Consider the TRS \( R_1 \) over the signature \( F = \{f, \circ, e\} \) taken from [76]:

\[
\begin{align*}
    f(x) \circ (y \circ z) &\rightarrow x \circ (f(y) \circ z) \\
    f(x) \circ (y \circ (z \circ w)) &\rightarrow x \circ (z \circ (y \circ w)) \\
    f(x) &\rightarrow x
\end{align*}
\]

The TRS \( R_1 \) is terminating. However we cannot find a precedence \( > \) such that \( R_1 \subseteq >_{mpo} \). This is due to the fact that \( R_1 \) encodes the Ackermann function as we will see in the next section.

In recent years the emphasis shifted towards transformation techniques like the dependency pair method [9], its extension the dependency pair framework [138] or semantic labeling [149, 150]. The advantage—in particular of the dependency pair method—is that they are easily automatable.

Here we briefly recall the basics of the dependency pair method. Below on page 2.2.1 we will also recall the semantic labeling technique. In the presentation of the dependency pair method we follow [9, 70]. Let \( t \) be a term. We set \( t^\# := t \) if \( t \in V \), and \( t^\# := f^\#(t_1, \ldots, t_n) \) if \( t = f(t_1, \ldots, t_n) \). Here \( f^\# \) is a new \( n \)-ary function symbol called dependency pair symbol. For a signature \( F \), we define \( F^\# = F \cup \{ f^\# \mid f \in F \} \). The set \( DP(R) \) of dependency pairs of a TRS \( R \) is defined as the set of pairs \( l^\# \rightarrow u^\# \), where \( l \rightarrow r \in R \) and \( u \) is a subterm of \( r \), whose root symbol is a defined function symbol. Moreover \( u \) is not a proper subterm of \( l \). 

It is not difficult to see that a TRS \( R \) is terminating if and only if there exists no infinite derivation of the following form

\[
t_1^\# \rightarrow_R^* t_2^\# \rightarrow_{DP(R)}^* t_3^\# \rightarrow_R^* \ldots ,
\]

where for all \( i > 0 \), \( t_i^\# \) is terminating with respect to \( R \). This is a key observation in the formulation of the basic setting of the dependency pair method. We obtain the following characterisation of termination due to Arts and Giesl:

- A TRS \( R \) is terminating if and only if there exist a reduction pair \((\succ, \triangleright)\) such that \( DP(R) \subseteq \succ > \) and \( R \subseteq \triangleright \).
2.1 Status of Research

Here a reduction pair \((\succeq, \succ)\) consists of a rewrite preorder \(\succeq\) and a compatible well-founded order \(\succ\) which is closed under substitutions; compatibility means the inclusion \(\succeq \cdot \succ \subseteq \succ \).

Another development is the use of automata techniques to prove termination \([54, 55, 98]\). Moreover the technique to show termination by building an order-preserving mapping into a well-founded domain has received renewed attention \([81, 48, 95]\).

These methods, among others, are used in several recent software tools that aim to prove termination automatically. We mention AProVE \([60]\), CiME \([42]\), Jambox \([48]\), Matchbox \([143]\), MU-TERM \([107]\), TPA \([93]\), TTT2 \([99]\). In the termination competition (a subset of) these provers compete against each other in order to prove termination of TRSs automatically, see

\[
\text{http://termcomp.uibk.ac.at},
\]

for this ongoing event.

2.1.2 Complexity of Rewrite Systems

In order to assess the complexity of a TRS it is natural to look at the maximal length of derivation sequences, a program that has already been suggested in \([79]\). See also \([35]\) for a complementary study of the complexity of term rewrite systems.

The derivational complexity function with respect to a (terminating) TRS \(R\) relates the length of a longest derivation sequence to the size of the initial term. Observe that the derivational complexity function is conceivable as a measure of proof complexity. Suppose an equational theory is representable as a convergent (i.e., a confluent and terminating) TRS, then rewriting to normal form induces an effective procedure to decide whether two terms are equal over a given equational theory. Thus the derivational complexity with respect to a convergent TRS amounts to the proof complexity of this proof of identity.

In order to make further discussion more concrete, we present the central definitions. Let \(R\) denote a finitely branching and terminating TRSs. The derivation length function \(dl(t, \rightarrow)\) of a term \(t\) with respect to a rewrite relation \(\rightarrow\) is defined as

\[
dl(t, \rightarrow) = \max\{n \mid \exists(t_0, \ldots, t_n): t = t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_n\}\]

To make the notion of derivation length independent of the choice of \(t\) one defines the derivational complexity function (with respect to \(R\)):

\[
dc_R(n) = \max(\{dl(t, \rightarrow_R) \mid |t| \leq n\}) ,
\]

where \(|\cdot|\) denotes a suitable term-complexity measure of \(t\), e.g. the number of symbols in \(t\). Observe that even for terminating and finitely branching TRS \(R\), the induced derivational complexity function \(dc_R\) is only well-defined, if either the signature \(F\) is finite or \(R\) is finite. Hofbauer and Lautemann \([79]\) showed that for finite TRS

- a termination proof by polynomial interpretations implies a double-exponential upper bound on the derivational complexity.
With respect to syntactically defined termination orders, Hofbauer \cite{76,77} established that for finite TRSs

- a termination proof via the multiset path order implies that there is a primitive recursive bound on the derivational complexity.

It is this result that explains why we cannot find a multiset path order compatible with the TRS $R_1$; recall that this TRS essentially encodes the Ackermann function, which in turn implies that its derivational complexity is at least the Ackermann function.

Weiermann \cite{145}, and Lepper \cite{103} established that for finite TRSs

- a termination proof via the lexicographic path order (LPO for short) induces a multiple recursive bound on the derivational complexity, and

- if termination is shown by the Knuth-Bendix order (KBO for short), then the derivational complexity function is a member of $\text{Ack}(O(n), 0)$, where $\text{Ack}$ denotes the binary Ackermann function.

In all mentioned cases the upper bounds are optimal, i.e., it is possible to provide TRSs, whose derivational complexity function form a tight lower bound on the established upper bounds.

For a specific TRS $R$ the mentioned results yield precise upper bounds on the derivational complexity of $R$, i.e., depending on $R$, one can compute exact upper bounds. Therefore, these results constitute an a priori complexity analysis of TRSs provably terminating by polynomial interpretations, MPOs, LPOs, or KBOs. As term rewriting forms the basis of declarative programming, such complexity results transcend naturally to (worst-case) complexity results on declarative programs.

It is well-known that all mentioned termination methods are incomparable in the sense that there exist TRSs whose termination can be shown by one of the thes techniques, but not by any of the other. Still it is a deep and interesting question, how to characterise the strength of termination methods in the large. Dershowitz and Okada argued that the order type of the employed reduction order would constitute a suitable uniform measure, cf. \cite{47}. Clearly the complexity induced by a termination method serves equally well (or perhaps better) as such a measure, cf. \cite{79}. See \cite{76,30,78,101,140} for further reading on this subject.

In the remainder of this thesis I refer to investigations on (derivational) complexities of TRSs as complexity analysis of term rewrite systems.

To conclude this section, let me apply Lepper’s result to the motivating TRS $R$. I kindly refer the reader to Chapter \cite{5} for a formal definition of KBO. Consider the precedence $>$ defined by $f > o > e$ together with the weight function $w(f) = w(o) = 0$ and $w(e) = 1$. Let $>_{kbo}$ be the KBO induced by $>$ and $w$. Then for all rules $l \rightarrow r \in R$, $l >_{kbo} r$. As $R$ is compatible with $>_{kbo}$, we infer that the derivational complexity of $R$ is bounded from above by the Ackermann function. Moreover this upper bound is tight, cf. \cite{76}. Hence we conclude that the derivational complexity function $dc_{R_1}$ features the same growth rate as the Ackermann function.
2.1.3 Proof-Theoretic Analysis of Complexity Analysis

The study of the length of derivations of TRSs has stirred some attention in proof theory, cf. [144, 30, 6, 50]. This proof-theoretic interest in termination proofs and their induced complexity is not surprising. After all, the conceptual theme here is an old theme of proof theory. We can suit a conception question dedicated to Kreisel to the present context and ask:

What more do we know from a termination proof, than the mere fact of termination?

While Hofbauer, Weiermann, and Lepper gave answers to this question for specific instances of termination proofs it was Buchholz who delivered a direct proof-theoretic analysis. In order to explain this result, we need a few definitions.

As usual Peano Arithmetic refers to the first-order axiomatisation of number theory and $I\Sigma_1$ is the fragment of Peano Arithmetic, where the axiom of mathematical induction is restricted to existential induction formulas. The fragment $I\Sigma_1$ is relatively weak, but strong enough to prove the totality of primitive recursive functions.

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function on the naturals and let $\langle f(x) = y\rangle$ denote a suitable chosen computation predicate for the function $f$. Then $f$ is called provably recursive (in the theory $T$) if $T \vdash \forall x \exists y \langle f(x) = y\rangle$ holds, i.e., the totality of $f$ is provable in $T$.

Note that the provably recursive functions of the theory $I\Sigma_1$ are exactly the primitive recursive functions. Let $\succ$ denote a simplification order like MPO, LPO, or KBO and let $W$ denote the accessible part of $\succ$ on $T(F, \mathcal{V})$, i.e.,

$$W = \bigcap \{X \subseteq T(F, \mathcal{V}) | \forall t(\forall s (s < t \rightarrow s \in X) \rightarrow t \in X)\}.$$

Then well-foundedness of $\succ$ can be shown using (second-order) induction over $W$ (see [30] but also [66]). In proof one uses the axioms ($\ddagger$) : $\forall t(\forall s (s < t \rightarrow s \in X) \leftrightarrow t \in W)$ and ($\ddagger$) : $\forall t \in W(\forall s (s < t \rightarrow F(s)) \rightarrow F(t)) \rightarrow \forall t \in WF(t)$, together with the definition of $\succ$.

Based on this well-foundedness proof, Buchholz observes that Hofbauer’s result is a consequence of the following meta-theorem, cf. [30].

If $\succ$ is a primitive recursive relation on $T(F, \mathcal{V})$ such that $I\Sigma_1 \vdash \langle s \succ t \rangle$ holds and $W$ is a $\Sigma_1$-set such that $I\Sigma_1$ proves axioms ($\ddagger$) and ($\ddagger$) for all $\Sigma_1$-formulas, then well-foundedness of $\succ$ is $I\Sigma_1$-provable.

To see this, Buchholz defines finite approximations $\succ_k$ of $\succ$ so that the assertions of the meta-theorem are fulfilled. Furthermore he proves that compatibility of a TRS $\mathcal{R}$ with $\succ_\text{mpo}$ implies that $\mathcal{R} \subseteq \succ_k$ for some $k$ depending only on $\mathcal{R}$. Conclusively the meta-theorem asserts that well-foundedness of $\succ_k$ is provable in $I\Sigma_1$. Hence the derivational complexity function $dc_\mathcal{R}$ is contained in the class of provably recursive functions of $I\Sigma_1$.

And thus the derivational complexity function $dc_\mathcal{R}$ is primitive recursive.

The definition of these approximations is surprisingly simple: It suffices to guarantee that $\succ_k$ fulfils the definition of MPO above and additionally $s \succ_k t$ implies $|s| + k \geq |t|$, where
2 Summary

cf. [30]. A similar argument works for LPO, i.e., Hofbauer’s and Weiermann’s result can both be obtained directly by a proof-theoretic analysis. It is worthy of note that Buchholz’s proof is more general than the combinatorial arguments of Hofbauer and Weiermann. First observe that in any case, we cannot dispense the assumption that the set \( \{ |r| \mid l \to r \in R \} \) is bounded. However in [77] interpretations into the natural number are employed that crucially rest on the cardinality of \( \mathcal{F} \) and the maximal arity of the symbols in \( \mathcal{F} \). In contrast to this [30] makes only use of the finiteness of the signature.

Observe that a proof-theoretical analysis of KBO along this lines is not possible. Although well-foundedness of finite approximations of KBO are formalisable in Peano Arithmetic, the needed fragment is too strong to provide an optimal complexity analysis. On the other hand proof theory can be used successfully to extend Lepper’s result (see Chapter 5).

The connection of rewriting and proof theory is also addressed in [36] (see also Chapter 3) where Cichon emphasises a connection between the order type of a given reduction order \( \succ \) and the induced derivational complexity. More precisely, the so-called Cichon’s principle can be formulated as follows.

The (worst-case) complexity of a rewrite system for which termination is provable using a reduction order of order type \( \alpha \) is eventually dominated by a function from the slow-growing hierarchy along \( \alpha \).

Here the slow-growing hierarchy denotes a hierarchy of number theoretic functions \( G_\alpha \), indexed by transfinite ordinals \( \alpha \), whose growth rate is relatively slow: for example \( G_\omega(x) = x + 1 \), where \( \omega \) denotes the first limit ordinal. This function hierarchy is sometimes called point-wise in the literature, cf. [63]. See [32, Chapter 3] for further reading.

It ought to be stressed that this principle is false in general. According to Cichon’s principle, for any simply terminating TRS \( R \), the derivational complexity function \( dc_R \) should be majorised by a multiple-recursive function \([125]\). This however, is not true. In [139] Touzet introduced a rewrite system \( R \) coding a restrained version of the (standard) Hydra Battle [92] such that no multiple-recursive function can majorise the derivational complexity function. Furthermore, \( R \) is simply terminating.

Thus the principle fails even for simply terminating rewrite systems. Motivated by these negative results Touzet asserts that the Hardy hierarchy, a hierarchy of rather fast growing functions, index by ordinals is the right tool to connect the order type and derivation lengths. This point is enforced by later results due to Lepper, cf. [105].

However, note that Cichon’s principle is correct for two instances of simplification orders mentioned above: MPO and LPO. Essentially this follows from the mentioned results by Hofbauer and Weiermann, cf. [77] [144]. Buchholz’s proof-theoretic analysis provides some explanation. Namely for \( R \) compatible with MPO, or LPO, the termination proof does not make full use of the order type of (the class of) MPOs or LPOs, but only in a point-wise way. Note that the termination proof can even be formalised in a provability relation that makes use of ordinals only in a point-wise way (see Arai [5] for a more precise account of this connection).
Let me conclude this section by mentioning that Cichon’s principle underlies the open
problem # 23 in the list of open problems in rewriting (RTALooP for short, see http://rtaloop.mancoosi.univ-paris-diderot.fr/).

Must any termination order used for proving termination of the Battle of
Hydra and Hercules-system have the Howard ordinal\(^1\) as its order type?

In [116] (see also [46]) I resolve this problem by answering it in the negative.

2.2 Research Program

The goal of my research is to make complexity analysis of term rewrite systems:

- **modern** - by studying the complexities induced by modern termination techniques,
- **useful** - by establishing refinements of existing termination techniques guaranteeing
  that the induced complexity is bounded by functions of low computational
  complexity, for example polytime computable function,
- **broad** - by analysing the complexities of higher-order rewrite systems and for TRSs
  based on particular rewrite strategies.

Further I want to ensure that the results that are developed in these three areas provide
computable and precise bounds. To this avail I am working (together with Avanzini
and Schnabl) on a software tool: the Tyrolean Complexity Tool (TCT for short) that
incorporates the most powerful techniques to analyse the complexity of rewrite systems
that are currently at hand.

In order to test the competitive capability of TCT, a specialised category for complexity
analysers has been integrated into the international termination competition; see http://termcomp.uibk.ac.at or Section 2.6 for further details.

2.2.1 Modern Termination Techniques

Modern termination provers rarely employ base orders as those mentioned above directly.
To the contrary almost all modern termination provers use variants of the dependency
pair method to prove termination. Hence, we cannot easily combine the result of a
modern termination prover and the above results to get insights on the complexity of a
given terminating TRS \(\mathcal{R}\).

To improve the situation, I investigated modern termination techniques, as are usu-
ally employed in termination provers, in particular I studied the complexities induced
by the dependency pair method, cf. Section 2.3.1. As indicated below, a complexity
analysis based on the dependency pair method is and (in its full generality) remains a
challenging task. See [118] for recent developments in the complexity analysis of this

---

\(^1\) The Howard-Bachmann ordinal is the proof theoretic ordinal of the arithmetical theory of one inductive
definition, see [32, Chapter 3]. Note that the Howard-Bachmann ordinal easily dwarfs the proof-
theoretical ordinal of Peano Arithmetic \(\epsilon_0\).
Consider a sorting algorithm \( P \) like insertion sort and its implementation in a functional programming language like OCaml:

```ocaml
let rec insert x =
  | [] -> [x]
  | y :: ys -> if x <= y then x :: y :: ys
              else y :: insert x ys ;;

let rec sort =
  | [] -> []
  | x :: xs -> insert x (sort xs) ;;
```

It is not difficult to translate this program into a TRS \( R \), such that termination of \( R \) implies termination of \( P \). With ease, termination of \( R \) can be verified automatically. If we can extend automatic termination proofs by expressive certificates on the complexity of \( R \), we obtain an automatic complexity analysis on \( P \). I.e., the prover gives us in addition to the assertion that \( R \) is terminating, an upper bound on the complexity of \( R \) (and therefore of \( P \)).

Of course this goal requires theoretical and practical work: Firstly deep theoretical considerations on the complexity induced by modern termination techniques are necessary and secondly modern termination provers have to be extended suitably to render the sought certificates automatically. Finally the complexity preservation of the transformation from the program \( P \) into the TRS \( R \) has to be established. Adapting transformation techniques as mentioned in the Introduction, it seems possible to extend this approach to imperative programming languages like Java or C without too much difficulties. See [124, 52] for current work on the termination analysis of imperative programs via rewriting.

In the following I discuss the challenges of this endeavour for the key examples of the dependency pair method and semantic labeling. In particular a complete analysis of the former is of utmost importance as this technique has extended the termination proving power of automatic tools significantly. To clarify my point, I briefly state the central observations and apply the dependency pair method to the example given in the introduction. For further information on the concepts and definitions employed, I kindly refer the reader to [9, 70]; further refinements can be found e.g. in [72, 61].

- A TRS \( R \) is terminating if and only if for every cycle \( C \) in the dependency graph \( DG(\mathcal{R}) \) there are no \( C \)-minimal rewrite sequences.

- If there exists an argument filtering \( \pi \) and a reduction pair \((\succeq, >)\) so that 
  \( \pi(R) \subseteq \succeq, \pi(C) \subseteq \succeq \cup >, \) and \( \pi(C) \cap > \neq \emptyset \), then there are no \( C \)-minimal rewrite sequences.

Note that this result is a refinement of the characterisation of termination mentioned on page 4 above. The dependency graph \( DG(\mathcal{R}) \) essentially plays the role of a call graph in program analysis.
Efficient implementations of the dependency pair method consider maximal cycles instead of cycles\footnote{In the literature maximal cycles are sometimes called \textit{strongly connected components}. We use this notion in its original graph-theoretic definitions later on, see Chapter \ref{ch:graph_theory}. Hence I refrain from following this convention.}. Moreover the stated criteria are applied recursively, by disregarding dependency pairs that are already strictly decreasing, cf. \cite{70}. Consider the TRS $R_1$, defined on page 4. In the first step the rules (1)–(3) are extended by the TRS $\text{DP}(R_1)$:

\begin{align}
    f(x) \circ^\sharp (y \circ z) &\rightarrow x \circ^\sharp (f(f(y)) \circ z) & (4) \\
    f(x) \circ^\sharp (y \circ z) &\rightarrow f(f(y)) \circ^\sharp y & (5) \\
    f(x) \circ^\sharp (y \circ z) &\rightarrow f^\sharp(f(y)) & (6) \\
    f(x) \circ^\sharp (y \circ z) &\rightarrow f^\sharp(y) & (7) \\
    f(x) \circ^\sharp (y \circ (z \circ w)) &\rightarrow x \circ^\sharp (z \circ (y \circ w)) & (8) \\
    f(x) \circ^\sharp (y \circ (z \circ w)) &\rightarrow z \circ^\sharp (y \circ w) & (9) \\
    f(x) \circ^\sharp (y \circ (z \circ w)) &\rightarrow y \circ^\sharp w & (10)
\end{align}

The next step is to compute an approximated dependency graph for $\text{DP}(R_1)$, presented in Figure \ref{fig:dependency_graph_dp_r1}.

This graph contains only one maximal cycle comprising the rules \{4, 5, 8, 9, 10\}. By taking the polynomial interpretation $f_N(x) = x$ and $\circ_N(x, y) = \circ_N^\sharp(x, y) = x + y + 1$ the rules in \{1 – 4, 8\} are weakly decreasing and the rules in \{5, 9, 10\} are strictly decreasing. The remaining maximal cycle (4, 8) is handled by the subterm criterion \cite{72}.

This simple example should clarify the challenge of the dependency pair method in the context of complexity analysis. Recall that the TRS $R_1$ (essentially) represents the binary Ackermann function. Hence the complexity cannot be bounded by a primitive recursive function. However, the only information we can directly gather from the given

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dependency_graph_dp_r1.png}
\caption{The approximated dependency graph $\text{DP}(R_1)$.}
\end{figure}
proof is the use of polynomial interpretations and the subterm criterion. Neither of these methods is individually—i.e., as a basic termination technique—of sufficient strength to yield an upper bound on the complexity of \( \mathcal{R} \).

**Semantic Labeling**

Similar to the dependency pair method, semantic labeling is a transformation technique. Its central idea is to employ semantic information on the given TRS \( \mathcal{R} \) to transform \( \mathcal{R} \) into a labeled TRS \( \mathcal{R}_{\text{lab}} \) such that \( \mathcal{R} \) is terminating if \( \mathcal{R}_{\text{lab}} \) is terminating. The obtained annotated TRS is typically larger and may even be infinite, but the structure may get simpler. If this is indeed the case then TRSs whose termination proof is challenging can be handled with relatively simple methods.

Let \( \mathcal{A} \) be a model of the TRS \( \mathcal{R} \). A labeling \( \ell \) for \( \mathcal{A} \) consists of a set of labels \( L_f \) together with mappings \( \ell_f : A^n \rightarrow L_f \) for every \( f \in \mathcal{F} \), \( f \) \( n \)-ary, where \( A \) is the domain of the model \( \mathcal{A} \). For every assignment \( \alpha : V \rightarrow A \), let \( \text{lab}_\alpha \) denote a mapping from terms to terms defined as follows:

\[
\text{lab}_\alpha(t) := \begin{cases} 
  t & \text{if } t \in V, \\
  f(\text{lab}_\alpha(t_1), \ldots, \text{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \emptyset, \\
  f_a(\text{lab}_\alpha(t_1), \ldots, \text{lab}_\alpha(t_n)) & \text{otherwise}.
\end{cases}
\]

The label \( a \) in the last case is defined as \( \ell_f([\alpha]_A t_1, \ldots, [\alpha]_A t_n) \), where \( [\alpha]_A t \) denotes the evaluation of term \( t \) with respect to the model \( \mathcal{A} \) and the assignment \( \alpha \). The labeled TRS \( \mathcal{R}_{\text{lab}} \) is defined as

\[
\{ \text{lab}_\alpha(l) \rightarrow \text{lab}_\alpha(r) \mid l \rightarrow r \in \mathcal{R} \text{ and } \alpha \text{ an assignment} \}.
\]

Below I state the central result for one variant of semantic labeling, for further refinements see \[149, 150, 148, 96, 71\].

- Let \( \mathcal{R} \) be a TRS, \( \mathcal{A} \) a model for \( \mathcal{R} \), and \( \ell \) a labeling for \( \mathcal{A} \). Then \( \mathcal{R} \) is terminating if and only if the labelled TRS \( \mathcal{R}_{\text{lab}} \) (with respect to \( \ell \)) is terminating.

Note that the model \( \mathcal{A} \) of the TRS \( \mathcal{R} \) is used to represent the semantic information of \( \mathcal{R} \).

Semantic labeling turns out to be a promising candidate for complexity analysis. It is not difficult to see that the derivation length of each term with respect to \( \mathcal{R} \) equals its derivation length with respect to \( \mathcal{R}_{\text{lab}} \). Therefore results on the complexity of \( \mathcal{R}_{\text{lab}} \) are transferable to the original system. The latter remains true, if refinements of the semantic labeling technique are used.

Still there is plenty of room for research, as often the transformed system \( \mathcal{R}_{\text{lab}} \) is infinite. However, the mentioned results on derivational complexities in Section \( \text{2.1.2} \) do not necessarily carry over to infinite TRSs. Indeed in the case of a TRS compatible with either MPO or LPO it is easily verified that the results become false for any computable upper bound, cf. Section \( \text{2.3.1} \).
Modern Direct Termination Techniques

Let me briefly mention known results on the complexities of modern termination techniques that we have not yet treated. First, we consider the match-bound technique a new method for automatically proving termination of left-linear term rewriting systems. In linear derivational complexity for linear match-bounded TRSs is established, but exponential lower bounds exist for top-bounded TRSs. This result extends to non-linear, but non-duplicating TRSs. For non-left-linear (non-duplicating) TRSs, the notion of match-boundedness has to be replaced by match-raise-boundedness. The latter technique is introduced in . Employing it is not difficult to argue that any non-duplicating, right-linear, and match-raise bounded TRS induces at most linear derivational complexity. In the context of derivational complexity analysis the restriction to non-duplicating TRSs is harmless, as any duplicating TRS induces at least exponential derivational complexity, see Section 2.3 for further details.

Secondly, consider the matrix interpretation method . In general the existence of a matrix interpretation for a given TRS induces exponential upper bounds on the derivational complexity function . However, two specific instances of the matrix interpretation method have recently been studied in the literature: the arctic matrix method and triangular matrices.

The arctic matrix method employs matrices over the arctic semi-ring that employs as domain the set together with the operations maximisation and addition, see . This technique induces linear derivational complexity for string rewrite systems if employed as a direct termination technique, cf. . On the other hand, triangular matrix interpretations restrict the form of matrices (defined over the natural numbers to the usual operations) to upper triangular form. In we establish that the induced derivational complexity is polynomial, where the degree of the polynomial is the dimension of the employed matrices.

2.2.2 Low-Complexity Bounding Functions

The greatest hindrance in exploiting the majority of results on derivational complexities is the fact that the obtained upper bounds for general TRS are only of theoretical value. Even the smallest bound, i.e., the double-exponential bound, mentioned in Section 2.1.2, cannot be considered computationally feasible. Note that, while this upper bound is tight for the class of polynomially terminating TRSs, it is not difficult to find polynomially terminating TRSs whose derivational complexity functions grow much slower than double-exponentially. In the same spirit we easily find TRSs compatible with MPO, LPO, or KBO, respectively that do not exhibit the theoretical upper bound on the derivational complexity presented in Section 2.1.2.

With respect to polynomial interpretations, this observation led for example to the development of context-dependent interpretations, that allow a finer analysis of the derivational complexity and with respect to LPO-termination this leads to a refined analysis of the above mentioned result that LPO induces multiple recursive upper

3 String rewrite systems are a specific class of TRSs, such that all function symbols have unary arity.
bounds, cf. [6]. Therefore, one would want a more careful calibration of the results mentioned in Section 2.1.2 so that the analysis of complexities of TRSs becomes more versatile. Below I will take this further by striving for bounding functions on the complexity that are feasible, or at least belong to one of the lower classes in the polynomial hierarchy.

This is a non-trivial task: Even in the case of linear termination, where we restrict the interpretation functions to linear polynomials, the derivational complexity is optimally bounded by an exponential function, cf. [79]. Moreover, although we can characterise the class of polytime computable functions (FP for short) by carefully controlling the way the successor symbols are interpreted, feasible upper bounds on the derivation length requires new ideas, see for example [37, 26, 78] but also [8] (Chapter 4) and [11] (Chapter 6).

The quest for low-complexity bounding functions highlights a shortcoming of the traditional notion of derivational complexity that I will discuss now. While the derivational complexity function $d_{\mathcal{R}}$ is well-motivated if we are mainly concerned with the strength of (direct) termination techniques (see [79, 76, 36, 78]), its usability becomes more questionable in the wider perspective we take here.

Consider a TRS $\mathcal{R}$, encoding a functional program $P$. Then the applicability of our results in program analysis hinges on the fact that results on the complexity of $\mathcal{R}$ are meaningful measures of the (runtime-)complexity of the program $P$. For example, consider the version of insertion sort introduced in Section 2.2.1. Typically, we will not call the function $\text{sort}$ iteratively, but $\text{sort}$ will be given values as arguments. Consequently we are only interested in the runtime complexity of such a function call, but the above given definition of derivational complexity may overestimate this complexity. In particular, the following example indicates that the derivational complexity function may overestimate the “real” complexity of a computed function for purely syntactic reasons.

Consider the TRS $\mathcal{R}_2$

\begin{align*}
\text{is_empty}(\text{nil}) & \rightarrow \top \\
\text{is_empty}(x::y) & \rightarrow \bot \\
\text{hd}(x::y) & \rightarrow x \\
\text{tl}(x::y) & \rightarrow y \\
\text{append}(x,y) & \rightarrow \text{ifappend}(x,y,x) \\
\text{ifappend}(x,y,u::v) & \rightarrow u::\text{append}(v,y) .
\end{align*}

Although the functions computed by $\mathcal{R}_2$ are obviously feasible this is not reflected in the derivational complexity of $\mathcal{R}$. Consider rule (\star), which I abbreviate as $C[x] \rightarrow D[x,x]$. Since the maximal derivation length starting with $C^n[x]$ equals $2^{n-1}$ for all $n > 0$, $\mathcal{R}$ admits (at least) exponential derivational complexity. A possible solution how to overcome this obstacle, is discussed in Section 2.3 below.

### 2.2.3 Strategies and Higher-Order Rewriting

Reduction strategies in rewriting and programming have attracted increasing attention within the last years. New types of reduction strategies have been invented and investi-
2.3 Contributions

A rewrite strategy for a TRS is a mapping \( S \) that assigns to every term \( s \) not in normal form a non-empty set of finite non-empty derivations starting from \( s \). We say that \( s \) rewrites under the strategy \( S \) to the term \( t \), if \( s \rightarrow^+ R t \in S( t ) \). Typically strategies are defined by selecting the redexes which are to be contracted in each step. Examples of such strategies are the leftmost outermost rewrite strategy, where always the leftmost outermost redex is selected. Likewise, the leftmost innermost strategy contracts the leftmost of the innermost redexes. Other examples of strategies are the parallel innermost, parallel outermost, the full substitution and the call-by-need strategy.

Strategies allow us to efficiently compute normal forms of weakly normalising rewrite systems. Thus considering complexities for TRSs governed by rewrite strategies immediately broadens the applicability of complexity investigations. The more pressing reason, why we want to investigate strategies is that rewrite strategies allow far more efficient computations of normal forms. Through strategies the best-case behaviour of a termination method, described as the shortest derivation length, becomes accessible. Hence, considering strategies appears to be one step forward to obtain feasible upper bounds on the complexities of rewrite systems.

Reduction strategies in rewriting are one way to broaden the applicability of complexity results. Another extension stems more directly from programming. Consider the following OCaml program \( P \) encoding the definition of the higher-order function \( \text{map} \):

\[
\begin{align*}
\text{let rec map f l} & = \text{function} \\
& | \[] \rightarrow \[] \\
& | \text{hd :: tl} \rightarrow f \text{ hd :: map f tl};;
\end{align*}
\]

Higher-order programs like \( P \) can either be represented as \( S \)-expression rewrite systems \([141, 142]\) or as applicative systems, employing a binary applicative symbol \( \circ \), cf. \([91, 114, 59]\). While \( S \)-expression rewrite systems and in particular applicative systems have been studied extensively, relative little effort has been spent to prove termination of higher-order rewrite systems directly, see \([88, 89, 67, 25, 90, 87]\).

Currently the complexity analysis of higher-order systems via rewriting has not yet attracted much attention. Although there is long established interest in the functional programming community to automatically verify complexity properties of programs, see for example \([4, 22, 65, 129]\), no results along the lines presented here can be found in the literature. Future research will overcome this restriction as the applicability of complexity analysis to rewrite systems in the context of (functional) programs is of utmost importance to the sustainability of this research, see Section 2.5.

2.3 Contributions

Above I define the derivation length function \( \text{dl}(t, \rightarrow) \) of term \( t \) with respect to a rewrite relation \( \rightarrow \) as the longest possible derivation (with respect to \( \rightarrow \)) starting with \( t \). Based on the derivation length, Hofbauer and Lautemann defined the derivational complexity function \( \text{dc}_R \) with respect to the (full) rewrite relation \( \rightarrow_R \). Instead I propose the
following generalisation of this concept to arbitrary relations. Additionally this concept allows for better control on the set of admitted start terms of a given computation. Let $\mathcal{R}$ be a TRS and $T \subseteq T(\mathcal{F}, \mathcal{V})$ be a set of terms. The \textit{runtime complexity function with respect to a relation} $\rightarrow$ on $T$ is defined as follows:

$$rc(n, T, \rightarrow) = \max\{dl(t, \rightarrow) \mid t \in T \text{ and } |t| \leq n\}. \quad (2.1)$$

Based on this notion the derivational complexity function becomes definable as follows: $dc_{\mathcal{R}}(n) = rc(n, T(\mathcal{F}, \mathcal{V}), \rightarrow \mathcal{R})$. Currently four instances of $(2.1)$ are most prominent in research:

- the \textit{derivational complexity function} $dc_{\mathcal{R}}$, as defined above.
- the \textit{innermost derivational complexity function} $dc_{\mathcal{R}}^{i} = rc(n, T(\mathcal{F}, \mathcal{V}), \mathcal{I} \rightarrow \mathcal{R})$.
- the \textit{runtime complexity function} $rc_{\mathcal{R}} = rc(n, T_{b}, \rightarrow)$.
- the \textit{innermost runtime complexity function} $rc_{\mathcal{R}}^{i} = rc(n, T_{b}, \mathcal{I} \rightarrow)$.

Here $\mathcal{I} \rightarrow \mathcal{R}$ denotes the innermost rewrite relation with respect to $\mathcal{R}$ and $T_{b} \subseteq T(\mathcal{F}, \mathcal{V})$ denotes the set of constructor-based terms, cf. [15]. A constructor based term directly represents a function call with values as argument. Hence this notion corresponds nicely to the typical use of \textit{runtime complexity} in the literature on functional programming, cf. [24]. Note that the runtime complexity of a TRS extends the notion of the \textit{cost} of (constructor based) term as introduced in [35]. Technically the broader definition has the advantage that the syntactic restriction to non-duplicating TRSs mentioned in Section 2.2.2 can be overcome. Indeed, as expected, the (innermost) runtime complexity function $rc_{\mathcal{R}}^{i}$ of the TRS $\mathcal{R}_{2}$ given on page 14 is linear, and this can be verified automatically, see Chapter 8.

2.3.1 Modern Termination Techniques

The starting point of my research into the complexity of rewrite systems was an investigation of Hofbauer’s and Weiermann’s results on the derivational complexity induced by MPO and LPO, see Section 2.1.2. More precisely, in Chapter 3 a \textit{generalised system of fundamental sequences} is introduced and its associated slow-growing hierarchy is defined. These notions provide the tools to establish a modernised (and correct) treatment of Cichon’s principle for the simplification orders MPO and LPO.

In order to state the central results precisely, I introduce some further definitions (see [32], Chapter 4 for additional background information). Let $\Lambda$ denote the small Veblen ordinal [133] and let $otype(\succ)$ denote the order type of a well-founded relation $\succ$. It is well-known that $\Lambda = \sup\{otype(\succ) \mid \succ \text{ is a lexicographic path order}\}$, cf. [130]. Let $\succ$ denote a precedence on the signature $\mathcal{F}$, let $\succ_{lpo}$ denote the induced LPO and let $\pi: T(\mathcal{F}) \rightarrow \Lambda$ denote an interpretation from the set of ground terms into the ordinals less than $\Lambda$. The central results of [119] (see Chapter 3) can be paraphrased as follows:
• There exists a generalised system of fundamental sequences for ordinals below $\Lambda$ that allows the definition of a point-wise relation $>(x)$. Roughly speaking $>(x)$ denotes the descent along the $x^{th}$ branch of these fundamental sequences.

• If $\mathcal{R}$ denotes a finite TRS compatible with $>_{lpo}$, then there exists a number $k$, such that for any rule $l \rightarrow r \in \mathcal{R}$ and any ground substitution $\rho$, we have $\pi(l\rho) > (k) \pi(r\rho)$.

• There exists a slow-growing hierarchy of sub-recursive function $\bigcup_{\alpha<\Lambda} \tilde{G}_\alpha$ such that if $\alpha > (x) \beta$, then $\tilde{G}_\alpha(x) > \tilde{G}_\beta(x)$.

As the hierarchy $\bigcup_{\alpha<\Lambda} E(\tilde{G}_\alpha)$ characterises exactly the multiple-recursive functions, we re-obtain the above mentioned result that LPO induces multiple-recursive derivational complexity. (Here $E(f)$ denotes the elementary closure of function $f$.)

In subsequent research I generalised the introduced concepts suitably to analyse the derivational complexity induced by the Knuth-Bendix order (see [115]). This substantiated and clarified claims made in [119] that the provided concepts are genuinely related to the classification of the complexity of rewrite systems for which termination is provable by a simplification order. In Chapter 5 the derivational complexity of TRSs $\mathcal{R}$ compatible with KBO is studied, where the signature of $\mathcal{R}$ may be infinite. It is shown that Lepper’s result on the derivational complexity with respect to finite TRS is essentially preserved, see [115] (cf. Chapter 5) for further details.

• Let $\mathcal{R}$ be a TRS based on a signature $\mathcal{F}$ with bounded arities that is compatible with a KBO $>_{\text{kbo}}$ and let some weak assumption on $\mathcal{R}$ be fulfilled. Then for any term $t$: $\text{dl}(t, \rightarrow_{\mathcal{R}}) \leq \text{Ack}(2^{O(n)}, 0)$, where the constant hidden in the big-Oh notation, depends only on syntactic properties of the function symbols in $t$, the TRS $\mathcal{R}$ and the instance $>_{\text{kbo}}$ used. Note that $\mathcal{F}$ need not be finite. As a corollary to this result I re-obtain the 2-recursive upper-bound on the derivational complexity of finite rewrite systems $\mathcal{R}$ compatible with KBO.

It seems worthy of note that the material presented in Chapter 5 provides the first in-depth derivational complexity analysis of semantic labeling. Recall from Section 2.2.1 that the central idea of semantic labeling is to transform the given TRS $\mathcal{R}$ into a system $\mathcal{R}_{\text{lab}}$ such that $\mathcal{R}$ is terminating if and only if $\mathcal{R}_{\text{lab}}$ is terminating. Furthermore showing termination of $\mathcal{R}_{\text{lab}}$ should be easier than showing termination of $\mathcal{R}$.

As indicated, semantic information (i.e., a model of $\mathcal{R}$) is used to define the new system $\mathcal{R}_{\text{lab}}$. If this model is finite, then the complexity certificates for $\mathcal{R}_{\text{lab}}$ are trivially transferable into complexity certificates for $\mathcal{R}$. However, often infinite models would be more suitable, which changes the picture completely. See [96, 71, 94] for further reading on semantic labeling with infinite models.

The main problem is that classic results on complexities of simplification orders (see Section 2.1.2) not necessarily extend to infinite signatures. It is not difficult to see that the complexity results on MPO and LPO mentioned in Section 2.1.2 cannot be extended to infinite signatures, cf. [114]. On the other hand the above result shows that for KBO,
complexity results are transferable, even if the underlying model is infinite. Observe that the weak restrictions mentioned, typically hold for systems obtained via the semantic labeling transformation, see [115] or Chapter 5.

In Section 2.2.1 I indicated the challenges posed, if we aim for a classification of the complexities of TRSs, whose termination is shown by the dependency pair method. In order to tackle these difficulties recent efforts in this direction (see [74, 75]) concentrate on estimates for (innermost) runtime complexities. In this context we are most interested in techniques that induce polynomial (innermost) runtime complexities.

In [74, 75] a variant of the dependency pair method for analysing runtime complexities has been introduced (cf. Chapter 8 and 9). We show how natural improvements of the dependency pair method, like usable rules, reduction pairs, argument filterings, and dependency graphs become applicable in this context. More precisely, we have established a notion of dependency pairs, called weak dependency pairs that are applicable in the context of complexity analysis. This notion provides us with the following method to analyse runtime complexity:

- Let $R$ be a TRS, let $A$ be a restricted polynomial interpretation, essentially expressing a weight function, let $(\succeq, >)$ denote a reduction pair (not necessarily based on $A$) that fulfills some additional conditions and let $P$ denote the set of weak dependency pairs of $R$ such that $P$ is non-duplicating. Suppose the usable rules $U(P)$ of $P$ are contained in $\succeq$ and $P \subseteq >$. Moreover, suppose $U(P) \subseteq >A$.

Then the runtime complexity function $rc_R$ with respect to $R$ depends linearly on the rank of the order $>$. Here the rank of a well-founded order is defined as usual. Observe that it is very easy to verify the mentioned additional restriction on the reduction pair $(\succeq, >)$, if $(\succeq, >)$ is based on a polynomial interpretation $B$, cf. Chapter 8. These results can be adapted for the special case of innermost rewriting. Here we replace the full rewrite relation $\rightarrow_R$ in the definition of runtime complexity by the innermost rewriting relation $i\rightarrow_R$. The established techniques are fully automatable and easy to implement.

Let me reformulate this important result in a slightly more concrete setting. Suppose $A$ is defined as above and assume $B$ denotes a polynomial interpretation, fulfilling the restriction that constructors are interpreted as weights. Then it is easy to see that if a TRS $R$ is compatible with such an interpretation $B$ the runtime complexity of $R$ is polynomial (see Chapter 8 but also [26]). As a corollary to the above result we obtain:

- Let $R$ be a TRS, let $P$ be the set of weak dependency pairs of $R$, and let $A$ and $B$ be defined as above. Suppose $(\succeq_B, >_B)$ forms a reduction pair and in addition: $U(P) \subseteq \succeq_B$ and $P \subseteq >_B$, where $P$ is supposed to be non-duplicating. If $U(P) \subseteq >_A$ then the runtime complexity function $rc_R$ with respect to $R$ is polynomial.

This result significantly extends the analytic power of existing direct methods. Moreover this entails the first method to analyse the derivation length induced by the (standard) dependency pair method for innermost rewriting, cf. Chapter 8.
2.3 Contributions

2.3.2 Low-Complexity Bounding Functions

As already observed in Section 2.2.2 it is not difficult to find polynomially terminating TRSs, whose derivational complexity functions grow significantly slower than double-exponentially. I.e., polynomial interpretations typically overestimate the induced derivational complexity. In [78] Hofbauer introduced context-dependent interpretations as a remedy. Consequently these interpretation provided a starting point in the analysis of termination methods that induce polynomial derivational complexity. Indeed in [117] (see Chapter 7) such an analysis is conducted and a new method to automatically conclude polynomial (even quadratic) derivational complexity is given such that we obtain the following result:

- Let $\mathcal{R}$ be compatible with a specific restriction of a context-dependent interpretation, called $\Delta$-restricted interpretation. Then $d_{\mathcal{R}}(n) = O(n^2)$. Moreover there exists a TRS $\mathcal{R}$ such that $d_{\mathcal{R}}(n) = \Omega(n^2)$.

Moreover, subsequent research revealed the existence of a tight correspondence between a subclass of context-dependent interpretations and restricted triangular matrix interpretations, cf. [120]. On the one hand this correspondence allows for a much simpler and more powerful method to automatically deduce polynomial derivational complexity. On the other hand this result reveals a connection between seemingly very different termination techniques: matrix interpretations and context-dependent interpretations. Moreover this result would not have been observed if we had investigated these techniques directly and not the induced complexity. (Observe that no indication of this correspondence result could be found in the literature.)

Buchholz’s result (described in Section 2.1.3) suggests another approach. Conceptually [30] provides a new well-foundedness proof of MPO and LPO (by induction on the accessible parts of these orders) and miniaturises this proof in the context of termination analysis. This entails the idea to directly study miniaturisations of well-known reduction orders in such a way that infeasible growth rates are prohibited. Of course these miniaturisations have to be done carefully to prevent us from robbing the order from any real termination power.

To this avail we introduce in [8] (see Chapter 4) the path order for FP (POP for short). We could show that POP characterises the functions computable in polytime, i.e., the complexity class FP. In particular any function in FP is representable as a TRS compatible with POP. Moreover, we established the following result:

- A termination proof for a TRS $\mathcal{R}$ via POP implies that for any $f \in \mathcal{F}$ of arity $m$
  $dl(f(S^{n_1}(0), \ldots, S^{n_m}(0)), \rightarrow_{\mathcal{R}})$ is polynomially bounded in the sum of the (binary) length of the input $S^{n_1}(0), \ldots, S^{n_m}(0)$.

Still, in practice, the applicability of POP is limited. Many natural term-rewriting representations of polytime computable functions cannot be handled by POP as the imposed restrictions are sometimes not general enough. To remedy this situation I studied generalisations of POP that are more broadly applicable. These investigations resulted in the definition of a syntactic restriction of MPO, called POP*, and the following result, cf. [11] (see Chapter 6).
• A termination proof for a TRS $\mathcal{R}$ via POP* implies that the innermost runtime complexity function $rc^i_\mathcal{R}$ is polynomially bounded.

Moreover POP* is complete for FP. It should be stressed that as characterisations of complexity classes the orders POP and POP* are closely related. However, with respect to direct applicability and in particular automatisation the latter result is a lot stronger.

It is worth noting that our result in [11] depends on the careful combination of the miniaturisation of the multiset path order together with a specific strategy. Hence [11] provides an important indication of the need to consider rewrite strategies in complexity analysis, see Section 2.2.3.

2.4 Related Work

I mention here only work that is not already cited in Sections 2.2.1–2.2.3. Concerning low-complexity bounding functions, I want to mention the connection between the complexity analysis of a TRS $\mathcal{R}$ and the computability of $\mathcal{R}$. Roughly speaking a function $f$ is computable by a terminating TRS $\mathcal{R}$ if there are function symbols $F, S, P, 0, 0'$ such that

$$F(S^{n_1}(0), \ldots, S^{n_m}(0)) \rightarrow^*_{\mathcal{R}} P^{f(n_1, \ldots, n_m)}(0'),$$

holds for all $n_1, \ldots, n_m$, cf. [84]. The distinction between the input successor $S$ and the output successor $P$, as well as between $0$ and $0'$ is sometimes necessary to allow finer distinctions.

We say a function $f$ is computable with respect to a termination method $M$, if $f$ is computable by a TRS that is $M$-terminating. For large complexity classes, as for example the primitive recursive functions, the derivational complexity induced by a termination method implies its computability, cf. [40]. For example the class of functions computable with respect to MPO equals the primitive recursive functions, cf. [76]. For small complexity classes this equivalence is lost. Consider the class of polytime computable function FP. The class FP is representable as the set of functions computable by TRSs $\mathcal{R}$ that are compatible with restricted polynomial interpretations $\mathcal{A}$, cf. [26]. On the other hand, the derivational complexity induced by $\mathcal{A}$ is double-exponentially, cf. Section 2.1.2.

This seems to strengthen the argument made above that the derivational complexity function $dc_\mathcal{R}$ is not always a suitable measure of the complexity of a TRS. Kindly observe that the runtime complexity $rc_\mathcal{R}$ with respect to $\mathcal{R}$ induced by the interpretations $\mathcal{A}$ is polynomial. Still, we cannot equate (runtime) complexity and computability in general. The fact that a given polytime computable function $f$ is computable by a TRS $\mathcal{R}$ need not imply that $rc_\mathcal{R}$ is indeed polynomial (see [8] but also [19]).

The study of the computability of a given function $f$ with respect to a termination method as outlined above is clearly connected to the investigations in implicit computational complexity theory. In the analysis of the implicit computational complexity of programs, one is interested in the analysis of the complexity of a given program rather than the study of the complexity of the function computed, or of the problem solved.
Much attention is directed towards the characterisation of “nice” classes of programs that define complexity classes in the polynomial hierarchy, most prominently the class of polytime computable functions $\text{FP}$. In particular, I want to mention related work employing term rewriting as an abstract model of computation and consequently using existing techniques from rewriting to characterise several computational complexity classes. Interesting techniques in this context comprise the miniaturisation of simplification orders like MPO and LPO, by Cichon and Marion, cf. [38, 109], as well as the use of quasi-interpretations or sup-interpretations to characterise complexity classes by Bonfante, Marion, Moyen, Péchoux and others, cf. [110, 28, 29].

On a more general level I want to mention additional work on tiering or ramification concepts by Leivant, Marion, and Pfennig, cf. [101, 100, 102, 126]. Moreover, I cite Hofmann’s seminal work [82, 33] as well as related results by Aehlig, Schwichtenberg, and others, cf. [1, 2, 134, 18]. In addition, there is highly interesting work on resource bounds of imperative programs by Niggl, Jones, Kristansen, and others, see [121, 122, 23, 86].

2.5 Future Research

In Section 2.2.3 I discussed the general aim to extend existing work on complexity analysis for first-order rewriting to the higher-order case. As already mentioned, one way to represent higher-order programs like the $\text{map}$ function defined in Section 2.2.3 are $S$-expression rewrite systems. For clarity, we recall the definition from [141, 142]. Let $C$ be a set of constants, $V$ be a set of variables such that $V \cap C = \emptyset$, and $\circ \notin C \cup V$ a variadic function symbol. We define the set $S(C, V)$ of $S$-expressions built from $C$ and $V$ as $T(C \cup \{\circ\}, V)$. We write $(s_1 \cdots s_n)$ instead of $\circ(s_1, \ldots, s_n)$. An $S$-expression rewrite system (SRS for short) is a TRS with the property that the left- and right-hand sides of all rewrite rules are $S$-expressions.

Applying transformation steps, like case analysis and rewriting of right-hand sides, the function $\text{map}$, as defined in Section 2.2.3, becomes representable as the following SRS:

\[(\text{map } f \text{ nil}) \rightarrow \text{nil} \]
\[(\text{map } f (\text{cons } x \text{ xs})) \rightarrow (\text{cons } (f \ x) \ (\text{map } f \text{ xs}))\]

In recent work together with Avanzini, Hirokawa and Middeldorp (see [13]) we study the runtime complexity of (a subset of) Scheme programs by a translation into SRSs. Scheme is a statically scoped and properly tail-recursive dialect of the Lisp programming language invented by Guy Lewis Steele Jr. and Gerald Jay Sussman, cf. [135]. Due to its clear and simple semantics, Scheme appears as an ideal candidate to apply our results on the complexity analysis of TRSs in the context of functional programming.

By designing the translation to be complexity preserving (or at least closed under polynomial functions) the complexity of the initial Scheme program can be estimated by analysing the complexity of the resulting SRS. Here we indicate how the above result on $\text{POP}^*$ is applicable to (a subset of) $S$-expression rewrite systems.
2 Summary

Let \( S \) be an SRS over \( S(C, V) \) and let \( C = D \cup K \) such that \( D \cap K = \emptyset \). We call the elements of \( K \) constructor constants and the elements of \( D \) defined constants. We define the notion of value in the context of SRSs. The set of values \( \text{Val}(S) \) of \( S \) with respect to \( K \) is inductively defined as follows: (i) if \( v \in C \) then \( v \in \text{Val}(S) \), (ii) if \( v_1, \ldots, v_n \in \text{Val}(S) \) and \( c \in K \) then \( (c, v_1, \ldots, v_n) \in \text{Val}(S) \).

Observe that (defined) constants are values, this reflects that in Scheme procedures are values, cf. \([13] \) and allows for a representation of higher-order programs. Scheme programs are conceivable as SRSs, allowing conditional if expressions in conjunction with an eager, i.e., innermost rewrite strategy. Thus we can delineate a class of SRSs that easily accommodates a relative large subset of Scheme programs, called constructor SRSs in \([13] \). Based on Toyama’s observation that recursive path orders can be successfully employed to prove termination of SRSs, we invented an automatic complexity analyser for Scheme programs, cf. \([13] \). The main theoretical contribution of this work can be paraphrased as follows:

- Let \( S \) be a constructor SRS compatible with POP*. Then the innermost runtime complexity function \( \text{rc}_R^d \) (suitably adapted to constructor SRSs) is polynomially bounded.

In conjunction with the fact that the transformation of Scheme programs into SRS is complexity preserving this result provides us with a complexity analysis of Scheme programs that is fully automatable. Still, this is only a partial result as the considered subset of Scheme programs is only of limited practical interest. In particular we cannot yet handle integer values. This will be subject to future research.

2.6 Conclusion

In order to assess the complexity of a TRS it is natural to look at the maximal length of derivation sequences, a program that has been suggested by Hofbauer and Lautemann in \([79] \). This concept has given rise to the area of derivational complexity analysis that produced a number of deep insights into the strength of direct termination methods, described in Section 2.1.2 and 2.1.3.

The goal of my subsequent research was and still is to make the (derivational) complexity analysis of rewrite systems modern, useful, and broad. For that purpose I have analysed the established results in order to assess their applicability in the context of modern termination provers. These investigations (notably in \([119, 115] \)) resulted in an improved understanding and clarification of the used concepts that often allowed the deduction of more general results.

During this research it became apparent that the “standard” notion of derivational complexity with respect to a given TRS was not the right tool to modernise complexity analysis. Instead its generalisation to the above introduced runtime complexity function with respect to a TRS and a given rewrite strategy proved (up-to now) as the most useful.
2.6 Conclusion

Based on this conceptional advance I was able (together with various co-authors) to modernise (derivational) complexity analysis to accommodate modern termination techniques like context-dependent interpretations, match-bounds, matrix interpretations, semantic labeling and dependency pairs, as documented in \([117, 74, 75]\).

Moreover, through the research published in \([8, 11]\) the viewpoint of (derivational) complexity analysis, is today much more focused on feasible bounding functions than in earlier research. This has important consequences for the applicability of this research. Earlier investigations were mainly conducted to reveal the strength of termination methods, while my research pushed the interest towards the strength or complexity of rewrite systems, proper. This opens the door to exciting applications in (automated) program analysis.

Lastly my research in this direction aims at the automation of the introduced techniques. To this avail I am building (together with Avanzini and Schnabl) the software tool \(TCT\) to analyse the complexity of rewrite systems automatically. In this context a specialised category for complexity analysers has been integrated into the termination competition, see [http://termcomp.uibk.ac.at](http://termcomp.uibk.ac.at).

The goal of this competition is twofold. On one hand the most advanced techniques become comparable in a direct contest. Hence different tools compete to provide for each system the best possible complexity certificate. For example, if we consider estimation of upper bounds, then the tool that provides the tightest bound, gets the highest score. On the other hand this competition provides a forum that allows to publicise the gained results and insights. A necessity if we want to apply these results outside rewriting.

I anticipate that the research described here will considerably advance the field of term rewriting. Moreover, I anticipate impact on the fields of implicit computational complexity theory and proof theory.

In the context of implicit computational complexity theory (see Section 2.4) my main interest lies in studies that employ term rewriting as abstract model of computation and consequently use existing techniques from rewriting to characterise several computational complexity classes, as described in Section 2.4. Here I highlight the latter approach to implicit computational complexity. In Section 2.2.1 we considered a functional program \(P\) that implements insertion sort. Interestingly \(P\) is a challenge for implicit computational complexity as its obvious polynomial runtime complexity cannot be easily verified. This was first observed by Caseiro \([33]\), see also \([82, 1]\). Observe that program \(P\) can be easily transformed into the following TRS \(R_S\):

\[
\begin{align*}
\text{if}(\top, x, y) & \rightarrow x & \text{insert}(x, \text{nil}) & \rightarrow x::\text{nil} \\
\text{if}(\bot, x, y) & \rightarrow y & \text{insert}(x, y::z) & \rightarrow \text{if}(x \leq y, x::y::z, y::\text{insert}(x, z)) \\
0 & \leq s(y) & \rightarrow \top & \text{sort}(\text{nil}) & \rightarrow \text{nil} \\
x & \leq 0 & \rightarrow \bot & \text{sort}(x::z) & \rightarrow \text{insert}(x, \text{sort}(z)) \\
s(x) & \leq s(y) & \rightarrow x \leq y
\end{align*}
\]

23
It is easy to see that $\mathcal{R}_3$ is MPO-terminating. Moreover, there exists a weakly monotone max-polynomial interpretation $\mathcal{A}$ such that the interpretation of constructor symbols is restricted to weight functions. The induced order $\succeq_\mathcal{A}$ weakly orients all rules, cf. Bonfante et al. [29]. Hence $\mathcal{R}_3$ belongs to a specific subclass of rewrite systems studied in [29] such that each function computed by such a TRS is polytime computable.\footnote{Note that this does not imply that the runtime complexity function $rc_{\mathcal{R}_3}$ is polynomial, but that the function computed is polytime computable (in the usual sense).}

In my research I am genuinely interested in “applicable” upper bounds on the complexities of rewrite systems and therefore I am less concerned with the classification of computational complexity classes. Moreover, it seems a not too important statement that insertion sort is a polytime computable function. Instead the exciting question is whether a given implementation $P$ of insertion sort admits (at most) polynomial runtime complexity. We thus have to clarify what exactly we accept as an implementation or program. I would argue that in this context term rewriting systems would be a good choice and the complexity of $P$ ought to be measured in the natural way for computation model. Unfortunately, we cannot conclude polynomial runtime complexity of $\mathcal{R}_3$ from the results by Bonfante et al. (see [29] but also Chapter 4).

Still, there are many connections between complexity analysis of term rewrite systems as discussed here and implicit computational complexity theory. For example the use of rewriting techniques opens the way for automatisation. Recently, Avanzini, Schnabl and myself implemented a fully automated system that incorporates the majority of these techniques. See [14] for the findings of this experimental comparison.

Furthermore derivational complexity studies have stirred some attention in proof theory, cf. [144, 30, 6, 50]. Clearly my research has implications for proof theory, see Section 2.1.3. Here I want to emphasise that we are implicitly dealing with the connection of partial orders and the growth-rate of functions defined by induction on these orders: We say that a TRS $\mathcal{R}$ is $\alpha$-terminating if $\mathcal{R}$ is compatible with an $\mathcal{F}$-algebra $(\alpha, >)$, where $>$ denotes ordinal comparison. Any function computable by an $\alpha$-terminating $\mathcal{R}$ gives rise to a function defined by transfinite induction up-to $\alpha$.

A related connection was first observed by Cichon, who conjectured that the slow-growing hierarchies would connect the order type of a termination order compatible with $\mathcal{R}$ with the derivational complexity of $\mathcal{R}$, cf. [36]. Unfortunately, this claim is incorrect, as shown by Touzet [139]. On the other hand, the principal connection refers to deep proof theoretic questions as for example the “naturalness” of a given ordinal notation system, cf. [19, 20], see also Section 2.1.3.
3

Relating Derivation Lengths with the Slow-Growing Hierarchy Directly

Publication Details

Ranking
The International Conference on Rewriting Techniques and Applications has been ranked A by the Computing Research and Education Association of Australasia (CORE for short) in 2007.

Abstract
In this article we introduce the notion of a generalized system of fundamental sequences and we define its associated slow-growing hierarchy. We claim that these concepts are genuinely related to the classification of the complexity—the derivation length—of rewrite systems for which termination is provable by a standard termination ordering.

To substantiate this claim, we re-obtain multiple recursive bounds on the derivation length for rewrite systems terminating under lexicographic path ordering, originally established by the second author.

3.1 Introduction

To show termination of a rewrite system $R$ one usually shows that the induced reduction relation $\rightarrow_R$ is contained in some abstract ordering known to be well-founded. One way to assess the strength of such a termination ordering is to calculate its order type, cf. [47].

\footnote{http://www.core.edu.au/}
There appears to be a subtle relationship between these order types and the complexity of the rewrite system $R$ considered. Cichon [36] discussed (and investigated) whether the complexity of a rewrite system for which termination is provable using a termination ordering of order type $\alpha$ is eventually dominated by a function from the slow-growing hierarchy along $\alpha$. It turned out that this principle—henceforth referred to as (CP)—is valid for the (i) multiset path ordering ($\succ_{\text{mpo}}$) and the (ii) lexicographic path ordering ($\succ_{\text{lpo}}$).

More precisely, Hofbauer [77] proved that $\succ_{\text{mpo}}$ as termination ordering implies primitive recursive derivation length, while the second author showed that $\succ_{\text{lpo}}$ as termination ordering implies multiply-recursive derivation length [144]. If one regards the order types of $\succ_{\text{mpo}}$ and $\succ_{\text{lpo}}$, respectively, then these results imply the correctness of (CP) for (i) and (ii). Buchholz [30] has given an alternative proof of (CP) for (i) and (ii). His proof avoids the (sometimes lengthy) calculations with functions from subrecursive hierarchies in [77, 144]. Instead a clever application of proof-theoretic results is used. Although this proof is of striking beauty, one might miss the link to term rewriting theory that is provided in [77, 144].

The mentioned proofs [77, 144, 30] of (CP)—with respect to (i) and (ii)—are indirect. I.e. without direct reference to the slow-growing hierarchy. By now, we know from the work of Touzet [139] and Lepper [103, 105] that (CP) fails to hold in general. However, our interest in (CP) is motivated by our strong belief that there exist reliable ties between proof theory and term rewriting theory. Ties which become particularly apparent if one studies those termination orderings for which (CP) holds.

To articulate this belief we give yet another direct proof of (CP) (with respect to (i) and (ii)). To this avail we introduce the notion of a generalized system of fundamental sequences and we define its associated slow-growing hierarchy. These concepts are genuinely related to classifying derivation lengths for rewrite systems for which termination is proved by a standard termination ordering. To emphasize this let us present the general outline of the proof method.

Let terms $s = t_0, t_1, \ldots, t_n$ be given, such that $s \rightarrow_R t_1 \rightarrow_R \cdots \rightarrow_R t_n$ holds, where $t_n$ is in normal form and term-depth of $s$ ($\tau(s)$) is $\leq n$. Assume $\rightarrow_R$ is contained in a termination ordering $\succ$. Hence $s \succ t_1 \succ \cdots \succ t_n$ holds. Assume further the sequence $(s, t_1, \ldots, t_n)$ is chosen so that $n$ is maximal. Then in the realm of classifications of derivation lengths one usually defines an interpretation $I: T(\Sigma, V) \rightarrow \mathbb{N}$ such that $I(s) > I(t_1) > \cdots > I(t_n)$ holds. ($T(\Sigma, V)$ denotes the term algebra over the signature $\Sigma$ and the set of variables $V$.) The existence of such an interpretation then directly yields a bound on the derivation length.

The problem with this approach is to guess the right interpretation from the beginning. More often than not this is not at all obvious. Therefore we want to generate the interpretation function directly from the termination ordering in an intrinsic way. To this avail we proceed as follows. We separate $I$ into an ordinal interpretation $\pi: T(\Sigma) \rightarrow T$ and an ordinal theoretic function $g: T \rightarrow \mathbb{N}$. ($T$ denotes a suitable chosen set of terms representing an initial segment of the ordinals, cf. Definition 3.3.1). This works smoothly. Firstly, we can employ the connection between the termination ordering $\succ$ and the ordering on the notation system $T$. This connection was already observed by Dershowitz...
and Okada, cf. [47]. Secondly, it turns out that \( g \) can be defined in terms of the slow-growing function \( G_x: T \to \mathbb{N}; x \in \mathbb{N} \). (Note that we have swapped the usual denotation of arguments, see Definition 3.4.2 and Definition 3.6.1.)

To simplify the presentation we restrict our attention to a rewrite system \( R \) whose termination can be shown by a lexicographic path ordering \( \succ \text{lpo} \). It will become apparent later that the proof presented below is (relative) easily adaptable to the case where the rewrite relation \( \rightarrow R \) is contained in a multiset path ordering \( \succ \text{mpo} \). We assume the signature \( \Sigma \) contains at least one constant \( c \).

Let \( R \) be a rewrite system over \( T(\Sigma, V) \) such that \( \rightarrow R \) is contained in a lexicographic path ordering. Let terms \( s = t_0, t_1, \ldots, t_n \) be given, such that \( s \rightarrow R t_1 \rightarrow R \cdots \rightarrow R t_n \) holds, where \( t_n \) is in normal form and \( \tau(s) \leq m \). By our choice of \( R \) this implies

\[
 s \succ \text{lpo} t_1 \succ \text{lpo} \cdots \succ \text{lpo} t_n. \tag{3.1}
\]

We define a ground substitution \( \rho: \rho(x) = c \), for all \( x \in V \). Let \( > \) denote a suitable defined (well-founded) ordering relation on the ordinal notation system \( T \). Let \( l, r \in T(\Sigma, V) \). Depending on \( m \) and properties of \( R \), we show the existence of a natural number \( h \) such that \( l \succ \text{lpo} r \) implies \( \pi(l\rho) > \pi(r\rho) \) and \( G_h(\pi(l\rho)) > G_h(\pi(r\rho)) \), respectively. Employing this form of an Interpretation Theorem we conclude from (3.1) for some \( \alpha \in T \)

\[
 \alpha > \pi(s\rho) > \pi(t_1\rho) > \cdots > \pi(t_n\rho) \ .
\]

and consequently

\[
 G_h(\alpha) > G_h(\pi(s\rho)) > G_h(\pi(t_1\rho)) > \cdots > G_h(\pi(t_n\rho)) \ .
\]

Thus \( G_h(\alpha) \) calculates an upper bound for \( n \). Therefore the complexity of \( R \) can be measured in terms of the slow-growing hierarchy along the order type of \( T \).

To see that this method calculates an optimal bound, it remains to relate the function \( G_x: T \to \mathbb{N} \) to the multiply-recursive functions. We employ Girard’s Hierarchy Comparison Theorem [64]. Due to (a variant) of this theorem any multiple-recursive function can be majorized by functions from the slow-growing hierarchy and vice versa [2]. (For further details see Section 3.4.)

Contrary to the original proof in [144], we can thus circumvent technical calculations with the \( F \)-hierarchy (the fast-growing hierarchy) and can shed light on the way the slow-growing hierarchy relates the order type of the termination ordering \( \succ \) to the bound on the length of reduction sequences along \( \rightarrow R \).

### 3.2 The Lexicographic Path Ordering

We assume familiarity with the basic concepts of term rewriting. However, we fix some notations. Let \( \Sigma = \{f_1, \ldots, f_K\} \) denote a finite signature such that any function symbol \( f \in \Sigma \) has a unique arity, denoted as \( \text{ar}(f) \). The cardinality \( K \) is assumed to be fixed in

---

2 A \( k \)-ary function \( g \) is said to be majorized by a unary function \( f \) if there exists a number \( n < \omega \) such that \( g(x_1, \ldots, x_k) < f(\max\{x_1, \ldots, x_k\}) \), whenever \( \max\{x_1, \ldots, x_k\} \geq n \).
3 Relating Derivation Lengths with the Slow-Growing Hierarchy Directly

the sequel. To avoid trivialities we demand that $\Sigma$ is non-empty and contains at least one constant, i.e. a function symbol of arity 0. We set $N := \max\{\ar(f) : f \in \Sigma\}$.

The set of terms over $\Sigma$ and the countably infinite set of variables $\mathcal{V}$ is denoted as $\mathcal{T}(\Sigma, \mathcal{V})$. We will use the meta-symbols $l, r, s, t, u, \ldots$ to denote terms. The set of variables occurring in a term $t$ is denoted as $\var(t)$. A term $t$ is called ground or closed if $\var(t) = \emptyset$. The set of ground terms over $\Sigma$ is denoted as $\mathcal{T}(\Sigma)$. If no confusion can arise, the reference to the signature $\Sigma$ and the set of variables $\mathcal{V}$ is dropped. With $\tau(s)$ we denote the term depth of $s$, defined as $\tau(s) := 0$, if $s \in \mathcal{V}$ or $s \in \Sigma$ and otherwise $\tau(f(s_1, \ldots, s_m)) := \max\{\tau(s_i) : 1 \leq i \leq m\} + 1$. A substitution $\sigma : \mathcal{V} \rightarrow \mathcal{T}$ is a mapping from the set of variables to the set of terms. The application of a substitution $\sigma$ to a term $t$ is (usually) written as $t\sigma$ instead of $\sigma(t)$.

A term rewriting system (or rewrite system) $R$ over $\mathcal{T}$ is a finite set of rewrite rules $(l, r)$. The rewrite relation $\rightarrow_R$ on $\mathcal{T}$ is the least binary relation on $\mathcal{T}$ containing $R$ such that (i) if $s \rightarrow_R t$ and $\sigma$ a substitution, then $s\sigma \rightarrow_R t\sigma$ holds, and (ii) if $s \rightarrow_R t$, then $f(\ldots, s, \ldots) \rightarrow_R f(\ldots, t, \ldots)$. A rewrite system $R$ is terminating if there is no infinite sequence $\langle t_i : i \in \mathbb{N} \rangle$ of terms such that $t_1 \rightarrow_R t_2 \rightarrow_R \cdots \rightarrow_R t_m \rightarrow_R \cdots$. Let $\succ$ denote a total order on $\Sigma$ such that $f_j \succ f_i \iff j > i$ for $i, j \in \{1, \ldots, K\}$. The lexicographic path ordering $\succ_{\text{lpo}}$ on $\mathcal{T}$ (induced by $\succ$) is defined as follows, cf. [15].

**Definition 3.2.1.** $s \succ_{\text{lpo}} t$ iff

1. $t \in \var(s)$ and $s \neq t$, or
2. $s = f_j(s_1, \ldots, s_m), t = f_i(t_1, \ldots, t_n)$, and
   - there exists $k (1 \leq k \leq m)$ with $s_k \succ_{\text{lpo}} t$, or
   - $j > i$ and $s \succ_{\text{lpo}} t_l$ for all $l = 1, \ldots, n$, or
   - $i = j$ and $s \succ_{\text{lpo}} t_l$ for all $l = 1, \ldots, n$, and there exists an $i_0 (1 \leq i_0 \leq m)$ such that $s_1 = t_1, \ldots, s_{i_0-1} = t_{i_0-1}$ and $s_{i_0} \succ_{\text{lpo}} t_{i_0}$.

**Proposition 3.2.1.** (Kamin-Levy).

1. If $s \succ_{\text{lpo}} t$, then $\var(t) \subseteq \var(s)$.
2. For any total order $\prec$ on $\Sigma$, the induced lexicographic order $\succ_{\text{lpo}}$ is a simplification order on $\mathcal{T}$.
3. If $R$ is a rewrite system such that $\rightarrow_R$ is contained in a lexicographic path ordering, then $R$ is terminating.

**Proof.** Folklore.

3.3 Ordinal Terms and the Lexicographic Path Ordering

Let $N$ be defined as in the previous section. In this section we define a set of terms $\mathcal{T}$ (and a subset $P \subseteq \mathcal{T}$) together with a well-ordering $\prec$ on $\mathcal{T}$. The elements of $\mathcal{T}$ are built from 0, + and the $(N + 1)$-ary function symbol $\psi$. It is important to note that the elements
of \( T \) are terms not ordinals. Although these terms can serve as representations of an initial segment of the set of ordinals \( \text{On} \), we will not make any use of this interpretation. In particular the reader not familiar with proof theory should have no difficulties to understand the definitions and propositions of this section. However some basic amount of understanding in proof theory may be useful to grasp the origin and meaning of the presented concepts, cf. [47, 105, 133]. For the reader familiar with proof theory: Note that \( P \) corresponds to the set of additive principal numbers in \( T \), while \( \psi \) represents the (set-theoretical) fixed-point free Veblen function, cf. [133, 105].

**Definition 3.3.1.** Recursive definition of a set \( T \) of ordinal terms, a subset \( P \subset T \), and a binary relation \( > \) on \( T \).

(i) \( 0 \in T \).

(ii) If \( \alpha_1, \ldots, \alpha_m \in P \) and \( \alpha_1 \geq \cdots \geq \alpha_m \), then \( \alpha_1 + \cdots + \alpha_m \in T \).

(iii) If \( \alpha_1, \ldots, \alpha_{N+1} \in T \), then \( \psi(\alpha_1, \ldots, \alpha_{N+1}) \in P \) and \( \psi(\alpha_1, \ldots, \alpha_{N+1}) \in T \).

(iv) \( \alpha \neq 0 \) implies \( \alpha > 0 \).

(v) \( \alpha > \beta_1, \ldots, \beta_m \) and \( \alpha \in P \) implies \( \alpha > \beta_1 + \cdots + \beta_m \).

(vi) Let \( \alpha = \alpha_1 + \cdots + \alpha_m, \beta = \beta_1 + \cdots + \beta_n \). Then \( \alpha > \beta \) iff

- \( m > n \), and for all \( i \ (i \in \{1, \ldots, n\}) \) \( \alpha_i = \beta_i \), or
- there exists \( i \ (i \in \{1, \ldots, m\}) \) such that \( \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1} \), and \( \alpha_i > \beta_i \).

(vii) Let \( \alpha = \psi(\alpha_1, \ldots, \alpha_{N+1}), \beta = \psi(\beta_1, \ldots, \beta_{N+1}) \). Then \( \alpha > \beta \) iff

- there exists \( k \ (1 \leq k \leq N+1) \) with \( \alpha_k \geq \beta \), or
- \( \alpha > \beta_l \) for all \( l = 1, \ldots, N+1 \) and there exists an \( i_0 \ (1 \leq i_0 \leq N+1) \) such that \( \alpha_1 = \beta_1, \ldots, \alpha_{i_0-1} = \beta_{i_0-1} \) and \( \alpha_{i_0} > \beta_{i_0} \).

We use lower-case Greek letters to denote the elements of \( T \). Furthermore we formally define \( \alpha + 0 = 0 + \alpha = \alpha \) for all \( \alpha \in T \).

We sometimes abbreviate sequences of (ordinal) terms like \( \alpha_1, \ldots, \alpha_n \) by \( \overline{\alpha} \). Hence, instead of \( \psi(\alpha_1, \ldots, \alpha_{N+1}) \) we may write \( \psi(\overline{\alpha}) \). To relate the elements of \( T \) to more expressive ordinal notations, we define \( 1 := \psi(\overline{0}), \omega := \psi(\overline{0}, 1) \), and \( \epsilon_0 := \psi(\overline{0}, 1, 0) \). Let \( \text{Lim} \) be the set of elements in \( T \) which are neither 0 nor of the form \( \alpha + 1 \). Elements of \( \text{Lim} \) are called limit ordinal terms.

**Proposition 3.3.1.** Let \( (T, <) \) be defined as above. Then \( (T, <) \) is a well-ordering.

*Proof.* Let \( |\alpha| \) denote the number of symbols in the ordinal term \( \alpha \). Exploiting induction on \( |\alpha| \) one easily verifies that the ordering \( (T, <) \) is well-defined. To show well-foundedness one uses induction on the lexicographic path ordering \( <_{\text{lpo}} \), exploiting the close connection between Definition 3.2.1.ii in Section 3.2 and Definition 3.3.1.vii above. \( \square \)
In the following proposition we want to relate the order type of the well-ordering \((T, <)\)
and the well-partial ordering \(\prec_{\text{lpo}}\). Concerning the latter it is best to momentarily
restrict our attention to the well-ordering \((T(\Sigma), \prec_{\text{lpo}})\). We indicate the arity of the
function symbol \(\psi\) employed in Definition 3.3.1. We write \((T(N + 1), <)\) instead of
\((T, <)\). Similarly we write \((T(\Sigma(N)), \prec_{\text{lpo}})\) to indicate the maximal arity of function
symbols in the finite signature \(\Sigma\). Let \(\Theta_{\Omega\pi}(0)\) denote the small Veblen ordinal \([133]\) and let \(\text{otyp}(M)\) denote the order type of a well-ordering \(M\).

**Proposition 3.3.2.** (i) For any number \(k\), there exists an order isomorphic embedding
from \((T(\Sigma(k)), \prec_{\text{lpo}})\) into \((T(k + 1), <)\).

(ii) For any number \(k > 2\), there exists an order isomorphic embedding from \((T(k), <)\)
into \((T(\Sigma(k)), \prec_{\text{lpo}})\).

(iii) \(\sup_{k < \omega}(\text{otyp}((T(k), <))) = \sup_{k < \omega}(\text{otyp}((T(\Sigma(k)), \prec_{\text{lpo}})))) = \Theta_{\Omega\pi}(0)\).

**Proof.** The first two assertions are a consequence of the well-ordering proof of \((T, <)\).
We only comment on the stated lower bound in the second one. The statement fails for
\((T(2), <)\) and \((T(\Sigma(2)), \prec_{\text{lpo}})\). The presence of the binary function symbol + in \(T(2)\)
can make the ordering < more expressive than \(\prec_{\text{lpo}}\). This difference vanishes for \(k \geq 3\).
The third assertion follows from \([130]\). \(\square\)

### 3.4 Fundamental Sequences and Sub-recursive Hierarchies

To each ordinal term \(\alpha \in T\) we assign a canonical sequence of ordinal terms \(\langle \alpha[x] : x \in \mathbb{N} \rangle\), the fundamental sequence. The concept of fundamental sequences is a crucial one in (ordinal) proof theory. The main idea of utilizing fundamental sequences in term rewriting, is that the descent along the branches of such a sequence can, informally speaking, code rewriting steps. We have to wade through some technical definitions.

We define the set \(\text{IS}_n(\gamma)\), the set of interesting subterms of \(\gamma\) (relative to \(\alpha\)) by induction on \(\gamma\). We set \(\text{IS}_n(0) := \emptyset\), \(\text{IS}_n(\gamma_1 + \cdots + \gamma_m) := \bigcup_{i=1}^{m} \text{IS}_n(\gamma_i)\), and finally

\[
\text{IS}_n(\psi(\gamma_1, \ldots, \gamma_{N+1})) := \left\{ \begin{array}{ll}
\{\psi(\gamma)\} & \text{if } (\gamma_1, \ldots, \gamma_N) \geq_{\text{lex}} (\alpha_1, \ldots, \alpha_N) \\
\bigcup_{i=1}^{N+1} \text{IS}_n(\gamma_i) & \text{otherwise}.
\end{array} \right.
\]

The (relative to \(\alpha\)) maximal interesting subterm \(\text{MS}_n(\gamma_1, \ldots, \gamma_n)\) of a non-empty sequence \((\gamma_1, \ldots, \gamma_n)\) is defined as the maximum of the terms occurring in \(\text{IS}_n(\gamma_i)\). Let \(\triangleright_{\text{lex}}\) denote the lexicographic ordering on sequences of ordinal terms induced by \(\triangleright\). Let \(\alpha = \alpha_1, \ldots, \alpha_N \in T\) and \(\beta \in T\). Then set

\[
\text{Fix}(\alpha) := \{\psi(\gamma, \delta) : \gamma >_{\text{lex}} \alpha \text{ and } \psi(\gamma, \delta) \triangleright \alpha_i \text{ for all } i = 1, \ldots, N\}.
\]

For a unary function symbol \(f\) we define the \(n\)th iteration \(f^n\) inductively as (i) \(f^0(x) := x\), and (ii) \(f^{n+1}(x) := f(f^n(x))\). We will make use of this notation for functions of higher arity by assuming that all but one argument remain fixed. We use \(\cdot\) to indicate the free position. In the sequel \(\lambda\) (possibly extended by a subscript) will always denote a limit ordinal term.
3.4 Fundamental Sequences and Sub-recursive Hierarchies

Definition 3.4.1. Recursive definition of $\alpha[x]$ for $x < \omega$.  

\[
\begin{align*}
0[x] & := 0 \\
(\alpha_1 + \cdots + \alpha_m)[x] & := \alpha_1 + \cdots + \alpha_m[x], \quad m > 1, \alpha_1 \geq \cdots \geq \alpha_m \\
\psi(\overline{0})[x] & := 0 \\
\psi(\overline{0}, \beta + 1)[x] & := \psi(\overline{0}, \beta) \cdot (x + 1) \\
\psi(\overline{0}, \lambda)[x] & := \psi(\overline{0}, \lambda[x]) \quad \lambda \notin \text{Fix}(\overline{0}) \\
\psi(\overline{0}, \lambda)[x] & := \lambda \cdot (x + 1) \quad \lambda \in \text{Fix}(\overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, 0)[x] & := \psi(\alpha_1, \ldots, \alpha_i, \overline{0})^{x+1}(0) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \beta + 1)[x] & := \psi(\alpha_1, \ldots, \alpha_i, \overline{0})^{x+1}(\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \beta)) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \overline{0}, \lambda)[x] & := \psi(\alpha_1, \ldots, \alpha_i, \overline{0})^{x+1}(\lambda) \quad \lambda \notin \text{Fix}(\overline{0}, \overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i, \overline{0}, 0)[x] & := \psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \lambda, x) \quad \lambda \in \text{Fix}(\overline{0}, \overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \beta + 1)[x] & := \psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \lambda, x) \quad \lambda \notin \text{Fix}(\overline{0}, \overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \lambda)[x] & := \psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \lambda, x) \quad \lambda \in \text{Fix}(\overline{0}, \overline{0}) \\
\psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \overline{0})[x] & := \psi(\alpha_1, \ldots, \alpha_i, \overline{0}, \overline{0}, x) \quad \lambda \in \text{Fix}(\overline{0}, \overline{0})
\end{align*}
\]

The above definition is given in such a way as to simplify the comparison between the fundamental sequences for $T$ and the fundamental sequences for the set of ordinal terms $T(2)$ (built from $0$, $+$, and a 2-ary function symbol $\psi$) as presented in [146]. Note that our definition is equivalent to the more compact one presented in [105]. The following proposition is stated without proof. A proof (for a slightly different assignment of fundamental sequences) can be found in [31].

Proposition 3.4.1. Let $\alpha \in T$ be given; assume $x < \omega$. If $\alpha > 0$, then $\alpha > \alpha[x]$. For $\alpha > 1$ we get $\alpha[x] > 0$, and if $\alpha \in \text{Lim}$, then $\alpha[x + 1] > \alpha[x]$. Finally, if $\beta < \alpha \in \text{Lim}$, then there exists $x < \omega$, such that $\beta < \alpha[x]$ holds.

In the definition of $\psi(\alpha_1, \ldots, \lambda_i, \overline{0}, 0)[x]$ we introduce at the last position of $\psi$ the term $\text{MS}_{\overline{0}, \overline{0}}(\overline{0})$. We cannot simply dispense of this term. To see this, we alter the definition of the crucial case. We momentarily consider only 3-ary $\psi$-functions; we set $\Gamma_0 := \psi(1, 0, 0)$ and calculate $\psi(0, \Gamma_0, 0)[x]$:  

\[
\begin{align*}
\psi(0, \Gamma_0, 0)[x] & = \psi(0, \psi(1, 0, 0)[x], 0) \\
& = \psi(0, \psi(0, \psi(0, \psi(0, \psi(0, 0)))[x], 0), 0) \\
& = \psi(0, \psi(0, \psi(0, \psi(0, \psi(0, 0))[x], 0), 0), 0) \\
& < \psi(1, 0, 0).
\end{align*}
\]

Hence for every $x < \omega$: $\psi(0, \Gamma_0, 0)[x] < \Gamma_0$ holds. This contradicts the last assertion of the proposition as $\Gamma_0 < \psi(0, \Gamma_0, 0)$. As a side-remark we want to mention that the given assignment of fundamental sequences even fulfills the Bachmann property, see [17]. Utilizing Definition 3.4.1 we are now in the position to define sub-recursive hierarchies of ordinal functions.
Definition 3.4.2. (The slow-growing hierarchy.) Recursive definition of the function $G_\alpha: \omega \to \omega$ for $\alpha \in T$.

\begin{align*}
G_0(x) &:= 0 \\
G_{\alpha+1}(x) &:= G_\alpha(x) + 1 \\
G_\lambda(x) &:= G_{\lambda[x]}(x).
\end{align*}

Definition 3.4.3. (The fast-growing hierarchy.) Recursive definition of the function $F_\alpha: \omega \to \omega$ for $\alpha \in T$.

\begin{align*}
F_0(x) &:= x + 1 \\
F_{\alpha+1}(x) &:= F_{\alpha}^{x+1}(x) \\
F_\lambda(x) &:= F_{\lambda[x]}(x).
\end{align*}

It is easy to see that $G_\alpha(x) < F_\alpha(x)$ for all $\alpha > 0$. To see that the name of the hierarchy $\{G_\alpha: \alpha \in T\}$ is appropriate, it suffices to calculate some examples. Take e.g. $G_\omega: G_\omega(x) = G_{\psi(0)(x+1)}(x) = G_{x+1}(x) = G_x(x) + 1 = x + 1$.

Recall that a function $f$ is elementary (in a function $g$) if $f$ is definable explicitly from $0, 1, +, \cdot$ (and $g$), using bounded sum and product. $E(g)$ denotes the class of all such functions $f$. Then $G_\epsilon_0$ majorizes the elementary functions $E$. In contrast the function $F_\omega$ already majorizes the primitive recursive functions, i.e. its growth rate is comparable to the (binary) Ackermann function. Furthermore the class of multiple recursive functions can be characterized by the hierarchy $\{E(F_\gamma): \gamma < \omega^\omega\}$, cf. [125, 127].

However, the following theorem states a (surprising) connection between the slow- and fast-growing hierarchy. See e.g. [64, 99, 146] for further reading on the Hierarchy Comparison Theorem.

Theorem 3.4.1. (The Hierarchy Comparison Theorem.)

\[
\bigcup_{\alpha \in T} E(G_\alpha) = \bigcup_{\gamma < \omega^{N+1}} E(F_\gamma).
\]

Proof. We do not give a detailed proof, but only state the main idea. In [146] the hierarchy comparison theorem has been established for the set of ordinal terms $T(2)$ (built from $0, +, \cdot$ and the function symbol $\psi$, where $\text{ar}(\psi) = 2$). To extend the result to $T$ it suffices to follow the pattern of the proof in [146].

The difficult direction is to show that every function in the hierarchy $\{F_\gamma: \gamma < \omega^{N+1}\}$ is majorized by some $G_\alpha$. To show this one in particular needs to extend the proofs of Lemma 5 and Theorem 1 in [146] adequately. The reversed direction follows by standard techniques, cf. [39].

3.5 The Interpretation Theorem

For all $\alpha \in T$ there are uniquely determined ordinal terms $\alpha_1 \geq \cdots \geq \alpha_m \in P$ such that $\alpha = \alpha_1 + \cdots + \alpha_m$ holds. In addition, for every $\alpha \in P$ there exist unique $\alpha_1, \ldots, \alpha_{N+1}$ such that $\alpha = \psi(\alpha_1, \ldots, \alpha_{N+1})$. (This normal form property is trivial by definition.)
Now assume $\alpha, \beta \in T$ with $\alpha = \gamma_1 + \cdots + \gamma_{m_0}$, $\beta = \gamma_{m_0+1} + \cdots + \gamma_m$. Then the natural sum $\alpha \# \beta$ is defined as $\gamma_{\rho(1)} + \cdots + \gamma_{\rho(m)}$, where $\rho$ denotes a permutation on $\{1, \ldots, m\}$ such that $\gamma_{\rho(1)} \geq \cdots \geq \gamma_{\rho(m)}$ holds.

Let $R$ denote a finite rewrite system whose induced rewrite relation is contained in $\succ_{\text{LPO}}$.

**Definition 3.5.1.** Recursive definition of the interpretation function $\pi: T(\Sigma) \to T$.

Let $N$ denote the maximal arity of a function symbol in $\Sigma$. If $s = f_j \in \Sigma$, then set $\pi(s) := \psi(j, \emptyset)$. Otherwise, let $s = f_j(s_1, \ldots, s_m)$ and set

$$\pi(s) := \psi(j, \pi(s_1), \ldots, \pi(s_m) + 1, \emptyset).$$

In the sequel of this section we show that $\pi$ defines an interpretation for $R$ on $(T, <)$; i.e. we establish the following theorem.

**Theorem 3.5.1.** For all $s, t \in T(\Sigma)$ we have $s \rightarrow_R t$ implies $\pi(s) > \pi(t)$.

Unfortunately this is not strong enough. The problem being that $\alpha > \beta$ implies that $G_{\alpha}$ majorizes $G_{\beta}$, only. Whereas to proceed with our general program—see Section 3.1—we need an interpretation theorem for a binary relation $\succ$ on $T$, such that $\alpha \succ \beta \Rightarrow G_{\alpha}(x) > G_{\beta}(x)$ holds for all $x$. We introduce a notion of a generalized system of fundamental sequences. Based on this generalized notion, it is then possible to define a suitable ordering $\succ$.

**Definition 3.5.2.** (Generalized system of fundamental sequences for $(T, <)$.) Recursive definition of $(\alpha)^x$ for $x < \omega$.

(i) $(0)^x := \emptyset$

(ii) Assume $\alpha = \alpha_1 + \cdots + \alpha_m; m > 1$. Then $\beta \in (\alpha)^x$ if either

- $\beta = \alpha_1 \# \cdots \# \alpha_m$ and $\alpha_i^* \in (\alpha_i)^x$ holds, or
- $\beta = \alpha_i$.

(iii) Assume $\alpha = \psi(\overline{\alpha})$. Then $\beta \in (\alpha)^x$ if

- $\beta = \psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1})$, and $\alpha_i^* \in (\alpha_i)^x$, or
- $\beta = \alpha_i + x$, where $\alpha_i > 0$, or
- $\beta = \psi(\overline{\alpha})[x]$.

By recursion we define the transitive closure of the ownership $(\alpha)^x \ni \beta$: $(\alpha > (x) \beta) \leftrightarrow (\exists \gamma \in (\alpha)^x(\gamma > (x) \beta \lor \gamma = \beta))$. Let $\alpha, \beta \in T$. It is easy to verify that $\alpha > (x) \beta$ (for some $x < \omega$) implies $\alpha > \beta$. If no confusion can arise we write $\alpha^x$ instead of $(\alpha)^x$.

**Lemma 3.5.1.** (Subterm Property) Let $x < \omega$ be arbitrary.

(i) $\alpha < (x) \gamma_1 \# \cdots \# \gamma_m$.

(ii) $\alpha < (x) \psi(\gamma_1, \ldots, \gamma, \ldots, \gamma_{N+1})$. 

33
3 Relating Derivation Lengths with the Slow-Growing Hierarchy Directly

Proof. The first assertion is trivial. The second assertion follows by the definition of $<_(x)$ and assertion $\Box$

Lemma 3.5.2. (Monotonicity Property) Let $x < \omega$ be arbitrary.

(i) If $\alpha >_{(x)} \beta$, then $\gamma_1 \# \cdots \# \alpha \# \cdots \# \gamma_m >_{(x)} \gamma_1 \# \cdots \# \beta \# \cdots \# \gamma_m$.

(ii) If $\alpha >_{(x)} \beta$, then $\psi(\gamma_1, \ldots, \alpha, \ldots, \gamma_{N+1}) >_{(x)} \psi(\gamma_1, \ldots, \beta, \ldots, \gamma_{N+1})$.

Proof. We employ induction on $\alpha$ to prove $\Box$. We write (ih) for induction hypothesis. We may assume that $\alpha > 0$. By definition of $\alpha >_{(x)} \beta$ we either have (i) that there exist $\delta \in \alpha^x$ and $\delta >_{(x)} \beta$ or (ii) $\beta \in \alpha^x$. Firstly, one considers the latter case. Then $(\gamma_1 \# \cdots \# \gamma_m) \in (\gamma_1 \# \cdots \# \gamma_m)^x$ holds by Definition 3.5.2. Therefore $(\gamma_1 \# \cdots \# \gamma_m) <_{(x)} (\gamma_1 \# \cdots \# \gamma_m)$ follows. Now, we consider the first case. By assumption $\delta >_{(x)} \beta$ holds, by (ih) this implies $(\gamma_1 \# \cdots \# \gamma_m) >_{(x)} (\gamma_1 \# \cdots \# \gamma_m)^x$. Now $(\gamma_1 \# \cdots \# \gamma_m) >_{(x)} (\gamma_1 \# \cdots \# \gamma_m)$ follows by definition of $>_{(x)}$, if we replace $\beta$ by $\delta$ in the proof of the second case. This completely proves $\Box$.

To prove (i) we proceed by induction on $\alpha$. By definition of $\alpha >_{(x)} \beta$ we have either (i) $\delta \in \alpha^x$ and $\delta >_{(x)} \beta$ or (ii) $\beta \in \alpha^x$. It is sufficient to consider the latter case, the first case follows from the second as above. By Definition 3.5.2, $\beta \in \alpha^x$ implies $\psi(\gamma_1, \ldots, \beta, \ldots, \gamma_{N+1}) \in \psi(\gamma_1, \ldots, \alpha, \ldots, \gamma_{N+1})^x$. $\Box$

In the sequel we show the existence of a natural number $e$, such that for all $s, t \in T$, and any ground substitution $\rho$, $s \rightarrow_R t$ implies $\pi(s\rho) >_{(e)} \pi(t\rho)$. Theorem 3.5.1 follows then as a corollary. The proof is involved, and makes use of a sequence of lemmas.

Lemma 3.5.3. Assume $\alpha, \beta \in \text{Lim}$; $x \geq 1$. If $\alpha >_{(x)} \beta$, then $\alpha >_{(x+1)} \beta + 1$ holds.

To prove the lemma we exploit the following auxiliary lemma.

Lemma 3.5.4. We assume the assumptions and notation of Lemma 3.5.3; assume Lemma 3.5.2 holds for all $\gamma, \delta \in \text{Lim}$ with $\gamma, \delta < \alpha$. Then $\alpha >_{(x+1)} \alpha[x+1] \geq (x+1)$ $\alpha[x] + 1$.

Proof. The lemma follows by induction on the form of $\alpha$ by analyzing all cases of Definition 3.4.1. $\Box$

Proof. (of Lemma 3.5.3) The proof proceeds by induction on the form of $\alpha$. We consider only the case where $\alpha = \psi(\alpha_1, \ldots, \alpha_{N+1})$. The case where $\alpha = \alpha_1 + \cdots + \alpha_m$ is similar but simpler.

By definition of $\alpha >_{(x)} \beta$ we have either (i) $\gamma \in \alpha^x$ and $\gamma >_{(x)} \beta$ or (ii) $\beta \in \alpha^x$. Assume for $\gamma \in \alpha^x$ we have already shown that $\gamma + 1 <_{(x+1)} \alpha$. Then for $\beta <_{(x)} \gamma$, we conclude by (ih) and the Subterm Property $\beta + 1 <_{(x+1)} \gamma <_{(x+1)} \gamma + 1 <_{(x+1)} \alpha$. Hence, it suffices to consider the second case. We proceed by case distinction on the form of $\beta$.

Case $\beta = \psi(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{N+1})$ where $\alpha_i \in (\alpha_i)^x$ for some $i$ $(1 \leq i \leq N + 1)$. Note that $\alpha_i < \alpha$, hence (ih) is applicable to establish $\alpha_i^x + 1 <_{(x+1)} \alpha_i$. $\Box$
Furthermore by the Subterm Property follows $\alpha_i^* <_{(x+1)} \alpha_i + 1$ and therefore

$$\psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1}) <_{(x+1)} \psi(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_{N+1})$$

holds with Monotonicity. Applying (ih) with respect to $\psi(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_{N+1})$ we obtain

$$\psi(\alpha_1, \ldots, \alpha_i^*, \ldots, \alpha_{N+1}) + 1 <_{(x+1)} \psi(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_{N+1}) <_{(x+1)} \psi(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{N+1}) = \alpha.$$  

The last inequality follows again by an application of the Monotonicity Property.

Case $\beta = \alpha_i + x$: Then $(\alpha_i + x) + 1 = \alpha_i + (x + 1) <_{(x+1)} \alpha$.

Case $\beta = \psi(\overline{\tau})[x]$. Clearly $\beta \in \text{Lim}$. Then the auxiliary lemma becomes applicable. Thus $\psi(\overline{\tau})[x] + 1 \leq _{(x+1)} \alpha[x + 1] <_{(x+1)} \alpha$. \hfill $\square$

**Lemma 3.5.5.** Let $t \in T(\Sigma)$ be given. Assume $\tau(t) \leq d$, and $f_j \in \Sigma$. If $f_j \succ_{\text{lpo}} t$, then $\pi(f_j) >_{(2d)} \pi(t)$.

**Proof.** We proceed by induction on $\tau(t)$. In the presentation of the argument, we will frequently employ the Subterm and the Monotonicity Property without further notice. Set $\alpha := \pi(f_j)$, and $\beta := \pi(t)$. Furthermore it is a crucial observation that $0 <_{(x)} \alpha$ holds for any $x < \omega$. $\alpha \in T$. (This follows by a simple induction on $\alpha$.)

Case $\tau(t) = 0$: Then by assumption $t = f_i \in \Sigma$, $i < j$. Hence $i <_{(2d)} j$ holds and we conclude $\pi(t) = \psi(i, \overline{\alpha}) <_{(2d)} \psi(j, \overline{\alpha}) = \pi(f_j)$.

Case $\tau(t) > 0$: Let $t = f_i(t_1, \ldots, t_n)$. Set $\beta_l := \pi(t_l)$ for all $l = 1, \ldots, n$. By (ih) one obtains $\beta_l <_{(2d-1)} \alpha$ for all $l$. For all $l$, we need only consider the case where $\beta_l \in \alpha^{2(d-1)}$. We consider $\psi(j, \overline{\alpha})[2d]$ and apply the following sequence of descents via $>_{(2d)}$:

$$\psi(j, \overline{\alpha})[2d] = \psi(j - 1, \cdot, \overline{\alpha})^{2d+1}(0)$$

$$= \psi(j - 1, \psi(j - 1, \cdot, \overline{\alpha})^{2d}(0), \overline{\alpha})$$

$$>_{(2d)} \psi(j - 1, \psi(j - 1, \cdot, \overline{\alpha})^{2d-1}(0) + 1, \overline{\alpha})$$

$$>_{(2d)} \psi(j - 1, \psi(j - 1, \cdot, \overline{\alpha})^{2d-1}(0), \overline{\alpha})^{2d+1}(0)$$

We define $\gamma_1 := \psi(j, \overline{\alpha})[2(d-1)]$ and $\gamma_{k+1} := \psi(j - 1, \gamma_1, \ldots, \gamma_k + 1, \overline{\alpha})[2(d-1)]$. By iteration of the above descent, we see

$$\alpha[2d] = \psi(j, \overline{\alpha})[2d]$$

$$>_{(2d)} \psi(j - 1, \gamma_1, \ldots, \gamma_n + 1, \overline{\alpha})$$

$$>_{(2d)} \psi(j - 1, \alpha[2(d-1)], \ldots, \alpha[2(d-1)] + 1, \overline{\alpha})$$

Let $l$ ($1 \leq l \leq n$) be fixed. By assumption we have $\beta_l \in \alpha^{2(d-1)}$. We proceed by case distinction on the definition of $\beta_l$. 

35
Assume $\beta_l = \psi(j, \overline{0})[2(d - 1)]$. Then $\delta = \psi(j - 1, \alpha[2(d - 1)], \ldots, \beta_l, \ldots, \alpha[2(d - 1)] + 1, \overline{0})$. Assume $\beta_l = \psi(j^*, \overline{0})$, where $j^* \in (j)^{2(d - 1)}$, i.e. $j^* \leq_{(2d)} j - 1 <_{(2d)} j$. Therefore $\alpha[2(d - 1)] >_{(2d)} \psi(j - 1, \overline{0})$. Hence $\delta >_{(2d)} \psi(j - 1, \alpha[2(d - 1)], \ldots, \beta_l, \ldots, \alpha[2(d - 1)] + 1, \overline{0})$.

Finally assume $\beta_l = j + 2(d - 1)$. Then $\beta_l <_{(2d - 1)} \psi(j, \overline{0}) <_{(2d - 1)} \psi(j - 1, \overline{0})^{2d - 1}(0) = \alpha[2(d - 1)]$. Hence $\beta_l <_{(2d)} \alpha[2(d - 1)]$ by Lemma 3.5.3 and therefore $\delta >_{(2d)} \psi(j - 1, \alpha[2(d - 1)], \ldots, \beta_l, \ldots, \alpha[2(d - 1)] + 1, \overline{0})$.

As $l$ was fixed but arbitrary, the above construction is valid for all $l$. And the lemma follows.

**Lemma 3.5.6.** Let $f_i(t_1,\ldots,t_n), f_j(s_1,\ldots,s_m) \in \mathcal{T}(\Sigma)$ be given; let $d > 0$. Then

(i) If $i < j$, $\pi(f_j(\overline{x})) >_{(2d - 1)} \pi(f_i(\overline{t}))$ for all $l = 1, \ldots, n$. Then $\pi(f_j(\overline{x})) >_{(2d)} \pi(f_i(\overline{t}))$ holds.

(ii) If $s_1 = t_1, \ldots, s_{i_0} = t_{i_0 - 1}, \pi(s_{i_0}) >_{(2d - 1)} \pi(t_{i_0})$, and $\pi(f_j(\overline{x})) >_{(2d - 1)} \pi(f_i(\overline{t}))$, for all $l = i_0 + 1, \ldots, n$, then $\pi(f_j(\overline{x})) >_{(2d)} \pi(f_i(\overline{t}))$ holds.

**Proof.** The proof of assertion [i] is similar to the proof of assertion [ii] but simpler. Hence, we concentrate on [ii]. Set $\alpha := \pi(f_j(\overline{x})); \beta := \pi(f_i(\overline{t}));$ finally set $\alpha_i := \pi(s_i)$ for all $i = 1, \ldots, m$, and $\beta_i := \pi(t : i)$ for all $i = 1, \ldots, n$. As above, we consider only the case where $\beta_i \in (\alpha)^{2(d - 1)}$. The other case follows easily.

$$\alpha[2d] = \psi(j, \alpha_1, \ldots, \alpha_m + 1, \overline{0})[2d] = \psi(j, \alpha_1, \ldots, \alpha_m, \psi(j, \alpha_1, \ldots, \alpha_m, \ldots, \overline{0})^{2d - 1}(0), \overline{0}) >_{(2d)} \psi(j, \alpha_1, \ldots, \alpha_m, \psi(j, \alpha_1, \ldots, \alpha_m, \ldots, \overline{0})^{2d - 1}(0) + 1, \overline{0}) = \psi(j, \alpha_1, \ldots, \alpha_m, \psi(j, \alpha_1, \ldots, \alpha_m + 1, \overline{0})[2(d - 1)] + 1, \overline{0}) \quad \alpha[2(d - 1)]$$

Similar to above, we define $\gamma_1 := \alpha[2(d - 1)] = \psi(j, \alpha_1, \ldots, \alpha_m + 1, \overline{0})[2(d - 1)]$ and $\gamma_{k + 1} := \psi(j, \alpha_1, \ldots, \alpha_m, \gamma_k + 1, \overline{0})[2(d - 1)]$ and obtain

$$\alpha[2d] >_{(2d)} \psi(j, \alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_{N - m} + 1) >_{(2d)} \psi(j, \alpha_1, \ldots, \alpha_m, \alpha[2(d - 1)], \ldots, \alpha[2(d - 1)] + 1) >_{(2d)} \psi(j, \alpha_1, \ldots, \alpha_{i_0}, \overline{0}, \alpha[2(d - 1)] + 1) .$$

By assumption $\beta_{i_0} <_{(2d - 1)} \alpha_{i_0}$ and by Lemma 3.5.3 this implies $\beta_{i_0} + 1 <_{(2d)} \alpha_{i_0}$. We set $\overline{\alpha} := \alpha_1, \ldots, \alpha_{i_0 - 1},$ then we obtain

$$\psi(j, \overline{\alpha}, \alpha_{i_0}, \overline{0}, \alpha[2(d - 1)] + 1) >_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0} + 1, \overline{0}, \alpha[2(d - 1)] + 1)$$

$$>_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0} + 1, \overline{0}, \alpha[2(d - 1)] + 1)[2d]$$

$$>_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0}, \psi(j, \overline{\alpha}, \beta_{i_0} + 1, \overline{0}, \alpha[2(d - 1)]), \overline{0})$$

$$>_{(2d)} \psi(j, \overline{\alpha}, \beta_{i_0}, \alpha[2(d - 1)] + 1, \overline{0}) = \psi(j, \beta_1, \ldots, \beta_{i_0}, \alpha[2(d - 1)] + 1, \overline{0}) .$$
As in the first part of the proof, we obtain \( \alpha[2d] >_{(2d)} \psi(j, \beta_1, \ldots, \beta_n, \alpha[2(d-1)] + 1) >_{(2d)} \psi(j, \beta_1, \ldots, \beta_n, \alpha[2(d-1)], \ldots, \alpha[2(d-1)] + 1, 0) \).

By assumption we have \( \beta_i <_{(2(d-1))} \alpha \) for all \( l = 1, \ldots, n \). It remains to prove that this implies \( \beta_i \leq_{(2d)} \gamma \). For this it is sufficient to consider the case where \( \beta_i \in (\alpha)^{2(d-1)} \).

The proof proceeds by case-distinction on the construction of \( \beta_i \). The proof is similar to the respective part in the proof of Lemma 3.5.5 and hence omitted.

**Lemma 3.5.7.** Let \( s, t \in T \) be given. Assume \( s = f_j(s_1, \ldots, s_m), \rho \) is a ground substitution, \( \tau(t) \leq d \). Assume further \( s_k \succeq_{\text{lp}} u \) and \( \tau(u) \leq d \) implies \( \pi(s_k \rho) >_{(2d)} \pi(u \rho) \) for all \( u \in T \). Then \( s \succeq_{\text{lp}} t \) implies \( \pi(s \rho) >_{(2d)} \pi(t \rho) \).

**Proof.** The proof is by induction on \( d \).

1. **Case \( d = 0 \):** Hence \( \tau(t) = 0 \); therefore \( t \in V \) or \( t = f_i \in \Sigma \). Consider \( t \in V \). Then \( t \) is a subterm of \( s \). Hence there exists \( k \) (\( 1 \leq k \leq m \)) s.t. \( t \) is subterm of \( s_k \). Hence \( s_k \succeq_{\text{lp}} t \), and by assumption this implies \( \pi(s_k \rho) >_{(2d)} \pi(t \rho) \), and therefore \( \pi(s \rho) >_{(2d)} \pi(t \rho) \) by the Subterm Property.

   Now assume \( t = f_i \in \Sigma \). As \( s \succeq_{\text{lp}} t \) by assumption either \( i < j \) or \( s_k \succeq_{\text{lp}} t \) holds.

   In the latter case, the assumptions render \( \pi(s_k \rho) >_{(2d)} \pi(t \rho) \); hence \( \pi(s \rho) >_{(2d)} \pi(t \rho) \).

   Otherwise, \( \pi(s \rho) = \psi(j, \pi(s_1 \rho), \ldots, \pi(s_m \rho) + 1, 0) \), while \( \pi(t \rho) = \psi(t) = \psi(i, 0) \). As \( \pi(s \rho) >_{(x)} 0 \) holds for arbitrary \( x < \omega \), we conclude \( \pi(s \rho) >_{(2d)} \pi(t \rho) \).

2. **Case \( d > 0 \):** Assume \( \tau(t) > 0 \). (Otherwise, the proof follows the pattern of the case \( d = 0 \).) Let \( t = f_i(t_1, \ldots, t_n) \), and clearly \( \tau(t_l) \leq (d - 1) \) for all \( l = 1, \ldots, n \). We start with the following observation: Assume there exists \( i_0 \) s.t. \( s \succeq_{\text{lp}} t_i \) holds for all \( l = i_0 + 1, \ldots, n \). Then by (ih) we have \( \pi(s \rho) >_{(2(d-1))} \pi(t_l) \).

   We proceed by case-distinction on \( s \succeq_{\text{lp}} t \). Assume firstly there exists \( k \) (\( 1 \leq k \leq m \)) s.t. \( s_k \succeq_{\text{lp}} t \). Utilizing the assumptions of the lemma, we conclude \( \pi(s \rho) >_{(2d)} \pi(t \rho) \).

   Now assume \( i < j \) and \( s \succeq_{\text{lp}} t \) for all \( l = 1, \ldots, n \). Clearly \( s \rho, t \rho \in T(\Sigma) \). By the observation \( \pi(s \rho) >_{(2(d-1))} \pi(t \rho) \) holds. Hence Lemma 3.5.6 becomes applicable and therefore \( \pi(s \rho) >_{(2d)} \pi(t \rho) \) holds true. Finally assume \( i = j; s_1 = t_1, \ldots, s_{i-1} = t_{i-1} \); \( s_{i_0} \succeq_{\text{lp}} t_{i_0}; s \succeq_{\text{lp}} t_l \); for all \( l = i_0 + 1, \ldots, m \). Utilizing the observation, we see that Lemma 3.5.6 becomes applicable and therefore \( \pi(s \rho) >_{(2d)} \pi(t \rho) \).

**Lemma 3.5.8.** Let \( t \in T(\Sigma) \) be given, assume \( \tau(t) \leq d \). Then \( \psi(K + 1, 0) >_{(2d)} \pi(t) \).

**Proof.** The inductive proof follows the pattern of the proof of Lemma 3.5.5.

**Theorem 3.5.2.** Let \( l, r \in T \) be given. Assume \( \rho \) is a ground substitution, \( \tau(t) \leq d \). Then \( l \succeq_{\text{lp}} r \) implies \( \pi(l \rho) >_{(2d)} \pi(r \rho) \).

**Proof.** We proceed by induction on \( \tau(s) \).

   **Case \( \tau(s) = 0 \):** Then \( s \) can either be a constant or a variable. As \( s \succeq_{\text{lp}} t \) holds, we can exclude the latter case. Hence assume \( s = f_j \). As \( f_j \succeq_{\text{lp}} t \), \( t \) is closed. Hence the assumptions of the theorem imply the assumptions of Lemma 3.5.7 and we conclude \( \pi(s \rho) = \pi(s) >_{(2d)} \pi(t) = \pi(t \rho) \).
CASE $\tau(s) > 0$: Then $s$ can be written as $f_j(s_1, \ldots, s_m)$. By (ih) $s_k \succ_{\text{LPO}} u$ and $\tau(u) \leq d$ imply $\pi(s_k \rho) \succ_{(2d)} \pi(t \rho)$. Therefore the present assumptions contain the assumptions of Lemma 3.5.7 and hence $\pi(s \rho) \succ_{(2d)} \pi(t \rho)$ follows.

**Theorem 3.5.3.** (The Interpretation Theorem.) Let $R$ denote a finite rewrite system whose induced rewrite relation is contained in $\succ_{\text{LPO}}$. Then there exists $k < \omega$, such that for all $l, r \in T$, and any ground substitution $\rho l \rightarrow_R r$ implies $\pi(l \rho) > (k) \pi(r \rho)$.

**Proof.** Set $d$ equal to $\max \{\tau(r) : \exists l, r \in R\}$. Then the theorem follows as a corollary to Theorem 3.5.2 if $k$ is set to $2d$.

### 3.6 Collapsing Theorem

We define a variant of the slow-growing hierarchy, cf. Definition 3.4.2, suitable for our purposes.

**Definition 3.6.1.** Recursive definition of the function $\tilde{G}_\alpha : \omega \rightarrow \omega$ for $\alpha \in T$.

\[
\tilde{G}_0(x) := 0 \\
\tilde{G}_\alpha(x) := \max\{\tilde{G}_\beta(x) : \beta \in (\alpha)^x\} + 1
\]

**Lemma 3.6.1.** Let $\alpha \in T$, $\alpha > 0$ be given. Assume $x < \omega$ is arbitrary.

(i) $\tilde{G}_\alpha$ is increasing. (Even strictly if $\alpha > \omega$.)

(ii) If $\alpha > (x) \beta$, then $\tilde{G}_\alpha(x) > \tilde{G}_\beta(x)$.

**Proof.** Both assertions follow by induction over $<$ on $\alpha$.

We need to know that this variant of the slow-growing hierarchy is indeed slow-growing. We show this by verifying that the hierarchies $\{\tilde{G}_\alpha : \alpha \in T\}$ and $\{G_\alpha : \alpha \in T\}$ coincide with respect to growth-rate. It is a triviality to verify that there exists $\beta \in T$ such that $\tilde{G}_\beta$ majorizes $G_\alpha$. (Simply set $\beta = \alpha$.) The other direction is less trivial. One first proves that for any $\alpha \in T$ there exists $\gamma < \omega^{N+1}$ such that $\tilde{G}_\alpha(x) \leq F_\gamma(x)$ for almost all $x$. Secondly one employs the Hierarchy Comparison Theorem once more to establish the existence of $\beta \in T$ such that $\tilde{G}_\alpha(x) \leq G_\beta(x)$ holds for almost all $x$.

**Theorem 3.6.1.**

\[
\bigcup_{\alpha \in T} E(G_\alpha) = \bigcup_{\alpha \in T} E(\tilde{G}_\alpha) = \bigcup_{\gamma < \omega^{N+1}} E(F_\gamma).
\]
3.7 Complexity Bounds

The complexity of a terminating finite rewrite system $R$ is measured by the derivation length function.

Definition 3.7.1. The derivation length function $Dl_R : \omega \to \omega$. Let $m < \omega$ be given. $Dl_R(m) := \max \{ n : \exists t_1, \ldots, t_n \in T \ ((t_1 \rightarrow_R \cdots \rightarrow_R t_n) \land (\tau(t_1) \leq m)) \}$.

Let $R$ be a rewrite system over $T$ such that $\rightarrow_R$ is contained in a lexicographic path ordering. Now assume that there exist $s = t_0, t_1, \ldots, t_n \in T$ with $\tau(s) \leq m$ such that

$s \rightarrow_R t_1 \rightarrow_R \cdots \rightarrow_R t_n$

holds. By our choice of $R$ this implies $s \succ_{\text{lpog}} t_1 \succ_{\text{lpog}} \cdots \succ_{\text{lpog}} t_n$. By assumption on $\Sigma$ there exists $c \in \Sigma$, with $\text{ar}(c) = 0$. We define a ground substitution $\rho : x \mapsto c$, for all $x \in V$. Let $k < \omega$ be defined as in Theorem 3.5.3. Recall that $K$ denotes the cardinality of $\Sigma$. We conclude from the Interpretation Theorem and Lemma 3.5.8, $\pi(s \rho) >_{\text{lpog}} \cdots >_{\text{lpog}} \pi(t_n \rho)$ and $\psi(K + 1, 0) >_{\text{lpog}} \pi(s \rho)$. Setting $h := \max \{ 2m, k \}$ and utilizing Lemma 3.5.8 we obtain $\psi(K + 1, 0) >_{\text{lpog}} \pi(s \rho) >_{\text{lpog}} \cdots >_{\text{lpog}} \pi(t_n \rho)$. An application of Lemma 3.6.1. yields

$\tilde{G}_{\psi(K+1,0)}(h) > \tilde{G}_{\pi(s \rho)}(h) > \cdots > \tilde{G}_{\pi(t_n \rho)}(h)$.

Employing Theorem 3.6.1 we conclude the existence of $\gamma < \omega^\omega$, such that

$F_\gamma(\max \{ 2m, k \}) \geq \tilde{G}_{\psi(K+1,0)}(\max \{ 2m, k \}) \geq Dl_R(m)$.

The class of multiply-recursive functions is captured by $\bigcup_{\gamma < \omega^\omega} E(F_\gamma)$, see [127]. Thus we have established a multiply-recursive upper bound for the derivation length of $R$ if $\rightarrow_R$ is contained in a lexicographic path ordering. Furthermore, this bound is essentially optimal, cf. [144].

3.8 Conclusion

The presented proof method is generally applicable. Let $R$ denote a rewrite system whose termination can be shown via $\succ_{\text{mpo}}$. To yield a primitive recursive upper bound for the complexity of $R$ the above proof can be employed. Firstly the definition of the interpretation function $\pi$ has to be changed as follows. If $s = f_j(s_1, \ldots, s_m)$, then we set

$\pi(s) := \psi(j, \pi(s_1)\# \cdots \# \pi(s_m)\#1)$.

Then the presented proof needs only partial changes. It suffices to reformulate (and reprove) Lemma 3.5.5, 3.5.6, 3.5.7, and 3.5.8 respectively.

Future work will be concerned with the Knuth-Bendix ordering. Due to the more complicated nature of this ordering the statement of the interpretation is not so simple. Still we believe that only mild alterations of the given proof are necessary.
4

Proofs of Termination of Rewrite Systems for Polytime Functions

Publication Details

Ranking
The Conference on Foundations of Software Technology and Theoretical Computer Science has been ranked A by CORE in 2007.

Abstract
We define a new path order $\prec_{\text{pop}}$ so that for a finite rewrite system $R$ compatible with $\prec_{\text{pop}}$, the complexity or derivation length function $Dl^f_R$ for each function symbol $f$ is guaranteed to be bounded by a polynomial in the length of the inputs. Our results yield a simplification and clarification of the results obtained by Beckmann and Weiermann (Archive for Mathematical Logic, 36:11–30, 1996).

4.1 Introduction
Suppose $C$ denotes an inductively defined class of recursive number-theoretic functions and suppose each $f \in C$ is defined via an equation (or more generally a system of equations) of the form
\[ f(x) = t(\lambda y. f(y), x), \]
where $t$ may involve previously defined functions. In a term-rewriting context these defining equations are oriented from left to right and the canonical term-rewriting characterisation $R_C$ of $C$ can be defined as follows: The signature $\Sigma$ of $R_C$ includes for each
function $f$ in $C$ a corresponding function symbol $f$. In order to represent natural numbers $\Sigma$ includes a constant 0 and a unary function symbol $S$. I.e. numbers are represented by their numerals. (Later we represent natural numbers in the form of binary strings.)

For each function $f \in C - \{0, S\}$, defined by (4.1), the rule

$$f(x) \to t(\lambda y.f(y), x),$$

is added to $R_C$. In all non-pathological cases the term rewrite system (TRS for short) $R_C$ is terminating and confluent. $R_C$ is best understood as a constructor TRS, where the constructors are 0 and $S$. Hence $R_C$ may be conceived as a functional program implementing the functions in $C$.

Term-rewriting characterisations have been studied e.g. in [19, 40, 123, 27]. The analysis of $R_C$ provides insight into the structure of $C$ or renders us with a delineation of a class of rewrite systems whose complexity (measured by the length of derivations) is guaranteed to belong to the class $C$. Term-rewriting characterisations turn the emphasis from the definition of a function $f$ to its computation. An essential property of term-rewriting characterisations $R_C$ is its feasibility: $R_C$ is called feasible, if for each $n$-ary function $f \in C$, there exists a function symbol $g$ in the signature of $R_C$ such that $g(m_1, \ldots, m_n)$ computes the value of $f(m_1, \ldots, m_n)$ and the derivation length of this computation is bounded by a function from $C$.

We study term-rewriting characterisations of the complexity class $\text{FP}$. In particular, our starting point is a clever characterisation $R'_B$ of $\text{FP}$ introduced by Beckmann and Weiermann. In [19] the feasibility of $R'_B$ is established and conclusively shown that any reduction strategy for $R'_B$ yields an algorithm for $f \in \text{FP}$ that runs in polytime. We provide a slight generalisation of the fact that $R'_B$ is feasible. Moreover, we flesh out the crucial ingredients of the TRS $R'_B$ by defining a path order for $\text{FP}$, denoted as $\prec_{\text{pop}}$. We show that for a finite TRS $R$, compatible with $\prec_{\text{pop}}$, the derivation length function $D_l^f$ is bounded by a polynomial in the length of the inputs for any defined function symbol $f$. Furthermore $\prec_{\text{pop}}$ is complete in the sense that for any function $f \in \text{FP}$, there exists a TRS $R$ computing $f$ such that termination of $R$ can be shown by $\prec_{\text{pop}}$.

### 4.2 A Rewrite System for $\text{FP}$

In the following we need some notions from term rewriting and assume (at least nodding) acquaintance with term rewriting. (For background information, please see [15].) Let $\mathcal{V}$ denote a countably infinite set of variables and $\Sigma$ a signature. The set of terms over $\Sigma$ and $\mathcal{V}$ is denoted as $T(\Sigma, \mathcal{V})$, while the set of ground terms is written as $T(\Sigma)$. The rewrite relation induced by a rewrite system $R$ is denoted as $\to_R$, and its transitive closure by $\to^*_R$. We write $\tau(t)$ to denote the size of a term $t$, i.e. the number of symbols in $t$.

**Conventions:** Terms are denoted by $r, s, t$, possibly extended by subscripts. We write $t$, to denote sequences of terms $t_1, \ldots, t_k \in T(\Sigma, \mathcal{V})$ and $g$ to denote sequences of function symbols $g_1, \ldots, g_k$, respectively. The letters $i, j, k, l, m, n$, possible extended by subscripts
will always refer to natural numbers. The set of natural numbers is denoted as usual by \( \mathbb{N} \).

We consider the class \( \text{FP} \) of \textit{polynomial computable functions}, i.e. those functions computable by a deterministic Turing machine \( M \), such that \( M \) runs in time \( \leq p(n) \) for all inputs of length \( n \), where \( p \) denotes a polynomial. We consider equivalent formulations of the class of polynomial computable functions in terms of recursion schemes.

Recursion schemes such as \textit{bounded recursion} due to Cobham [41] generate exactly the functions computable in polytime. In contrast to this, Bellantoni-Cook [21] introduce certain \textit{unbounded} recursion schemes that distinguish between arguments as to their position in a function. This separation of variables gives rise to the following definition of the \textit{predicative recursive functions} \( \mathcal{B} \); for further details see [21]. We fix a suitable signature of \textit{predicative recursive function symbols} \( \mathcal{B} \).

**Definition 4.2.1.** For \( k, l \in \mathbb{N} \) we define \( B^{k,l} \) inductively.

- \( S^{0,1}_i \in B^{0,1} \), where \( i \in [0, 1] \).
- \( O^{k,l} \in B^{k,l} \).
- \( U^{k,l}_r \in B^{k,l} \), for all \( r \in [1, k + l] \).
- \( P^{0,1} \in B^{0,1} \).
- \( C^{0,3} \in B^{0,3} \).
- If \( f \in B^{k',l'} \), \( g_1, \ldots, g_{k'} \in B^{k,0} \), and \( h_1, \ldots, h_{l'} \in B^{k,l} \), then \( \text{SUB}_{k',l'}^{i} \left[ f, g, h \right] \in B^{k,l} \).
- If \( g \in B^{k,l} \), \( h_0, h_1 \in B^{k+1,l+1} \), then \( \text{PREC}_{k+1,l+1}^{i} \left[ g, h_1, h_2 \right] \in B^{k+1,l} \).

Set \( B := \bigcup_{k,l \in \mathbb{N}} B^{k,l} \).

To simplify notation we usually drop the superscripts, when denoting predicative recursive function symbols. Occasionally, we even write \( \text{SUB} \), \( \text{PREC} \), instead of \( \text{SUB}^{i} \left[ f, g \right] \), \( \text{PREC}^{i} \left[ g, h \right] \). No confusion will arise from this.

The binary successor function \( m \mapsto 2m + i \), \( i \in \{0, 1\} \) is denoted as \( S_i \). Every natural number can be build up from 0 with repeated applications of \( S_i \). The binary length of a number \( m \) is defined as follows: \( |0| := 0 \) and \( |S_i(m)| := |m| + 1 \).

We write \( \mathbb{N}^{k,l} \) for \( \mathbb{N}^k \times \mathbb{N}^l \) and for \( f : \mathbb{N}^{k,l} \to \mathbb{N} \), write \( f(m_1, \ldots, m_k; n_1, \ldots, n_l) \) instead of \( f(\langle m_1, \ldots, m_k \rangle, \langle n_1, \ldots, n_l \rangle) \). The arguments occurring to the left of the semi-colon are called \textit{normal}, while the arguments to the right are called \textit{safe}. We define the following functions: \( S_i^{\Delta} \), \( i \in \{0, 1\} \) denotes the function \( \langle : m \rangle \mapsto 2m + i \). \( O^{k,l} \) denotes the function \( \langle : m; n \rangle \mapsto 0 \). \( U^{k,l}_r \) denotes the function \( \langle m_1, \ldots, m_k; m_{k+1}, \ldots, m_{k+l} \rangle \mapsto m_r \). \( P^{0,1} \) denotes the unique number-theoretic function satisfying the following equations: \( f(\langle 0 \rangle) = 0 \), \( f(\langle S_i(m) \rangle) = m \). \( C^{0,3} \) denotes the unique function satisfying: \( f(\langle 0, m_0, m_1 \rangle) = \).

43
If \( m_0, f(S_i(m), m_0, m_1) = m_i \). If \( f : \mathbb{N}^{k', l'} \rightarrow \mathbb{N}, g_i : \mathbb{N}^{k, 0} \rightarrow \mathbb{N} \) for \( i \in [1, k'] \), \( h_j : \mathbb{N}^{k, l} \rightarrow \mathbb{N} \) for \( j \in [1, l'] \), then \( \mathbb{SU}B_{k', l'}^{k, l}[f, g, h] \) denotes the function

\[
\langle m; n \rangle \mapsto f(g_1(m), \ldots, g_{k'}(m); h_1(m; n), \ldots, h_{l'}(m; n))
\]

If \( g : \mathbb{N}^{k, l} \rightarrow \mathbb{N}, h_i : \mathbb{N}^{k+1, l+1} \rightarrow \mathbb{N} \) for \( i \in [0, 1] \) then \( \mathbb{PRE}C^{k+1, l+1}[g, h_1, h_2] \) denotes the number-theoretic function \( f \) satisfying: \( f(0, m; n) = g(m; n) \) and \( f(S_i(m), m; n) = h_i(m, m; n, f(m, m; n)) \).

**Definition 4.2.2.** For \( k, l \in \mathbb{N} \) we define \( B^{k, l} \) inductively.

- \( S_i^{0, 1} \in B^{0, 1} \), where \( i \in [0, 1] \).
- \( C^{k, l} \in B^{k, l} \).
- \( U^{k, l} \in B^{k, l} \), for all \( r \in [1, k + l] \).
- \( P^{0, 1} \in B^{0, 1} \).
- \( C^{0, 3} \in B^{0, 3} \).
- If \( f \in B^{k', l'}, g_1, \ldots, g_{k'} \in B^{k, 0} \), and \( h_1, \ldots, h_{l'} \in B^{k, l} \), then \( \mathbb{SU}B_{k', l'}^{k, l}[f, g, h] \in B^{k, l} \).
- If \( g \in B^{k, l}, h_0, h_1 \in B^{k+1, l+1} \), then \( \mathbb{PRE}C^{k+1, l+1}[g, h_1, h_2] \in B^{k+1, l} \).

The set of **predicative recursive functions** is defined as \( B = \bigcup_{k, l} B^{k, l} \).

It follows from the definitions that for each \( f \in B \), there exists a unique predicative recursive function \( f^B \); the latter is called the interpretation of \( f \) in \( B \). For every number \( m \) we define its **numeral** \( \overline{m} \in T(B, V) \) as follows: \( \overline{0} := 0, \overline{S_i(m)} := S_i(\overline{m}) \) for \( i \in [0, 1] \). We write \( \overline{m} \) to denote a sequence of numerals \( \overline{m}_1, \ldots, \overline{m}_k \). Now the polytime computable functions \( \text{FP} \) can be defined as follows, see [21]:

\[
\text{FP} = \bigcup_k B^{k, 0}
\]

In [19] a clever **feasible** term-rewriting characterisation \( R'_B \) of the predicative recursive functions \( B \) is given. By Bellantoni’s result this yields a feasible term-rewriting characterisation of the class of polytime computable functions \( \text{FP} \). The (infinite) TRS is given in Table 1.

The TRS \( R'_B \) is terminating and confluent. Termination follows by the multiset path order. Confluence is a consequence of the fact that \( R'_B \) is orthogonal. Note the restriction in the rewrite rules for **safe composition** and **predicative recursion**. These rules only apply if all **safe** arguments are numerals, i.e. in normal-form. This peculiar restriction is necessary as the canonical term-rewriting characterisation \( R_B \) of \( B \), admits exponential lower-bounds, hence \( R_B \) is **non-feasible**, compare [19].

Let \( R \) denote a TRS. A **derivation** is a sequence of terms \( t_i, i \in \mathbb{N} \), such that for all \( i \), \( t_i \rightarrow_R t_{i+1} \). The \( (i + 1)\text{th} \) element of a sequence \( a \) is denoted as \( (a)_i \). We write \( \sim \) for
Table 4.1: A Feasible Term-Rewriting Characterisation of the Predicative-\(O^{k,l}(x;a)\) Recursive Functions

| \(U^{k,l}(x_1,\ldots,x_k; x_{k+1},\ldots,x_{k+l})\rightarrow x_r\) | [zero] |
| \(f^{0,1}(;0)\rightarrow 0\) | [projection] |
| \(f^{0,1}(;S_i(;a))\rightarrow a\) | [predecessor] |
| \(C^{0,3}(;0,a_0,a_1)\rightarrow a_0\) | [conditional] |
| \(C^{0,3}(;S_i(;a),a_1,a_0)\rightarrow a_{2-i}\) | |
| \(\text{SUB}^{k,l}[f,g,h](x;n)\rightarrow f(g(x);h(x;n))\) | [safe composition] |
| \(\text{PREC}^{k+1,l}[g,h_1,h_2](0,x;n)\rightarrow g(x;n)\) | [predicative recursion] |
| \(\text{PREC}^{k+1,l}[g,h_1,h_2](S_i(;b),x;n)\rightarrow h_i(b,x;n,\text{PREC}^{k+1,l}[g,h_1,h_2](b,x;n))\) | on notation |

We use the following notation: \(i \in [0,1]\) and \(r \in [1,k+l]\).

the concatenation of sequences and define the length \(|a|\) of a sequence \(a\) as usually. We define a partial order \(\subseteq\) on pairs of sequences. \(a \subseteq b\), if \(b\) is an extension of \(a\), i.e. \(|a| \leq |b|\) and for all \(i < |a|\) we have \((a)_i = (b)_i\). A derivation \(d\) with \((d)_0 = t\) is called derivation starting with \(t\). The derivation tree \(T_R(t)\) of \(t\) is defined as the structure \((T(t), \subseteq)\), where \(T(t) := \{d|d\) is a derivation starting with \(t\}\). The root of \(T_R(t)\) is denoted by \(t\) (instead of \((t)\)).

We measure the complexity or derivation length of the computation of \(f(\overline{m})\) by the height of \(T_R(f(\overline{m}))\), i.e., we define the derivation length function \(D_R^f: \Sigma \rightarrow \mathbb{N}:

\[D_R^f(\overline{m}) := \max\{n \mid \exists t_0,\ldots,t_n \in T(\Sigma) (t_n \leftarrow_R \ldots \leftarrow_R t_0 = f(\overline{m}))\}.\]

Based on these definitions we make the notion of feasible term-rewriting characterisation precise. A term-rewriting characterisation \(R_C\) of a function class \(C\) is called feasible, if for each \(n\)-ary function \(f \in C\), there exists a function symbol \(g\) in the signature of \(R_C\) such that \(g(\overline{m}_1,\ldots,\overline{m}_n)\) computes the value of \(f(m_1,\ldots,m_n)\) and \(D_R^f\) is bounded by a function from \(C\). For the rewrite system \(R_B^f\) we have the following proposition.

**Proposition 4.2.1.** For every \(f \in \mathcal{B}\), \(D_R^f\) is bounded by a monotone polynomial in the length of the normal inputs. Specifically for each \(f\) we can find a number \(\ell(f)\) so that \(D_R^f(\overline{m};\overline{n}) \leq (2 + |\overline{m}|)^{\ell(f)}\), where \(|\overline{m}|\) denotes the sum of the length normal inputs \(m_i\).

**Proof.** See \[7\] for a proof, essentially we employ the observation that the derivation trees \(T_{R_B^f}(f(\overline{m};\overline{n}))\) are isomorphic no matter how the safe input numerals \(\overline{n}\) vary, to drop the dependency on the length of the normal inputs. \(\Box\)

45
4.3 A Path Ordering for FP

To extend the above results and to facilitate the study of the polytime computable functions in a term-rewriting framework, we introduce in this section a new path order for FP, which is a miniaturisation of the recursive path order, cf. \cite{15}, see also \cite{30}.

In the definition we make use of an auxiliary varyadic function symbol ‘list’ of arbitrary, but finite arity, to denote sequences $s_0, \ldots, s_n$ of terms. Instead of list($s_0, \ldots, s_n$) we write $(s_0, \ldots, s_n)$. We write $a \sim b$ for sequences $a = (s_0, \ldots, s_n)$, $b = (s_{n+1}, \ldots, s_{n+m})$ to denote the concatenation $(s_0, \ldots, s_{n+m})$ of $a$ and $b$.

Let $\Sigma$ be a signature. We write $T^\ast(\Sigma, \mathcal{V})$ to denote the set of all finite sequences of terms in $T(\Sigma, \mathcal{V})$. To ensure that $T(\Sigma, \mathcal{V}) \subseteq T^\ast(\Sigma, \mathcal{V})$, any term is identified with the sequence list($t$) = ($t$). We denote sequences by $a, b, c$, both possible extended with subscripts. Sometimes we write $fa$ as abbreviations of $f(t_0, \ldots, t_n)$, if $a = (t_0, \ldots, t_n)$.

We suppose a partial well-founded relation on $S$, the precedence, denoted as $\prec$. We write $f \sim g$ if $(f \preceq g) \land (g \preceq f)$ and we write $f > g$ and $g < f$ interchangeably. Further, we suppose that the signature $\Sigma$ contains two unary symbols $S_0, S_1$ of lowest rank in the precedence, i.e. $\Sigma = \{S_0, S_1\} \cup \Sigma'$ and $S_0 \prec S_1$ and for all $f \in \Sigma'$, $S_0, S_1 < f$. Moreover, we define $0 := ()$. For every number $m$ we define its numeral $\overline{m} \in T(\Sigma, \mathcal{V})$ as follows: $\overline{0} := ()$; $\overline{S_i(m)} := S_i(\overline{m})$ for $i \in [0, 1]$.

The definition of the path order for FP (POP for short) $\prec_{\text{POP}}$ (induced by $\prec$) is based on an auxiliary order $\sqsubset$. The separation in two orders is necessary to break the strength of the recursive path order that induces primitive recursive derivation length, cf. \cite{77}.

**Definition 4.3.1.** Inductive definition of $\sqsubseteq$ induced by $\prec$.

(i) $\exists j \in [1, n] \ (s \sqsubseteq t_j) \implies s \sqsubseteq f(t_1, \ldots, t_n)$,

(ii) $t = f(t_1, \ldots, t_n) \land s = g(s_1, \ldots, s_m)$ with $g < f \land \forall i \in [1, m] (s_i \sqsubset t)$

$\implies s \sqsubseteq t$.

**Definition 4.3.2.** Inductive definition of $\prec_{\text{POP}}$ induced by $\prec$; $\prec_{\text{POP}}$ is based on $\sqsubset$.

(i) $s \sqsubset t \implies s \prec_{\text{POP}} t$,

(ii) $\exists j \in [1, n] \ (s \preceq_{\text{POP}} t_j) \implies s \prec_{\text{POP}} f(t_1, \ldots, t_n) \land s \prec_{\text{POP}} (t_1, \ldots, t_n)$,

(iii) $t = f(t_1, \ldots, t_n) \land (m = 0 \lor (\exists i_0 \ (\forall i \neq i_0 (s_i \sqsubset t) \land s_i \prec_{\text{POP}} t))$

$\implies (s_1, \ldots, s_m) \prec_{\text{POP}} t$,

(iv) $t = f(t_0, \ldots, t_n) \land s = g(s_0, \ldots, s_m)$ with $f \sim g \land (s_0, \ldots, s_m) \prec_{\text{POP}} (t_0, \ldots, t_n)$

$\implies s \prec_{\text{POP}} t$,

(v) $a \approx a_0 \cdot \ldots \cdot a_n \land \forall i \leq n \ (a_i \preceq_{\text{POP}} b_i) \land \exists i \leq n (a_i \prec_{\text{POP}} b_i)$

$\implies a \prec_{\text{POP}} (b_0, \ldots, b_n)$ if $n \geq 1$,

$a \approx a_0 \cdot \ldots \cdot a_n$ denotes the fact that the sequence $a$ of terms is obtained from the concatenated $a_0 \cdot \ldots \cdot a_n$ by permutation.

46
Note that due to rule \(\text{iii}()\) \(\prec_{\text{pop}}\) a for any sequence \(a \in T^*(\Sigma, V)\). Further, we write \(s \succ_{\text{pop}} t\) for \(t \prec_{\text{pop}} s\). It is not difficult to argue that \(\prec_{\text{pop}}\) is a reduction order. A number of relations are missing; we mention only the following:

\[-t = f(t_1, \ldots, t_n) \& s = g(s_1, \ldots, s_m) \text{ with } g < f \& \forall i \in [1, m](s_i \prec_{\text{pop}} t) \implies s \prec_{\text{pop}} t.\]

We indicate the reasons for the omission of this clause.

**Example 4.3.1.** Consider the following TRS, where \(\Sigma\) contains additionally the symbols \(a, g, h, f\) with precedence \(a, h < f, g < h\).

\[
f(0) \to a \quad f(S_i(x)) \to h(f(x)) \quad h(x) \to g(x, x).
\]

It is easy to see that \(\prec_{\text{pop}}\) cannot handle the TRS in the example, but would if rule above is included. However, note that the TRS admits an exponential lower-bound on the derivation length function.

We introduce suitable approximations \(\prec_k\) of \(\prec_{\text{pop}}\).

**Definition 4.3.3.** Inductive definition of \(\sqsubseteq^l_k\) induced by \(<\); we write \(\sqsubseteq_k\) to abbreviate \(\sqsubseteq^l_k\).

\[(i) \exists j \in [1,n](s \sqsubseteq_k^l t_j) \implies s \sqsubseteq_k f(t_0, \ldots, t_n),\]

\[(ii) t = f(t_0, \ldots, t_n) \& s = g(s_0, \ldots, s_m) \text{ with } g < f \& m < k \& \forall i (s_i \sqsubseteq_k l t) \implies s \sqsubseteq_k^{l+1} t.\]

**Definition 4.3.4.** Inductive definition of \(\prec_k\) induced by \(<\); \(\prec_k\) is based on \(\sqsubseteq_k\).

\[(i) s \sqsubseteq_k t \implies s \prec_k t,\]

\[(ii) \exists j \in [1,n](s \preceq_k t_j) \implies s \prec_k f(t_1, \ldots, t_n),\]

\[(iii) t = f(t_1, \ldots, t_n) \& \exists i_0 \in [1, m] \forall i \neq i_0 (s_i \sqsubseteq_k l t) \& s_{i_0} \prec_k t \implies s \prec_k t,\]

\[(iv) t = f(t_{i_0}, \ldots, t_n) \& s = g(s_{i_0}, \ldots, s_m) \prec_k (t_{i_0}, \ldots, t_n) \& m < \max\{k, n\} \implies s \prec_k t,\]

\[(v) a \approx a_0 \cdots \approx a_n \& \forall i \leq n (a_i \preceq_k b_i) \& \exists i \leq n (a_i \prec_k b_i) \implies a \prec_k (b_0, \ldots, b_n) \text{ if } n \geq 1.\]

In the following we prove that if for a finite rewrite system \(R, R \subseteq \prec_{\text{pop}},\) then it even holds that \(\rightarrow_R \subseteq \prec_k\), where \(k\) depends on \(R\) only.

**Lemma 4.3.1.** If \(s \prec_k t\) and \(k < l\), then \(s \prec_l t\).

We introduce the auxiliary measure \(|.|: T^*(\Sigma, V) \to \mathbb{N}: (i) |x| := 1, x \in V, (ii) |(s_1, \ldots, s_n)| := \max\{n, |s_1|, \ldots, |s_n|\}, (iii) |fa| := |a| + 1.\)

47
Lemma 4.3.2. If \( s \prec_{\text{pop}} t \), then for any substitution \( \sigma \), \( s\sigma \prec |s| t\sigma \).

Lemma 4.3.3. If \( f(t_1, \ldots, v, \ldots, t_n) = f(t_1, \ldots, u, \ldots, t_n) \) with \( u \prec_k v \), where \( k \geq \max\{\ar(f) : f \in \Sigma\} \), then \( s \prec_k t \).

Recall that \( \prec_{\text{pop}} \) is a reduction order. Hence the assumption \( R \subseteq \prec_{\text{pop}} \) implies \( t \rightarrow_R s \).

Lemma 4.3.4. Let \( k = \max\{\max\{\tau(r) | (l \rightarrow r) \in R\}, \max\{\ar(f) | f \in S\}\} \). Then \( t \rightarrow_R s \) implies \( s \prec_k t \).

We set
\[
G_k (\sigma) := \max\{n \in \mathbb{N} | \exists (a_0, \ldots, a_n) (a_n \prec_k \cdots \prec_k a_0 = a)\},
\]
\[
F_{k,p} (n) := \max\{G_k (fa) : \rk(f) = p \& G_k (a) \leq n\},
\]
where \( \rk(f) : \Sigma \rightarrow \mathbb{N} \) is defined inductively: \( \rk(f) := \max\{\rk(g) + 1 : g \in \Sigma \cup g < f\} \).

We collect some properties of the function \( G_k \) in the next lemma.

Lemma 4.3.5. (i) \( G_k ((s_0, \ldots, s_n)) = \sum_{i=0}^{n} G_k (a_i) \).

(ii) \( G_k (m) = |m| \) for any natural number \( m \).

Lemma 4.3.6. Inductively we define \( d_{k,0} := 2 \) and \( d_{k,p-1} := (d_{k,p})^k + 1 \). Then there exists a constant \( c \) (depending only on \( k \) and \( p \)) such that \( F_{k,p} (n) \leq c \cdot n^{d_{k,p}} + c \).

Proof. The lemma is proven by main induction on \( p \) and side induction on \( \sigma \).

Set \( a := (t_0, \ldots, t_n) \) and let \( w \prec_k f(t_0, \ldots, t_n) =: t, \rk(f) = p \) and \( w \) maximal. By assumption \( G_k (a) \leq n \). We prove
\[
G_k (w) < cn^{d_{k,p}} \quad \text{for almost all } n,
\]
by case-distinction on the definition of \( \prec_k \). Without loss of generality, we only consider the case \( w = (r_0, \ldots, r_m) \).

Case. \( p = 0 \) and \( \forall i \leq m (r_i \preceq_k t) \). By definition of \( \prec_{\text{pop}} \) we have \( \forall i \leq m \exists j \leq n (r_i \preceq_k t_j) \). Then \( G_k (w) \leq G_k (a) = n \). Hence
\[
G_k (w) \leq kn < cn^2,
\]
where we set \( c := k \).

Case. \( p = 0, \forall i \neq i_0 (r_i \preceq_k t) \), and \( r_{i_0} \prec_k t \). By definition of \( \prec_{\text{pop}} \) we have \( \forall i \leq m \exists j \leq n (r_i \preceq_k t_j) \) and \( r_{i_0} = f(s_0, \ldots, s_t), \rk(f) = 0, \) with \( (s_0, \ldots, s_t) \prec_k a \). Hence by induction hypothesis on \( a \), there exists a constant \( c \), such that \( G_k (r_{i_0}) \leq c(n - 1)^2 \) a.e. Employing Lemma 4.3.5, we obtain:
\[
G_k (w) = G_k ((r_0, \ldots, r_m)) = \sum_{i=0}^{m} G_k (r_i) \leq c(n - 1)^2 + (k - 1)n < cn^2,
\]

48
4.3 A Path Ordering for FP

as we can assume $c > k$.

CASE. $p > 0$ and $\forall i \leq m \; (r_i \sqsubseteq_k t)$. Let $i$ be arbitrary. We can assume $r_i = g(s_0, \ldots, s_t)$, $g \prec f$, and $\forall i \leq l \; (s_i \sqsubseteq_{k-1} t)$. Otherwise, if $r_i = g(s_0, \ldots, s_t)$ with $g \prec f$ s.t. there $\exists j \leq n \; (r_i \sqsubseteq t_j)$ we proceed as in the first case. By induction hypothesis there exists $c$ and $d = d_{k,p}$ s.t. $F_k(p(n)) \leq cn^d$ a.e.

We show the existence of a constant $c'$ s.t. $F_k,p+1(n) \leq c'n^d$, where $d' = d_{k,p+1}$. We define $f(a) := ca^d$ and $g^{(0)}(a) := a$, $g^{(l+1)}(a) = f(g^{(l)}(a) \cdot k)$; we obtain:

$$Dl(f) = \max \{l \mid \exists \; t_0, \ldots, t_n \in T(\Sigma) \; (t_n \leftarrow_R \ldots \leftarrow_R t_0 = f(\langle m \rangle))\}$$

We have established the following theorem.

**Theorem 4.3.1.** If for a finite TRS $R$ defined over $T(\Sigma, \Psi)$, $R \subseteq \prec_{pop}$, then for each $f \in \Sigma$, $Dl(f)_R$ is bounded by a monotone polynomial in the sum of the binary length of the inputs.

**Proof.** Let $R$ be a finite TRS defined over $T(\Sigma, \Psi)$, such that for every rule $(l \rightarrow r) \in R$, $r \prec_{pop} l$ holds. This implies that for any two terms $t, s \rightarrow_R s$ implies $s \prec_{pop} t$. Hence by Lemma 4.3.4 there exists $k \in \mathbb{N}$, s.t. $\rightarrow_l \subseteq_k$. Suppose $f$ is an $n$-ary function symbol and set $t := f(\langle m_1, \ldots, m_n \rangle)$. By definition it follows that

$$Dl(f)_{R}(\langle m_1, \ldots, m_n \rangle) \leq G_k(f(\langle m_1, \ldots, m_n \rangle)).$$

By Lemma 4.3.6 there exists a polynomial $p$, depending only on $k$ and the rank of $f$, s.t.

$$G_k(f(\langle m_1, \ldots, m_n \rangle)) \leq p(G_k((\langle m_1, \ldots, m_n \rangle))).$$

Employing with Lemma 4.3.5 we obtain $Dl(f)_{R}(\langle m_1, \ldots, m_n \rangle) \leq p(\sum_{i=1}^{n} |m_i|)$. 

\square
4.4 Predicative Recursion and POP

In the previous section we have shown that if for a finite TRS $R$, defined over $T^+(\Sigma, V)$, $R \subseteq \prec_{\text{pop}}$, then the derivation length function $Dl_R^t$ is bounded by a monotone polynomial in the binary length of the inputs. As an application of Theorem 4.3.1 we prove in this section that $Dl_R^t$ is bounded by a monotone polynomial in the binary length of the normal inputs. I.e. we give an alternative proof of Prop. 4.2.1. As $R'_B$ exactly characterises the functions in $\text{FP}$, this yields that $\prec_{\text{pop}}$—via the mapping $S$ defined below—exactly characterises the class of polytime computable functions $\text{FP}$.

It suffices to define a mapping $S$: $T(B) \to T^+(\Sigma)$, such that $S$ is a monotone interpretation such that $S(l\sigma) \succ_{\text{pop}} S(r\sigma)$ holds for all $(l \to r) \in R'_B$. We suppose the signature $\Sigma$ is defined such that for any function symbol $f \in B^{k,l}$ there is a function symbol $f' \in \Sigma$ of arity $k$. Moreover, $\Sigma$ includes two constants $S_0, S_1$ and a varyadic function symbol $\star$ of lowest rank. We need a few auxiliary notions: $sn(\overline{\pi}) := n$ for numerals $\pi$; $sn(f(t; s)) = \sum_j(sn(s_j))$, otherwise. For every number $m$ we define its representation $\overline{m} \in T(\Sigma, V)$ as follows: $\overline{0} := \star; \overline{s_i(m)} := \star \overline{s_i} * \overline{m}$ for $i \in [0, 1]$, where $\star(s_0, \ldots, s_i) \star \star(s_{i+1}, \ldots, s_n) := \star(s_0, \ldots, s_n)$. We define $S$: $T(B) \to T^+(\Sigma)$ by mutual induction together with the interpretation $N$: $T(B) \to T^+(\Sigma)$.

Definition 4.4.1.

- $S(\overline{\pi}) := ()$ and $S(S_i(t); t) := (S_i) \circ S(t)$ for $t \neq \overline{\pi}$ (i.e. $t$ is not a numeral).
- For $f \neq S_i$, define $S(f(t; s)) := (f(N(t_0), \ldots, N(t_n)), S(s_0), \ldots, S(s_m))$.
- $N(t) := \star S(t) \star sn(t)$.

First we show that for $Q \in \{S, N\}$, $Q(l\sigma) \succ_{\text{pop}} Q(r\sigma)$. More precisely we show the following lemma.

Lemma 4.4.1. Let $(l \to r) \in R'_B$, $\sigma$ a ground substitution, such that $l\sigma, r\sigma \in T(B)$. Then there exists $k$, depending on the rule $(l \to r)$, such that $Q(r\sigma) \prec_k Q(l\sigma)$.

Proof. Let $(l \to r)$ and $\sigma$ as in the assumptions of the lemma. We sketch the proof by considering the rule:

$$\text{PREC}^{p+1,q}[g, h_1, h_2](S_i(t); t; n) \to h_i(t, t; n, \text{PREC}[g, h_1, h_2](t; n)).$$

We abbreviate $F := \text{PREC}^{p+1,q}[g, h_1, h_2]$ and set $k := 1 + \max\{3, p + 1, q + 1\}$. Let $lh(f)$, $f \in B$ be defined as follows: $lh(f) := 1$, for $f \in \{S_i, O, U, P\}$. $lh($SUB$[f, g, h]) := 1 + lh(f) + lh(g_1) + \cdots + lh(g_v) + lh(h_1) + \cdots + lh(h_v)$. $lh($PREC$[g, h_1, h_2]) := 1 + lh(g) + lh(h_1) + lh(h_2)$. Then we define the precedence $< \prec$ over $\Sigma$ compatible with $lh$, i.e. $f' < g'$ if $lh(f) < lh(g)$. For $Q = S$, we employ the following sequence of comparisons:

$$S(F(S_i(t); t; n))$$
$$= (F'(N(S_i(t); t), N(t_1), \ldots, N(t_p)), S(\overline{n_1}), \ldots, S(\overline{n_q}))$$
$$= F'(N(S_i(t); t), N(t_1), \ldots, N(t_p))$$
$$= F'(\star S_i \star N(t), N(t_1), \ldots, N(t_p)) .$$
By definition \( S(\tau_i) = () \) and for each \( t \in T(\Sigma, V) \), \( t = (t) \). Moreover it is a direct consequence of the definitions that \( N(S_i(;t)) = \bullet(S_i) * N(t) \). Further:

\[
\begin{align*}
F'(\bullet(S_i) * N(t), N(t_1), \ldots, N(t_p)) \\
\succ_k (h'_i(N(t), N(t_1), \ldots, N(t_p)), F'(N(t), N(t_1), \ldots, N(t_p)))
\end{align*}
\]

By Definition 4.3.4(iii) we obtain \( \bullet(S_i) * N(t) \succ_k N(t) \). This yields by rule 4.3.4(iv) and rule 4.3.3(ii), using \( k > p + 1 \):

\[
F'(\bullet(S_i) * N(t), N(t_1), \ldots, N(t_p)) \succ_k F'(N(t), N(t_1), \ldots, N(t_p)).
\]

Finally applying Definition 4.3.4(iii) together with rule 4.3.4(iii) and 4.3.3(ii) yields the inequality. In these rule applications we employ \( k > q + 1 \) and \( F' > h'_i \).

Finally, it is easy to see that \( N(F(S_i(;t), t; n)) \succ_k N(h_i(t; t; n, F(t; t; n)). \) We established the lemma for the rule \( F(S_i(;t), t; n) \rightarrow h_i(t; t; n, F(t; t; n)). \) The other rules follow similar.

Note that the definition of \( k \) in all cases depends on the arity-information encoded in the head function symbol on the left-hand side. Moreover at most 3 iterated applications of \( \sqcup_k \) are necessary.

The next lemma establish monotonicity for the interpretations \( S, N \).

**Lemma 4.4.2.** For \( k \in \mathbb{N} \) and for \( u, v \in T(\Sigma) \), \( Q(u) \prec_k Q(v) \) for \( Q \in \{S, N\} \). Suppose \( f \in B^{p,q} \) and \( t, \pi \in T(\Sigma) \). Then

- \( Q(f(t_1, \ldots, u, \ldots, t_p; \pi) \prec_k Q(f(t_1, \ldots, v, \ldots, t_p; \pi)) \) for \( Q \in \{S, N\} \), and
- \( Q(f(t; s_1, \ldots, u, \ldots, s_q) \prec_k Q(f(t; s_1, \ldots, v, \ldots, s_q)) \) for \( Q \in \{S, N\} \).

We define the derivation length function \( \text{Di}_{R_B}^f \) over the ground term-set \( T(\Sigma) \):

\[
\text{Di}_{R_B}^f (m; n) := \max\{n \mid \exists t_0, \ldots, t_n \in T(B) \left( t_n \leftarrow_{R'_B} \ldots \leftarrow_{R'_B} t_0 = f(m; n) \right) \}\}
\]

Recall the definition of the derivation tree \( T_{R_B}' \). Note that for each \( t \in T(B, V) \), \( T_{R_B}(t) \) is finite. This follows from the fact that \( R'_B \) is terminating and \( T_{R_B}(t) \) is finitely branching. The latter is shown by well-founded induction on \( \rightarrow_{R'_B} \). Let \( f \in B \) be a fixed predicative recursive function symbol. As the derivation tree \( T_{R_B}'(f(m; n)) \) is finite only finitely many function symbols occur in \( T_{R_B}'(f(m; n)) \). This allows to define a finite subset \( F \subset B \), such that all terms occurring in \( T_{R_B}'(f(m; n)) \) belong to \( T(F) \). We define

\[
k := 1 + \max\{3 \cup \{p, q + 1\}| f^{p,q} \in B \text{ occurs in } T_{R_B}'(f(m; n))\}.
\]

Let \( R' \) denote the restriction of \( R_B' \) to \( T(F) \). Then, we have \( \text{Di}_{R_B}^f (m; n) = \text{Di}_{R'}(m; n) \). From these observations together with Lemma 4.4.1 and 4.4.2 we conclude
Lemma 4.4.3. Let \( s, t \in T(F) \) such that \( t \rightarrow_R s \). Then \( S(s) \prec_k S(t) \).

In summary we obtain, by following the pattern of the proof of Thm. 4.3.1:

Theorem 4.4.1. For every \( f \in B \), \( \text{Di}^f_{\text{F}R} (\bar{m}_1, \ldots, \bar{m}_p; \bar{n}_1, \ldots, \bar{n}_q) \) is bounded by a monotone polynomial in the sum of the length of the normal inputs \( m_1, \ldots, m_p \).

4.5 Conclusion

The main contribution of this paper is the definition of a \( \text{path order for } \text{FP} \), denoted as \( \prec_{\text{pop}} \). This path order has the property that for a finite TRS \( R \) compatible with \( \prec_{\text{pop}} \), the \( \text{derivation length function } \text{Di}^f_R \) is bounded by a polynomial in the length of the inputs for any defined function symbol \( f \) in the signature of \( R \). Moreover \( \prec_{\text{pop}} \) is \( \text{complete} \) in the sense that for a function \( f \in \text{FP} \), there exists a TRS \( R \) computing \( f \) such that such that termination of \( R \) follows by \( \prec_{\text{pop}} \). Another feature of \( \prec_{\text{pop}} \) is, that its definition is devoid of the separation of normal and safe arguments, present in the definition of the predicative recursive functions and therefore in the definition of the term-rewriting characterisation \( \text{R}_B' \).

We briefly relate our findings to the notion of the \( \text{light multiset path order} \), denoted as \( \prec_{\text{lmpo}} \), introduced by Marion in [109]. It is possible to define a variant of \( \prec_{\text{pop}} \)—denoted as \( \prec_{\text{popV}} \)—such that Theorem 4.3.1 remains true for \( \prec_{\text{popV}} \) when suitably reformulated. While Definition 4.3.1 and 4.3.2 are based on an arbitrary signature, the definition of \( \prec_{\text{popV}} \) assumes that normal and safe arguments are separated as in Section 4.2. It is easy to see that \( \prec_{\text{popV}} \subset \prec_{\text{lmpo}} \) and this inclusion is strict as \( \prec_{\text{lmpo}} \) proves termination of the non-feasible rewrite system \( \text{R}_B \), while \( \prec_{\text{popV}} \) clearly does not. On the other hand let \( R \) be a functional rewrite program (i.e. a constructor TRS) computing a number-theoretic function \( f \). A termination proof of \( R \) via \( \prec_{\text{lmpo}} \) guarantees the existence of a polytime algorithm for \( f \). However, a termination proof of \( R \) via or the introduced path order \( \prec_{\text{popV}} \) (or \( \prec_{\text{pop}} \)) guarantees that \( R \) itself is already a polytime algorithm for \( f \). It seems clear to us that the latter property is of more practical value.
5

Derivational Complexity of Knuth-Bendix Orders revisited

Publication Details


Ranking

The Conference on Logic Programming and Automated Reasoning has been ranked A by CORE in 2007.

Abstract

We study the derivational complexity of rewrite systems $\mathcal{R}$ compatible with Knuth-Bendix orders (KBOs for short), if the signature of $\mathcal{R}$ is infinite. We show that the known bounds on the derivation height are preserved, if $\mathcal{R}$ fulfils some mild conditions. This allows us to obtain bounds on the derivational height of non simply terminating TRSs. Furthermore, we re-establish the 2-recursive upper-bound on the derivational complexity of finite rewrite systems $\mathcal{R}$ compatible with KBO.

5.1 Introduction

One of the main themes in rewriting is termination. Over the years powerful methods have been introduced to establish termination of a given term rewrite system (TRS) $\mathcal{R}$. Earlier research mainly concentrated on inventing suitable reduction orders—for example simplification orders, see Chapter 6, authored by Zantema in [137]—capable of proving termination directly. In recent years the emphasis shifted towards transformation techniques like the dependency pair method or semantic labelling, see [137]. The dependency pair method is easily automatable and lies at the heart of many successful termination provers like TTT [73] or AProVE [58]. Semantic labelling with infinitely
labels was conceived to be unsuitable for automation. Hence, only the variant with finitely many elements was incorporated (for example in AProVE [58] or TORPA [148]). Very recently this belief was proven wrong. TPA [93] implements semantic labelling with natural numbers, in combination with multiset path orders (MPOs) efficiently. As remarked in [96] a sensible extension of this implementation is the combination of semantic labelling with Knuth-Bendix orders (KBOs for short).

In order to assess the power and weaknesses of different termination techniques it is natural to look at the length of derivation sequences, induced by different techniques. This program has been suggested in [79]. The best known result is that for finite rewrite systems, MPO induces primitive recursive derivational complexity. This bound is essentially optimal, see [76, 77]. Similar optimal results have been obtained for lexicographic path orders (LPOs) and KBOs. Weiermann [145] showed that LPO induce multiply recursive derivational complexity. In [103] Lepper showed that for TRSs compatible with KBO, the derivational complexity is bounded by the Ackermann function.

These results not only assess different proof techniques for termination, but constitute an a priori complexity analysis for term rewrite systems (TRSs for short) provably terminating by MPO, LPO or KBO. The application of termination provers as basis for the termination analysis of logic or functional programs is currently a very hot topic. Applicability of an a priori complexity analysis for TRSs in this direction seems likely.

While the aforementioned program has spawned a number of impressive results, not much is known about the derivational complexity induced by the dependency pair method or semantic labelling (for fixed base orders, obviously). We indicate the situation with an example.

**Example 5.1.1.** Consider the TRS \((\mathcal{F}, \mathcal{R})\) [16] consisting of the following rewrite rules:

\[
\begin{align*}
  f(h(x)) & \rightarrow f(i(x)) & h(a) & \rightarrow b \\
  g(i(x)) & \rightarrow g(h(x)) & i(a) & \rightarrow b .
\end{align*}
\]

It is not difficult to see that termination of \(\mathcal{R}\) cannot be established directly with path orders or KBOs. On the other hand, termination is easily shown via the dependency pair method or via semantic labelling. For the sake of the argument we show termination via semantic labelling with KBOs.

We use natural numbers as semantics and as labels. As interpretation for the function symbols we use \(a = b = g = f = 1, i(n) = n, \text{ and } h(n) = n + 1.\) The resulting algebra \((\mathbb{N}, >)\) is a quasi-model for \(\mathcal{R}\). It suffices to label the symbol \(f\). We define the labelling function \(\ell_f : \mathbb{N} \rightarrow \mathbb{N}\) as \(\ell_f(n) = n.\) Replacing

\[
\begin{align*}
  f(h(x)) & \rightarrow f(i(x)) \\
  g(i(x)) & \rightarrow g(h(x)) \\
  h(a) & \rightarrow b
\end{align*}
\]

by the infinitely many rules

\[
\begin{align*}
  f_{n+1}(h(x)) & \rightarrow f_n(i(x))
\end{align*}
\]

we obtain the labelled TRS, \((\mathcal{F}_{lab}, \mathcal{R}_{lab})\). Further the TRS \((\mathcal{F}_{lab}, \mathcal{Dec})\) consists of all rules

\[
\begin{align*}
  f_{n+1}(x) & \rightarrow f_n(x)
\end{align*}
\]
Now we can show termination of $R' := R_{lab} \cup Dec$ by an instance $\succ_{kbo}$ of KBO. We set the weight for all occurring function symbols to 1. Further, the precedence is defined as

$$f_{n+1} \succ f_n \succ \cdots \succ f_0 \succ i \succ h \succ g \succ a \succ b.$$ 

It is easy to see that $R' \subseteq \succ_{kbo}$. Thus termination of $R$ is guaranteed.

As the rewrite system $R'$ is infinite we cannot directly apply the aforementioned result on the derivational complexity induced by Knuth-Bendix orders. A careful study of [103] reveals that the crucial problem is not that $R'$ is infinite, but that the signature $F_{lab}$ is infinite, as Lepper’s proof makes explicit use of the finiteness of the signature: To establish an upper-bound on the derivational complexity of a TRS $R$, compatible with KBO, an interpretation function $I$ is defined, where the cardinality of the underlying signature is hard-coded into $I$, cf. [103].

We study the situation by giving an alternative proof of Lepper’s result compare [103]. The outcome of this study is that the assumption of finiteness of the rewrite system can be weakened. By enforcing conditions that are still weak enough to treat interesting rewrite systems, we show that for (possibly infinite) TRSs $R$ over infinite signatures, compatible with KBO, the derivation height of $R$ can be bounded by the Ackermann function. Using an example that stems from [76] we show that this upper-bound is essentially optimal.

Specialised to Example 5.1.1, our results provide an upper bound on the derivation height function with respect to $R$: For every $t \in T(F)$ there exists a constant $c$ (depending only on $t$, $R'$, and $\succ_{kbo}$) such that the derivation height $d_R(t)$ with respect to $R$ is $\leq \text{Ack}(c^n, 0)$. As the constant $c$ can be made precise, the method is capable of automation.

This paper is organised as follows: In Section 5.2 and 5.3 some basic facts on rewriting, set theory and KBOs are recalled. In Section 5.4 we define an embedding from $\succ_{kbo}$ into $>_{\text{lex}}$, the lexicographic comparison of sequences of natural numbers. This embedding renders an alternative description of the derivation height of a term, based on the partial order $>_{\text{lex}}$. This description is discussed in Section 5.5 and linked to the Ackermann function in Section 5.6. The above mentioned central result is contained in Section 5.7. Moreover in Section 5.7 we apply our result to a non simply terminating TRS, whose derivational complexity cannot be primitive recursively bounded.

### 5.2 Preliminaries

We assume familiarity with term rewriting. For further details see [137]. Let $V$ denote a countably infinite set of variables and $F$ a signature. We assume that $F$ contains at least one constant. The set of terms over $F$ and $V$ is denoted as $T(F, V)$, while the set of ground terms is written as $T(F)$. The set of variables occurring in a term $t$ is denoted as $\text{Var}(t)$. The set of function symbols occurring in $t$ is denoted as $\text{FS}(t)$. The size of a term $t$, written as $\text{Size}(t)$, is the number of variables and function symbols in it. The number of occurrences of a symbol $a \in F \cup V$ in $t$ is denoted as $|t|_a$. A TRS $(F, R)$ over $T(F, V)$ is a set of rewrite rules. The smallest rewrite relation that contains
\( \mathcal{R} \) is denoted as \( \rightarrow_{\mathcal{R}} \). The transitive closure of \( \rightarrow_{\mathcal{R}} \) is denoted by \( \rightarrow_{\mathcal{R}}^* \), and its transitive and reflexive closure by \( \rightarrow_{\mathcal{R}}^\ast \). A TRS \((\mathcal{F}, \mathcal{R})\) is called terminating if there is no infinite rewrite sequence. As usual, we frequently drop the reference to the signature \( \mathcal{F} \).

A partial order \( > \) is an irreflexive and transitive relation. The converse of \( > \) is written as \(<\). A partial order \( > \) on a set \( A \) is well-founded if there exists no infinite descending sequence \( a_1 > a_2 > \cdots \) of elements of \( A \). A rewrite relation that is also a partial order is called rewrite order. A well-founded rewrite order is called reduction order. A TRS \( \mathcal{R} \) and a partial order \( > \) are compatible if \( \mathcal{R} \subseteq \succ > \). We also say that \( \mathcal{R} \) is compatible with \( > \) or vice versa. A TRS \( \mathcal{R} \) is terminating iff it is compatible with a reduction order \( \succ > \).

Let \((\mathcal{A}, >)\) denote a well-founded weakly monotone \( \mathcal{F} \)-algebra. \((\mathcal{A}, >)\) consists of a carrier \( \mathcal{A} \), interpretations \( f_\mathcal{A} \) for each function symbol in \( \mathcal{F} \), and a well-founded partial order \( > \) on \( \mathcal{A} \) such that every \( f_\mathcal{A} \) is weakly monotone in all arguments. We define a quasi-order \( \succeq_A \): \( s \succeq_A t \) if for all assignments \( \alpha: V \to A \) \([\alpha]_A(s) \geq [\alpha]_A(t)\). Here \( \geq \) denotes the reflexive closure of \( > \). The algebra \((\mathcal{A}, >)\) is a quasi-model of a TRS \( \mathcal{R} \), if \( \mathcal{R} \subseteq \succeq_A \).

A labelling \( \ell \) for \( \mathcal{A} \) consists of a set of labels \( L_f \) together with mappings \( \ell_f: A^n \to L_f \) for every \( f \in \mathcal{F} \), \( f \) n-ary. A labelling is called weakly monotone if all labelling functions \( \ell_f \) are weakly monotone in all arguments. The labelled signature \( \mathcal{F}_{\text{lab}} \) consists of \( n \)-ary functions symbols \( f_\mathcal{A} \) for every \( f \in \mathcal{F} \), \( a \in L_f \), together with all \( f \in \mathcal{F} \), such that \( L_f = \emptyset \). The TRS \( \mathcal{D}_{\text{dec}} \) consists of all rules

\[
\ell f a_{n+1}(x_1, \ldots, x_n) \rightarrow \ell f a(x_1, \ldots, x_n),
\]

for all \( f \in \mathcal{F} \). The \( x_i \) denote pairwise different variables. Our definition of \( \mathcal{D}_{\text{dec}} \) is motivated by a similar definition in [96]. Note that the rewrite relation \( \rightarrow_{\mathcal{D}_{\text{dec}}}^\ast \) is not changed by this modification of \( \mathcal{D}_{\text{dec}} \). For every assignment \( \alpha \), we inductively define a mapping \( \text{lab}_\alpha: \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}_{\text{lab}}, \mathcal{V}) \):

\[
\text{lab}_\alpha(t) := \begin{cases} 
  t & \text{if } t \in \mathcal{V}, \\
  \ell f (\text{lab}_\alpha(t_1), \ldots, \text{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \emptyset, \\
  f_\mathcal{A}(\text{lab}_\alpha(t_1), \ldots, \text{lab}_\alpha(t_n)) & \text{otherwise}.
\end{cases}
\]

The label \( a \) in the last case is defined as \( \ell f ([\alpha]_A(t_1), \ldots, [\alpha]_A(t_n)) \). The labelled TRS \( \mathcal{R}_{\text{lab}} \) over \( \mathcal{F}_{\text{lab}} \) is defined as

\[
\{ \text{lab}_\alpha(l) \to \text{lab}_\alpha(r) \mid l \to r \in \mathcal{R} \text{ and } \alpha \text{ an assignment} \}.
\]

Theorem 5.2.1 (Zantema [150]). Let \( \mathcal{R} \) be a TRS, \((\mathcal{A}, >)\) a well-founded weakly monotone quasi-model for \( \mathcal{R} \), and \( \ell \) a weakly monotone labelling for \((\mathcal{A}, >)\). Then \( \mathcal{R} \) is terminating iff \( \mathcal{R}_{\text{lab}} \cup \mathcal{D}_{\text{dec}} \) is terminating.

The proof of the theorem uses the following lemma.

Lemma 5.2.1. Let \( \mathcal{R} \) be a TRS, \((\mathcal{A}, >)\) a quasi-model of \( \mathcal{R} \), and \( \ell \) a weakly monotone labelling for \((\mathcal{A}, >)\). If \( s \rightarrow_{\mathcal{R}} \mathcal{R} t \), then \( \text{lab}_\alpha(s) \rightarrow_{\mathcal{D}_{\text{dec}}} \cdot \rightarrow_{\mathcal{R}_{\text{lab}}} \text{lab}_\alpha(t) \) for all assignments \( \alpha \).
We briefly review a few basic concepts from set-theory in particular ordinals, see \[85\]. We write \( > \) to denote the well-ordering of ordinals. Any ordinal \( \alpha \neq 0 \), smaller than \( \epsilon_0 \), can uniquely be represented by its Cantor Normal Form (CNF for short)

\[
\omega^{\alpha_1}n_1 + \cdots + \omega^{\alpha_k}n_k \quad \text{with} \quad \alpha_1 > \cdots > \alpha_k.
\]

To each well-founded partial order \( > \) on a set \( A \) we can associate a (set-theoretic) ordinal, its order type. First we associate an ordinal to each element \( a \) of \( A \) by setting \( \text{otype}_>(a) := \sup \{ \text{otype}_>(b) + 1 : b \in A \text{ and } b > a \} \). The order type of \( > \), denoted by \( \text{otype}(> \text{)}, \) is the supremum of \( \text{otype}_>(a) + 1 \) with \( a \in A \). For two partial orders \( > \) and \( >' \) on \( A \) and \( A' \), respectively, a mapping \( o : A \to A' \) embeds \( > \) into \( >' \) if for all \( p, q \in A \), \( p > q \) implies \( o(p) >' o(q) \). Such a mapping is an order-isomorphism if it is bijective and the partial orders \( > \) and \( >' \) are linear.

### 5.3 The Knuth Bendix Orders

A weight function for \( \mathcal{F} \) is a pair \((w, w_0)\) consisting of a function \( w : \mathcal{F} \to \mathbb{N} \) and a minimal weight \( w_0 \in \mathbb{N} \), \( w_0 > 0 \) such that \( w(c) \geq w_0 \) if \( c \) is a constant. A weight function \((w, w_0)\) is called admissible for a precedence \( > \) if \( f > g \) for all \( g \in \mathcal{F} \) different from \( f \), when \( f \) is unary with \( w(f) = 0 \). The function symbol \( f \) (if present) is called special. The weight of a term \( t \), denoted as \( w(t) \) is defined inductively. Assume \( t \) is a variable, then set \( w(t) := w_0 \), otherwise if \( t = g(t_1, \ldots, t_n) \), we define \( w(t) := w(g) + w(t_1) + \cdots + w(t_n) \).

The following definition of KBO is tailored to our purposes. It is taken from \[103\]. We write \( s = f^s s' \) if \( s = f^s(s') \) and the root symbol of \( s' \) is distinct from the special symbol \( f \). Let \( > \) be a precedence. The rank of a function symbol is defined as: \( \text{rk}(f) := \max \{ \text{rk}(g) + 1 | f > g \} \). (To assert well-definedness we stipulate \( \max(\emptyset) = 0 \).)

#### Definition 5.3.1

Let \((w, w_0)\) denote an admissible weight function for \( \mathcal{F} \) and let \( > \) denote a precedence on \( \mathcal{F} \). We write \( f \) for the special symbol. The Knuth Bendix order \( >_{\text{kbo}2} \) on \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \) is inductively defined as follows: \( s \succ_{\text{kbo}2} t \) if \( |s|_x \geq |t|_x \) for all \( x \in \mathcal{V} \) and

(i) \( w(s) > w(t) \), or

(ii) \( w(s) = w(t), \ s = f^s s', \ t = f^h t', \) where \( s' = g(s_1, \ldots, s_n), \ t' = h(t_1, \ldots, t_m) \), and one of the following cases holds.

a) \( a > b \), or

b) \( a = b \) and \( g > h \), or

c) \( a = b, \ g = h, \) and \( (s_1, \ldots, s_n) \succ_{\text{lex}} (t_1, \ldots, t_n) \).

Let \( \succ_{\text{kbo}} \) denote the KBO on terms in its usual definition, see \[137\]. The following lemma, taken from \[103\], states that both orders are interchangeable.

#### Lemma 5.3.1 (Lepper \[103\])

The orders \( \succ_{\text{kbo}} \) and \( \succ_{\text{kbo}2} \) coincide.

In the literature real-valued KBOs and other generalisations of KBOs are studied as well, cf. \[112\], \[45\]. However, as established in \[97\] any TRS shown to be terminating by a real-valued KBO can be shown to be terminating by a integer-valued KBO.
5.4 Exploiting the Order-Type of KBOs

We define \( \mathbb{N}^* \) to denote the set of finite sequences of natural numbers. Let \( p \in \mathbb{N}^* \), we write \( |p| \) for the \textit{length} of \( p \), i.e. the number of positions in the sequence \( p \). The \( i^\text{th} \) element of the sequence \( a \) is denoted as \( (p)_{i-1} \). We write \( p \bowtie q \) to denote the concatenation of the sequences \( p \) and \( q \). The next definition is standard but included here, for sake of completeness.

**Definition 5.4.1.** We define the lexicographic order on \( \mathbb{N}^* \). If \( p, q \in \mathbb{N}^* \), then \( p >_{\text{lex}} q \) if,

- \( |p| > |q| \), or
- \( |p| = |q| = n \) and there exists \( i \in [0, n-1] \), such that for all \( j \in [0, i-1] \) \((p)_j = (q)_j \) and \((p)_i > (q)_i \).

It is not difficult to see that \( \text{otype}(>_{\text{lex}}) = \omega^\omega \), moreover in [103] it is shown that \( \text{otype}(>_\text{kbo}) = \omega^\omega \). Hence \( \text{otype}(>_{\text{lex}}) = \text{otype}(>_\text{kbo}) \), a fact we exploit below. However, to make this work, we have to restrict our attention to signatures \( F \) with bounded arities. The maximal arity of \( F \) is denoted as \( \text{Ar}(F) \).

**Definition 5.4.2.** Let the signature \( F \) and a weight function \( (w, w_0) \) for \( F \) be fixed. We define an embedding \( \text{tw} : T(F, \mathcal{V}) \rightarrow \mathbb{N}^* \). Set \( b := \max\{\text{Ar}(F), 3\} + 1 \).

\[
\text{tw}(t) := \begin{cases} 
(w_0, a, 0) \bowtie 0^m & \text{if } t = f^a x, x \in \mathcal{V}, \\
(w(t), a, \text{rk}(g)) \bowtie \text{tw}(t_1) \bowtie \cdots \bowtie \text{tw}(t_n) \bowtie 0^m & \text{if } t = f^a g(t_1, \ldots, t_n). 
\end{cases}
\]

The number \( m \) is set suitably, so that \( |\text{tw}(t)| = b^{\text{tw}(t)+1} \).

The mapping \( \text{tw} \) flattens a term \( t \) by transforming it into a concatenation of triples. Each triple holds the weight of the considered subterm \( r \), the number of leading special symbols and the rank of the first non-special function symbol of \( r \). In this way all the information necessary to compare two terms via \(>_\text{kbo} \) is expressed as a very simple data structure: a list of natural numbers.

**Lemma 5.4.1.** \( \text{tw} \) embeds \(>_\text{kbo} \) into \(>_{\text{lex}}\): If \( s >_{\text{kbo}} t \), then \( \text{tw}(s) >_{\text{lex}} \text{tw}(t) \).

**Proof.** The proof follows the pattern of the proof of Lemma 9 in [103].

Firstly, we make sure that the mapping \( \text{tw} \) is \textit{well-defined}, i.e., we show that the length restriction can be met. We proceed by induction on \( t \); let \( t = f^a t' \). We consider two cases (i) \( t' \in \mathcal{V} \) or (ii) \( t' = g(t_1, \ldots, t_n) \). Suppose the former:

\[
|(w_0, a, 0)| = 3 \leq b^{\text{tw}(t)+1}.
\]

Now suppose case (ii): Let \( j = \text{rk}(g) \), we obtain

\[
|(w(t), a, j) \bowtie \text{tw}(t_1) \bowtie \cdots \bowtie \text{tw}(t_n)| = 3 + b^{\text{tw}(t_1)+1} + \cdots + b^{\text{tw}(t_n)+1} \\
\leq 3 + n \cdot b^{\text{tw}(t)} \leq b^{\text{tw}(t)+1}.
\]
Secondly, we show the following, slight generalisation of the lemma:

\[ s \succ_{\text{kbo}} t \land |\mathbf{tw}(s) \smallsetminus r| = |\mathbf{tw}(t) \smallsetminus r'| \implies \mathbf{tw}(s) \smallsetminus r >_{\text{lex}} \mathbf{tw}(t) \smallsetminus r'. \]  

(5.1)

To prove (5.1) we proceed by induction on \( s \succ_{\text{kbo}} t \). Set \( p = \mathbf{tw}(s) \smallsetminus r \), \( q = \mathbf{tw}(t) \smallsetminus r' \).

CASE \( w(s) > w(t) \): By definition of the mapping \( \mathbf{tw} \), we have: If \( w(s) > w(t) \), then \( (\mathbf{tw}(s))_0 > (\mathbf{tw}(t))_0 \). Thus \( p >_{\text{lex}} q \) follows.

CASE \( w(s) = w(t) \): We only consider the sub-case where \( s = f^a g(s_1, \ldots, s_n) \) and \( t = f^a g(t_1, \ldots, t_n) \) and there exists \( i \in [1, n] \) such that \( s_i = t_1, \ldots, s_{i-1} = t_{i-1} \), and \( s_i \succ_{\text{kbo}} t_i \). (The other cases are treated as in the case above.) The induction hypothesis expresses that if \( |\mathbf{tw}(s_i) \smallsetminus v| = |\mathbf{tw}(t_i) \smallsetminus v'| \), then \( \mathbf{tw}(s_i) \smallsetminus v >_{\text{lex}} \mathbf{tw}(t_i) \smallsetminus v' \). For \( j = \text{rk}(g) \), we obtain

\[
\begin{align*}
 p &= w(s) \smallsetminus \mathbf{tw}(s_1) \smallsetminus \cdots \smallsetminus \mathbf{tw}(s_{i-1}) \smallsetminus \mathbf{tw}(s_i) \smallsetminus \cdots \smallsetminus \mathbf{tw}(s_n) \smallsetminus r, \\
 q &= w(s) \smallsetminus \mathbf{tw}(s_1) \smallsetminus \cdots \smallsetminus \mathbf{tw}(s_{i-1}) \smallsetminus \mathbf{tw}(t_i) \smallsetminus \cdots \smallsetminus \mathbf{tw}(t_n) \smallsetminus r' . 
\end{align*}
\]

Due to \( |p| = |q| \), we conclude

\[ |\mathbf{tw}(s_1) \smallsetminus \cdots \smallsetminus \mathbf{tw}(s_n) \smallsetminus r| = |\mathbf{tw}(t_1) \smallsetminus \cdots \smallsetminus \mathbf{tw}(t_n) \smallsetminus r'| . \]

Hence induction hypothesis is applicable and we obtain

\[ \mathbf{tw}(s_1) \smallsetminus \cdots \smallsetminus \mathbf{tw}(s_n) \smallsetminus r >_{\text{lex}} \mathbf{tw}(t_1) \smallsetminus \cdots \smallsetminus \mathbf{tw}(t_n) \smallsetminus r' , \]

which yields \( p >_{\text{lex}} q \). This completes the proof of (5.1).

Finally, to establish the lemma, we assume \( s \succ_{\text{kbo}} t \). By definition either \( w(s) > w(t) \) or \( w(s) = w(t) \). In the latter case \( \mathbf{tw}(s) >_{\text{lex}} \mathbf{tw}(t) \) follows by (5.1). While in the former \( \mathbf{tw}(s) >_{\text{lex}} \mathbf{tw}(t) \) follows as \( w(s) > w(t) \) implies \( |\mathbf{tw}(s)| > |\mathbf{tw}(t)| \).

\[ \square \]

5.5 Derivation Height of Knuth-Bendix Orders

Let \( \mathcal{R} \) be a TRS and \( \succ_{\text{kbo}} \) a KBO such that \( \succ_{\text{kbo}} \) is compatible with \( \mathcal{R} \). The TRS \( \mathcal{R} \) and the KBO \( \succ_{\text{kbo}} \) are fixed for the remainder of the paper. We want to extract an upper-bound on the length of derivations in \( \mathcal{R} \). We recall the central definitions. Note that we can restrict the definition to the set ground terms. The derivation height function \( \text{dh}_\mathcal{R} \) (with respect to \( \mathcal{R} \) on \( \mathcal{T}(\mathcal{F}) \)) is defined as follows.

\[ \text{dh}_\mathcal{R}(t) := \max\{n \mid \exists (t_0, \ldots, t_n) \ t = t_0 \to_\mathcal{R} t_1 \to_\mathcal{R} \ldots \to_\mathcal{R} t_n\} . \]

We introduce a couple of measure functions for term and sequence complexities, respectively. The first measure \( \text{sp}: \mathcal{T}(\mathcal{F}, \mathcal{V}) \rightarrow \mathbb{N} \) bounds the maximal nesting of special symbols in the term:

\[ \text{sp}(t) := \begin{cases} 
 a & \text{if } t = f^a x, \ x \in \mathcal{V} , \\
 \max\{a \cup \{\text{sp}(t_j) \mid j \in [1, n]\}\} & \text{if } t = f^a g(t_1, \ldots, t_n) .
\end{cases} \]
The second and third measure $\text{rk}: \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathbb{N}$ and $\text{mrk}: \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathbb{N}$ collect information on the ranks of non special function symbols occurring:

$$
\text{rk}(t) := \begin{cases} 0 & \text{if } t = f^a x, \ x \in \mathcal{V}, \\ j & \text{if } t = f^a g(t_1, \ldots, t_n) \text{ and } \text{rk}(g) = j, \\
\end{cases}
$$

$$
\text{mrk}(t) := \begin{cases} 0 & \text{if } t = f^a x, \ x \in \mathcal{V}, \\ \max(\{j\} \cup \{\text{mrk}(t_i) \mid i \in [1, n]\}) & \text{if } t = f^a g(t_1, \ldots, t_n), \ \text{rk}(g) = j. \\
\end{cases}
$$

The fourth measure $\text{max}: \mathbb{N}^* \to \mathbb{N}$ considers sequences $p$ and bounds the maximal number occurring in $p$:

$$
\text{max}(p) := \max(\{(p)_i \mid i \in [0, |p| - 1]\}).
$$

It is immediate from the definitions that for any term $t$: $\text{sp}(t), \text{rk}(t), \text{mrk}(t) \leq \text{max}(\text{tw}(t))$. We write $r \subseteq t$ to denote the fact that $r$ is a subterm of $t$.

**Lemma 5.5.1.** If $r \subseteq t$, then $\text{max}(\text{tw}(t)) \geq \text{max}(\text{tw}(r))$.

We informally argue for the correctness of the lemma. Suppose $r$ is a subterm of $t$. Then clearly $w(r) \leq w(t)$. The maximal occurring nesting of special symbols in $r$ is smaller (or equal) than in $t$. And the maximal rank of a symbol in $r$ is smaller (or equal) than in $t$. The mapping $\text{tw}$ transforms $r$ to a sequence $p$ whose coefficients are less than $w(t)$, less than the maximal nesting of special symbols and less than the maximal rank of non-special function symbol in $r$. Hence $\text{max}(\text{tw}(t)) \geq \text{max}(\text{tw}(r))$ holds.

**Lemma 5.5.2.** If $p = \text{tw}(t)$ and $q = \text{tw}(f^a t)$, then $\text{max}(p) + a \geq \text{max}(q)$.

**Proof.** The proof of the lemma proceeds by a case distinction on $t$. \hfill \square

**Lemma 5.5.3.** We write $m \vdash n$ to denote $\max(\{m - n, 0\})$. Assume $s \succ_k \succ_{\text{b.o.}} t$ with $\text{sp}(t) \leq K$ and $(\text{mrk}(t) - \text{rk}(s)) \leq K$. Let $\sigma$ be a substitution and set $p = \text{tw}(s\sigma)$, $q = \text{tw}(t\sigma)$. Then $p \succ_{\text{lex}} q$ and $\text{max}(p) + K \geq \text{max}(q)$.

**Proof.** It suffices to show $\text{max}(p) + K \geq \text{max}(q)$ as $p \succ_{\text{lex}} q$ follows from Lemma 5.4.4. We proceed by induction on $t$; let $t = f^a t'$.

Case $t' \in \mathcal{V}$: Set $t' = x$. We consider two sub-cases: Either (i) $x\sigma = f^b y$, $y \in \mathcal{V}$ or (ii) $x\sigma = f^b g(u_1, \ldots, u_m)$. It suffices to consider sub-case (ii), as sub-case (i) is treated in a similar way. From $s \succ_k s\sigma$, we know that for all $y \in \mathcal{V}, |s|_y \geq |t|_y$, hence $x \in \text{Var}(s)$ and $x\sigma \subseteq s\sigma$. Let $l := \text{rk}(g)$; by Lemma 5.5.1 we conclude $\text{max}(\text{tw}(x\sigma)) \leq \text{max}(p)$. I.e. $b, l, \text{max}(\text{tw}(u_1)), \ldots, \text{max}(\text{tw}(u_m)) \leq \text{max}(p)$. We obtain

$$
\text{max}(q) = \max(\{w_0, a + b, l\} \cup \{\text{max}(\text{tw}(u_j)) \mid i \in [1, m]\}) \leq \max(\{w(s\sigma), \text{sp}(t) + \text{max}(p), \text{max}(p)\} \cup \{\text{max}(p)\}) \leq \max(\{w(s\sigma), \text{max}(p) + K\} \cup \{\text{max}(p)\}) = \text{max}(p) + K.
$$
With respect to the TRS \( R \), By definition of the rewrite relation there exists a context \( C \). Hence for all \( i \), \( sp(t_i) \leq K \) and \( mrk(t_i) \leq mrk(t) \). Thus induction hypothesis is applicable: For all \( i \), \( \max(tw(t_i)) \leq \max(p) + K \). By using the assumption \( (mrk(t) - rk(s)) \leq K \) we obtain:

\[
\max(q) = \max \{ \{ w(t_1), a, j \} \cup \{ \max(tw(t_i)) \mid i \in [1, n] \} \leq \max \{ \{ w(t), sp(t), rk(s) + K \} \cup \{ \max(p) + K \} \leq \max \{ \{ w(s), sp(t), rk(s) + K \} \cup \{ \max(p) + K \} \leq \max \{ \{ w(s), K, \max(p) + K \} \cup \{ \max(p) + K \} = \max(p) + K .
\]

In the following, we assume that the set

\[
M := \{ sp(r) \mid l \rightarrow r \in R \} \cup \{ (mrk(r) - rk(l)) \mid l \rightarrow r \in R \}
\]

is finite. We set \( K := \max(M) \) and let \( K \) be fixed for the remainder.

**Example 5.5.1.** With respect to the TRS \( R' := \mathcal{R}_{ab} \cup \mathcal{D} \) from Example 5.1.1 we have \( M = \{ (mrk(r) - rk(l)) \mid l \rightarrow r \in R' \} \). Note that the signature of \( R' \) doesn’t contain a special symbol.

Clearly \( M \) is finite and it is easy to see that \( \max(M) = 1 \). Exemplary, we consider the rule schemata \( f_{n+1}(w(x)) \rightarrow f_n(i(x)) \). Note that the rank of \( i \) equals 4, the rank of \( h \) is 3, and the rank of \( f_n \) is given by \( n + 5 \). Hence \( mrk(f_n(i(x))) = n + 5 \) and \( rk(f_{n+1}(w(x))) = n + 6 \). Clearly \( (n + 5 + n + 6) \leq 1 \).

**Lemma 5.5.4.** If \( s \rightarrow_R t, p = tw(s), q = tw(t) \), then \( p >_{\text{lex}} q \) and \( u(\max(p), K) \geq \max(q) \), where \( u \) denotes a monotone polynomial such that \( u(n, m) \geq 2n + m \).

**Proof.** By definition of the rewrite relation there exists a context \( C \), a substitution \( \sigma \) and a rule \( l \rightarrow r \in R \) such that \( s = C[l] \) and \( t = C[r] \). We prove \( \max(q) \leq u(\max(p), K) \) by induction on \( C \). Note that \( C \) can only have the form (i) \( C = f^a[\square] \) or (ii) \( C = f^a g(u_1, \ldots, C'[\square], \ldots, u_n) \).

**Case \( C = f^a[\square] \):** By Lemma 5.5.3 we see \( \max(tw(r)) \leq \max(tw(l)) + K \). Employing in addition Lemma 5.5.2 and Lemma 5.5.1 we obtain:

\[
\max(q) = \max(tw(f^a r)) \leq \max(tw(r)) + a \leq \max(tw(lr)) + K + a \leq \max(p) + K + \max(p) \leq u(\max(p), K) .
\]

**Case \( C = f^a g(u_1, \ldots, C'[\square], \ldots, u_n) \):** As \( C'[\sigma] \rightarrow_R C'[\sigma] \), induction hypothesis is applicable: Let \( p' = tw(C'[\sigma]) \), \( q' = tw(C'[\sigma]) \). Then \( \max(q') \leq u(\max(p'), K) \). For \( rk(g) = l \), we obtain by application of induction hypothesis and Lemma 5.5.1

\[
\max(q) = \max \{ \{ w(t), a, l \} \cup \{ \max(tw(u_1)), \ldots, \max(q'), \ldots, \max(tw(u_n)) \} \leq \max \{ \{ w(s), a, l \} \cup \\
\cup \{ \max(tw(u_1)), \ldots, u(\max(p'), K), \ldots, \max(tw(u_n)) \} \leq \max \{ \{ w(s), a, l \} \cup \{ \max(p), u(\max(p), K) \} \} = u(\max(p), K) .
\]
We define approximations of the partial order $\succ_{\text{lex}}$.

$$p \succ_{n}^{\text{lex}} q \iff p \succ_{\text{lex}} q \text{ and } u(\max(p), n) \geq \max(q),$$

where $u$ is defined as in Lemma 5.5.4. Now Lemma 5.5.3 can be concisely expressed as follows, for $K$ as above.

**Proposition 5.5.1.** If $s \rightarrow_{R} t$, then $\text{tw}(s) \succ_{K}^{\text{lex}} \text{tw}(t)$.

In the spirit of the definition of derivation height, we define a family of functions $A_{h_{n}} : \mathbb{N} \rightarrow \mathbb{N}$:

$$A_{h_{n}}(p) := \max\{m \mid \exists (p_{0}, \ldots, p_{m}) \ p = p_{0} \succ_{n}^{\text{lex}} p_{1} \succ_{n}^{\text{lex}} \cdots \succ_{n}^{\text{lex}} p_{m}\}.$$  

The following proposition is an easy consequence of the definitions and Proposition 5.5.1.

**Theorem 5.5.1.** Let $(F, R)$ be a TRS, compatible with KBO. Assume the set $M := \{sp(r) \mid l \rightarrow r \in R\} \cup \{(\text{mrk}(r) \div \text{rk}(l)) \mid l \rightarrow r \in R\}$ is finite and the arities in of the symbols in $F$ are bounded; set $K := \max(M)$. Then $dh_{R}(t) \leq A_{h_{K}}(\text{tw}(t))$.

In the next section we show that $A_{h_{n}}$ is bounded by the Ackermann function $\text{Ack}$. Thus providing the sought upper-bound on the derivation height of $R$.

### 5.6 Bounding the Growth of $A_{h_{n}}$

Instead of directly relating the functions $A_{h_{n}}$ to the Ackermann function, we make use of the fast-growing Hardy functions, cf. [128]. The Hardy functions form a hierarchy of unary functions $H_{\alpha} : \mathbb{N} \rightarrow \mathbb{N}$ indexed by ordinals. We will only be interested in a small part of this hierarchy, namely in the set of functions $\{H_{\alpha} \mid \alpha < \omega^{\omega}\}$.

**Definition 5.6.1.** We define the embedding $o : \mathbb{N}^{*} \rightarrow \omega^{\omega}$ as follows:

$$o(p) := \omega^{\ell-1}(p)_{0} + \cdots + \omega(p)_{\ell-2} + (p)_{\ell-1},$$

where $\ell = |p|$.

The next lemma follows directly from the definitions.

**Lemma 5.6.1.** If $p \succ_{\text{lex}} q$, then $o(p) > o(q)$.

We associate with every $\alpha < \omega^{\omega}$ in CNF an ordinal $\alpha_{n}$, where $n \in \mathbb{N}$. The sequence $(\alpha_{n})_{n}$ is called fundamental sequence of $\alpha$. (For the connection between rewriting and fundamental sequences see e.g. [119] or Chapter 3)

$$\alpha_{n} := \begin{cases} 0 & \text{if } \alpha = 0, \\ \beta & \text{if } \alpha = \beta + 1, \\ \beta + \omega^{\gamma+1} \cdot (k-1) + \omega^{\gamma} \cdot (n+1) & \text{if } \alpha = \beta + \omega^{\gamma+1} \cdot k. \end{cases}$$
Based on the definition of $\alpha_n$, we define $H_\alpha: \mathbb{N} \to \mathbb{N}$, for $\alpha < \omega$, by transfinite induction on $\alpha$:

$$H_0(n) := n \quad H_\alpha(n) := H_{\alpha \circ H}(n + 1).$$

Let $>_{(n)}$ denote the transitive closure of $(.)_n$, i.e. $\alpha >_{(n)} \beta$ iff $\alpha_n >_{(n)} \beta$ or $\alpha_n = \beta$. Suppose $\alpha, \beta < \omega$. Let $\alpha = \omega^{a_1}n_1 + \cdots + \omega^{a_k}n_k$ and $\beta = \omega^{b_1}m_1 + \cdots + \omega^{b_l}m_l$. Recall that any ordinal $\alpha \neq 0$ can be uniquely written in CNF, hence we can assume that $\alpha_1 > \cdots > \alpha_k$ and $\beta_1 > \cdots > \beta_l$. Furthermore by our assumption that $\alpha, \beta < \omega$, we have $\alpha_i, \beta_j \in \mathbb{N}$. We write $NF(\alpha, \beta)$ if $\alpha_k \leq \beta_1$.

Before we proceed in our estimation of the functions $A_{\alpha n}$, we state some simple facts that help us to calculate with the function $H_\alpha$.

**Lemma 5.6.2.**

(i) If $\alpha >_{(n)} \beta$, then $\alpha >_{(n+1)} \beta + 1$ or $\alpha = \beta + 1$.

(ii) If $\alpha >_{(n)} \beta$ and $n \geq m$, then $H_\alpha(n) > H_\beta(m)$.

(iii) If $n > m$, then $H_\alpha(n) > H_\alpha(m)$.

(iv) If $NF(\alpha, \beta)$, then $H_{\alpha + \beta}(n) = H_\alpha \circ H_\beta(n)$; $\circ$ denotes function composition.

We relate the Hardy functions with the Ackermann function. The stated upper-bound is a gross one, but a more careful estimation is not necessary here.

**Lemma 5.6.3.** For $n \geq 1$: $H_{\omega n}(m) \leq Ack(2n, m)$.

**Proof.** We recall the definition of the Ackermann function:

$$\begin{align*}
    Ack(0, m) &= m + 1 \\
    Ack(n + 1, 0) &= Ack(n, 1) \\
    Ack(n + 1, m + 1) &= Ack(n, Ack(n + 1, m))
\end{align*}$$

In the following we sometimes denote the Ackermann function as a unary function, indexed by its first argument: $Ack(n, m) = Ack_n(m)$. To prove the lemma, we proceed by induction on the lexicographic comparison of $n$ and $m$. We only present the case, where $n$ and $m$ are greater than 0. As preparation note that $m + 1 \leq H_{\omega n}(m)$ holds for any $n$ and $Ack_n^2(m + 1) \leq Ack_{n+1}(m + 1)$ holds for any $n, m$.

$$\begin{align*}
    H_{\omega n}(m + 1) &= H_{\omega n(m+2)}(m + 2) \\
    &\leq H_{\omega n(m+2)+\omega n}(m + 1) \quad \text{Lemma 5.6.2 [iii, iv]} \\
    &= H_{\omega n(m+1)}^{\omega n}(m + 1) \quad \text{Lemma 5.6.2 [v]} \\
    &= H_{\omega n}^2\omega_n^\omega(m + 1) \\
    &\leq Ack_{2n}^2Ack_{2(n+1)}(m) \quad \text{induction hypothesis} \\
    &\leq Ack_{2n+1}Ack_{2(n+1)}(m) \\
    &= Ack(2(n + 1), m + 1).
\end{align*}$$

\[\Box\]
Lemma 5.6.4. Assume \(u(m, n) \leq 2m + n\) and set \(\ell = |p|\). For all \(n \in \mathbb{N}\):
\[
\text{Ah}_n(p) \leq H_{\omega^2 \cdot o(p)}(u(\max(p), n) + 1) < H_{\omega^{\max(p) + n}} .
\] (5.3)

Proof. To prove the first half of (5.3), we make use of the following fact:
\[
p \leq \omega_q \wedge n \geq \max(q) \implies o(p) > n \implies o(q) .
\] (5.4)

To prove (5.4), one proceeds by induction on \(\omega_q\) and uses that the embedding \(o: \mathbb{N}^* \rightarrow \omega\) is essentially an order-isomorphism. We omit the details.

By definition, we have \(\text{Ah}_n(q) = \max(\{\text{Ah}_n(q) + 1 \mid p > \omega_q q\})\). Hence it suffices to prove
\[
p \leq \omega_q \wedge u(\max(q), n) \geq \max(q) \implies \text{Ah}_n(q) < H_{\omega^2 \cdot o(p)(u(\max(q), n) + 1)}
\] (5.5)

We fix \(p\) fulfilling the assumptions in (5.5): let \(\alpha = o(p), \beta = o(q), v = u(\max(q), n)\). We use (5.4) to obtain \(\alpha >_\omega \beta\). We proceed by induction on \(p\).

Consider the case \(\alpha_v = \beta\). As \(\omega_q \leq \omega\), we can employ induction hypothesis to conclude \(\text{Ah}_n(q) \leq H_{\omega^2 \cdot o(p)}(u(\max(q), n) + 1)\). It is not difficult to see that for any \(p \in \mathbb{N}^*\) and \(n \in \mathbb{N}\), \(4\max(p) + 2n + 1 \leq H_{\omega^2}(u(\max(p), n))\). In sum, we obtain:
\[
\text{Ah}_n(q) \leq H_{\omega^2 \cdot o(q)}(u(\max(q), n) + 1)
\leq H_{\omega^2 \cdot \omega_q}(u(u(\max(p), n), n) + 1)
\leq H_{\omega^2 \cdot \omega_q}(4\max(p) + 2n + 1)
\leq H_{\omega^2 \cdot \omega_q}(u(\max(p), n))
\leq H_{\omega^2 \cdot (\alpha_v + 1)}(u(\max(p), n) + 1)
\leq H_{\omega^2 \cdot \alpha}(u(\max(p), n) + 1)
\leq \max(q) \leq u(\max(p), n)
\]
\[
\text{Ah}_n(p) \leq H_{\omega^2 \cdot o(p)}(u(\max(p), n) + 1) \leq H_{\omega^{\max(p) + n}}(u(\max(p), n) + 1)
\]

The application of Lemma 5.6.2[i-v] in the last step is feasible as by definition \(\alpha >_\omega \alpha_v\). An application of Lemma 5.6.2[ii] yields \(\alpha_v + 1 \leq (\alpha_v + 1) \omega^{\max(p)}\). From which we deduce \(\omega^\ell \leq o(p)\).

Secondly, consider the case \(\alpha_v > (\alpha_v \beta\). In this case the proof follows the pattern of the above proof, but an additional application of Lemma 5.6.2[i] is required. This completes the proof of (5.5).

To prove the second part of (5.3), we proceed as follows: The fact that \(\omega^\ell \geq o(p)\) is immediate from the definitions. Induction on \(p\) reveals that even \(\omega^\ell \geq (\max(p)) o(p)\) holds. Thus in conjunction with the first part of (5.3), we obtain:
\[
\text{Ah}_n(p) \leq H_{\omega^2 \cdot o(p)}(u(\max(p), n) + 1) \leq H_{\omega^{\max(p) + n}}(u(\max(p), n) + 1)
\]

The last step follows as \(2\max(p) + n + 1 \leq H_{\omega^2}(\max(p) + n)\).

As a consequence of Lemma 5.6.3 and 5.6.4, we obtain the following proposition.

Theorem 5.6.1. For all \(n \geq 1\): If \(\ell = |p|\), then \(\text{Ah}_n(p) \leq \text{Ack}(2\ell + 8, \max(p) + n)\).
5.7 Derivation Height of TRSs over Infinite Signatures Compatible with KBOs

Based on Theorem [5.5.1] and [5.6.1], we obtain that the derivation height of \( t \in T(F) \) is bounded in the Ackermann function.

**Theorem 5.7.1.** Let \((F, R)\) be a TRS, compatible with KBO. Assume the set \( M := \{sp(r) \mid l \rightarrow r \in R\} \cup \{\text{mrk}(r) - \text{rk}(l) \mid l \rightarrow r \in R\} \) is finite and the arities of the symbols in \( F \) are bounded; set \( K := \max(M) \). Then \( dh_R(t) \leq \text{Ack}(O(|tw(t)|) + \max(|tw(t)|) + K, 0) \).

**Proof.** We set \( u(n, m) = 2n + m \) and keep the polynomial \( u \) fixed for the remainder. Let \( p = tw(t) \) and \( \ell = |p| \). Due to Theorem [5.5.1] we conclude that \( dh_R(t) \leq \text{Ah}_K(p) \).

It is easy to see that \( \text{Ack}(n, m) \leq \text{Ack}(n + m, 0) \). Using this fact and Theorem [5.6.1] we obtain: \( \text{Ah}_K(p) \leq \text{Ack}(O(\ell), \max(p) + K) \leq \text{Ack}(O(\ell) + \max(p) + K, 0) \). Thus the theorem follows. \[ \Box \]

For fixed \( t \in T(F) \) we can bound the argument of the Ackermann function in the above theorem in terms of the size of \( t \). We define

\[
\begin{align*}
  r_{\text{max}} &:= \text{mrk}(t) \\
  w_{\text{max}} &:= \max(\{w(u) \mid u \in FS(t) \cup \text{Var}(t)\}) .
\end{align*}
\]

**Lemma 5.7.1.** For \( t \in T(F) \), let \( r_{\text{max}}, w_{\text{max}} \) be as above. Let \( b := \max\{\text{Ar}(F), 3\} + 1 \), and set \( n := \text{Size}(r_{\text{max}}) \). Then \( w(t) \leq w_{\text{max}} \cdot n \), \( \text{sp}(t) \leq n \), \( \text{mrk}(t) \leq r_{\text{max}} \). Hence \( |tw(t)| \leq b^{w_{\text{max}}(n) - n + 1} \) and \( \max(|tw(t)|) \leq w_{\text{max}}(n) \cdot n + r_{\text{max}} \).

**Proof.** The proof proceeds by induction on \( t \). \[ \Box \]

**Corollary 5.7.1.** Let \((F, R)\) be a TRS, compatible with a KBO \( \succ_{\text{kbo}} \). Assume the set \( \{sp(r) \mid l \rightarrow r \in R\} \cup \{\text{mrk}(r) - \text{rk}(l) \mid l \rightarrow r \in R\} \) is finite and the arities of the symbols in \( F \) are bounded. Then for \( t \in T(F) \), there exists a constant \( c \)—depending on \( t, (F, R) \), and \( \succ_{\text{kbo}} \)—such that \( dh_R(t) \leq \text{Ack}(c^n, 0) \).

**Proof.** The corollary is a direct consequence of Theorem [5.7.1] and Lemma [5.7.1]. \[ \Box \]

**Remark 5.7.1.** Note that it is not straightforward to apply Theorem [5.6.1] to classify the derivational complexity of \( R \), over infinite signature, compatible with KBO. This is only possible in the (unlikely) case that for every term \( t \) the maximal rank \( \text{mrk}(t) \) and the weight \( w(t) \) of \( t \) can be bounded uniformly, i.e. independent of the size of \( t \).

We apply Corollary [5.7.1] to the motivating example introduced in Section 5.1.

**Example 5.7.1.** Recall the definition of \( R \) and \( R' := R_{\text{lab}} \cup \text{Dec} \) from Example [5.1.1] and [5.5.1] respectively. Let \( s \in T(F_{\text{lab}}) \) be fixed and set \( n := \text{Size}(s) \).

Clearly the arities of the symbols in \( F_{\text{lab}} \) are bounded. In Example [5.5.1] we indicated that the set \( M = \{\text{mrk}(r) - \text{rk}(l) \mid l \rightarrow r \in R'\} \) is finite. Hence, Corollary [5.7.1] is applicable to conclude the existence of \( c \in \mathbb{N} \) with \( dh_{R'}(s) \leq \text{Ack}(c^n, 0) \). In order
to bound the derivation height of $\mathcal{R}$, we employ Lemma 5.2.1 to observe that for all $t \in \mathcal{T}(\mathcal{F})$: $dh_{\mathcal{R}}(t) \leq dh_{\mathcal{R}}(\text{lab}_n(t))$, for arbitrary $\alpha$. As $\text{Size}(t) = \text{Size}(\text{lab}_n(t))$ the above calculation yields

$$dh_{\mathcal{R}}(t) \leq dh_{\mathcal{R}'}(\text{lab}_n(t)) \leq \text{Ack}(\epsilon^9, 0).$$

Note that $c$ depends only on $t$, $\mathcal{R}'$ and the KBO $\succ_k$ employed.

The main motivation of this work was to provide an alternative proof of Lepper’s result that the derivational complexity of any finite TRS, compatible with KBO, is bounded by the Ackermann function, see [103]. We recall the definition of the derivational complexity:

$$dc_{\mathcal{R}}(n) := \max\{dh_{\mathcal{R}}(t) \mid \text{Size}(t) \leq n\}.$$

**Corollary 5.7.2.** Let $(\mathcal{F}, \mathcal{R})$ be a TRS, compatible with KBO, such that $\mathcal{F}$ is finite. Then $dh_{\mathcal{R}}(n) \leq \text{Ack}(2^{O(n)}, 0)$.

**Proof.** As $\mathcal{F}$ is finite, the $K = \max\{(\text{mrk}(t) - \text{rk}(t)) \mid t \rightarrow t' \in \mathcal{R}'\}$ and $Ar(\mathcal{F})$ are obviously well-defined. Theorem 5.7.1 yields that $dh_{\mathcal{R}}(t) \leq \text{Ack}(O(\omega(t)) + \max(\omega(t)) + K, 0)$. Again due to the finiteness of $\mathcal{F}$, for any $t \in \mathcal{T}(\mathcal{F})$, $\text{mrk}(t)$ and $\omega(t)$ can be estimated independent of $t$. A similar argument calculation as in Lemma 5.7.1 thus yields $dh_{\mathcal{R}}(t) \leq \text{Ack}(2^{O(\text{Size}(t))}, 0)$. Hence the result follows.

**Remark 5.7.2.** Note that if we compare the above corollary to Corollary 19 in [103], we see that Lepper could even show that $dc_{\mathcal{R}}(n) \leq \text{Ack}(O(n), 0)$. On the other hand, as already remarked above, Lepper’s result is not admissible if the signature is infinite.

In concluding, we want to stress that the method is also applicable to obtain bounds on the derivational height of non simply terminating TRSs, a feature only shared by Hofbauer’s approach to utilise context-dependent interpretations, cf. [78].

**Example 5.7.2.** Consider the TRS consisting of the following rules:

$$f(x) \circ (y \circ z) \rightarrow x \circ (f^2(y) \circ z) \quad a(a(x)) \rightarrow a(b(a(x)))$$

$$f(x) \circ (y \circ (z \circ w)) \rightarrow x \circ (z \circ (y \circ w))$$

$$f(x) \rightarrow x$$

Let us call this TRS $\mathcal{R}$ in the following. Due to the rule $a(a(x)) \rightarrow a(b(a(x)))$, $\mathcal{R}$ is not simply terminating. And due to the three rules, presented on the left, the derivational complexity of $\mathcal{R}$ cannot be bounded by a primitive recursive function, compare [76].

Termination can be shown by semantic labelling, where the natural numbers are used as semantics and as labels. The interpretations $a_N(n) = n + 1$, $b_N(n) = \max\{0, n - 1\}$, $f_N(n) = n$, and $m_{\mathcal{C}} n = m + n$ give rise to a quasi-model. Using the labelling function $\ell_d(\alpha) = n$, termination of $\mathcal{R}' := \mathcal{R}_{\text{lab}} \cup \text{Dec}$ can be shown by an instance $\succ_k$ of KBO with weight function $(w, 1)$: $w(\circ) = w(f) = 0$, $w(b) = 1$, and $w(a_n) = n$ and precedence: $f \succ \circ \succ \ldots a_{n+1} \succ a_n \succ \cdots \succ a_0 \succ b$. The symbol $f$ is special. Clearly the arities of the symbols in $\mathcal{F}_{\text{lab}}$ are bounded. Further, it is not difficult to see that
the set $M = \{ sp(r) \mid l \to r \in \mathcal{R}' \} \cup \{ (mrk(r) - rk(l)) \mid l \to r \in \mathcal{R}' \}$ is finite and $K := \max(M) = 2$.

Proceeding as in Example 5.7.1 we see that for each $t \in T(\mathcal{F})$, there exists a constant $c$ (depending on $t$, $\mathcal{R}'$ and $\succ_{kbo}$) such that $d_{hR}(t) \leq Ack(c^n, 0)$. 
6

Complexity Analysis by Rewriting

Publication Details

Ranking
The International Symposium on Functional and Logic Programming has been ranked A by CORE in 2007.

Abstract
In this paper we introduce a restrictive version of the multiset path order, called polynomial path order. This recursive path order induces polynomial bounds on the maximal number of innermost rewrite steps. This result opens the way to automatically verify for a given program, written in an eager functional programming language, that the maximal number of evaluation steps starting from any function call is polynomial in the input size. To test the feasibility of our approach we have implemented this technique and compare its applicability to existing methods.

6.1 Introduction

Term rewriting is a conceptually simple but powerful abstract model of computation that underlies much of declarative programming. In rewriting, proving termination is an important research field. Powerful methods have been introduced to establish termination of a given term rewrite system. One of the most natural ways to proof termination is the use of interpretations. Consequentially this technique has been introduced quite early. Moreover, if one is interested in automatically proving termination, polynomial interpretations provide a natural starting point, cf. [13]. However, termination proofs via

---

1 This research was partially supported by FWF (Austrian Science Fund) project P20133.
polynomial interpretations are limited as the longest possible rewrite sequences admitted by rewrite systems compatible with a polynomial interpretation are double-exponential (in the size of the initial term), see [79]. Another well-studied (and direct) termination technique is the use of reduction orders—for example simplification orders. Still this technique is limited, which can again be shown by the analysis of the induced derivation length, cf. [77, 144, 103]. In recent years the emphasis shifted towards transformation techniques like the dependency pair method or semantic labeling. Transformation techniques have significantly increased the possibility to automatically prove termination.

Once we have established termination of a given rewrite system $R$, it seems natural to direct the attention to the analysis of the complexity of $R$. In rewriting the complexity of a rewrite system $R$ is measured as the maximal derivation length with respect to $R$. As mentioned above for direct termination methods a significant amount of investigations has been conducted, providing a suitable foundation for further research. Unfortunately, almost nothing is known about the length of derivations induced by state-of-the-art termination techniques like the dependency pair method or semantic labeling. For the dependency pair method no results on the induced derivation length are known. Partial result with respect to semantic labeling are reported in [115].

In this paper we introduce a restriction of the multiset path order, called polynomial path order (denoted as $\succ_{pop^*}$). Our main result states that this recursive path order induces polynomial bounds on the maximal length of innermost rewrite steps. As we have successfully implemented this technique, we thus can automatically verify for a given term rewrite system $R$ that $R$ admits at most polynomial innermost derivation length (on the set of constructor-based terms). This opens the way to automatically verify for a given program—written in an eager functional programming language—that its runtime complexity is polynomial (in the input size). The only restrictions in the applicability of the result are that (i) the functional program $P$ is transformable into a term rewrite system $R$ and (ii) a feasible (i.e., polynomial) derivation length with respect to $R$ gives rise to a feasible runtime complexity of $P$. In short the transformation has to be non-termination and complexity preserving.

The definition of polynomial path orders employs the idea of tiered recursion [21]. Syntactically this amounts to a separation of arguments into normal and safe argument. (Below this will be governed by the presence of mappings safe and nrm associating with each function symbol a list of argument positions.) We explain our approach by an example rewrite system that clearly admits at most polynomial derivation length.

**Example 6.1.1.** Consider the following rewrite system $R_{\text{mult}}$.

\[
\begin{align*}
\text{add}(x,0) & \rightarrow x \\
\text{add}(s(x),y) & \rightarrow s(\text{add}(x,y)) \\
\text{mult}(0,y) & \rightarrow 0 \\
\text{mult}(s(x),y) & \rightarrow \text{add}(y,\text{mult}(x,y))
\end{align*}
\]

We suppose that all arguments of the successor ($s$) are safe ($\text{safe}(s) = \{1\}$), that the second argument of addition ($\text{add}$) is safe ($\text{safe}(\text{add}) = \{2\}$) and that all arguments of multiplication ($\text{mult}$) are normal ($\text{safe}(\text{mult}) = \emptyset$). Furthermore let the (strict) precedence $>$ be defined as $\text{mult} > \text{add} > s$. Then $R_{\text{mult}}$ is compatible with $\succ_{pop^*}$ (see
6.1 Introduction

Definition 6.3.2 and as a consequence of our main theorem (see Section 6.3) we con-
clude that the number of rewrite steps starting from \( \text{mult}(s^n(0), s^m(0)) \) is polynomially
bounded in \( n \) and \( m \). (Here we write \( s^n(0) \) as abbreviation of \( s(\ldots(s(0)\ldots)) \) with \( n \)
ocurrences of the successor symbol \( s. \))

The polynomial path order is an extension of the path order for \( \text{FP} \) introduced by Arai
and the second author in [8] (see also Chapter 4). A central motivation of this research
is the observation that the direct application of the latter order is only successful on a
handful of (very simple) rewrite systems. The path order for \( \text{FP} \) gains only power if
additional transformations are performed. Unfortunately, such powerful transformations
are difficult to find automatically.

Further note that the polynomial path order is to some extent related to the light
multiset path order introduced by Marion [109]. Roughly speaking the light multiset
path order is a tamed version of the multiset path order, characterising the functions
computable in polytime. It seems important to stress that the below stated main theorem
fails for the light multiset path order. This can be easily seen from the next example.

Example 6.1.2. Consider the following rewrite system \( R_{\text{bin}} \). (This is Example 2.21
about binomial coefficients from [136].)

\[
\begin{align*}
\text{bin}(x, 0) & \rightarrow s(0) & \text{bin}(s(x), s(y)) & \rightarrow +\left(\text{bin}(x, s(y)), \text{bin}(x, y)\right) \\
\text{bin}(0, s(y)) & \rightarrow 0
\end{align*}
\]

For a precedence that fulfills \( \text{bin} > s \), \( \text{bin} > + \) and separations of arguments \( \text{safe(bin)} = \emptyset \),
\( \text{safe}(+) = \{1, 2\} \), we obtain that \( R_{\text{bin}} \) is compatible with the light multiset path order,
cf. [109]. However it is straightforward to verify that the (innermost) derivation height
of \( \text{bin}(s^n(0), s^m(0)) \) is exponential in \( n \).

To test the feasibility of our approach we have implemented a small complexity analy-
ser based on the polynomial path order and compare its applicability to existing tech-
niques. To do so, we also have implemented the light multiset path order and a restricted
form of polynomial interpretations, so-called additive polynomial interpretations, cf. [26].
Note that compatibility with addditive polynomial interpretations induces polynomial
derivation length for constructor-based terms, cf. [26].

The research in [26, 109] falls into the realm of implicit complexity theory. In this
context related work to our research is due to Bonfante et al. [29] but see also seminal
work by Hofmann [82] and Schwichtenberg [134]. While [82, 134] are incomparable to
our techniques, a comparison to [29] is also not straightforward. Our principal concern
is that the termination techniques employed allow for an complexity analysis of the
subjected program. On the other hand the crucial feature of quasi-interpretations (the
central contribution of [29]) is their weak monotonicity, hence termination can only
be shown in conjunction with other termination techniques. For example the class of
polytime computable functions can be characterised as the class of functions computable
by confluent constructor rewrite systems compatible with the multiset path order and
that admit only additive quasi-interpretations, cf. [29]. This interesting result renders an
insightful implicit characterisation of the polytime computable function, but it is of little help, if one wants to obtain a complexity analysis of a term rewrite system subjected to a modern termination prover. Recently an interesting application of quasi-interpretations has been reported by Lucas and Peña [108]. Here the dependency pair method is used in conjunction with quasi-interpretations to obtain bounds on the memory consumption of Safe programs. This method is easily automatable, but new ideas are necessary to yield bounds on the runtime behaviour of functional programs.

The remainder of this paper is organised as follows. In the next section we recall basic notions and starting points of this paper. In Section 6.3 we have collected our main results. In order to prove these results we extend results originally presented in [8]. Our findings in this direction are presented in Section 6.4. The central argument to prove the main theorem is then given in Section 6.5. In Section 6.6 we give the experimental results. In order to prove these results we extend results originally presented in [8]. Our notions and starting points of this paper. In Section 6.3 we have collected our main evidence mentioned above. In Section 6.7 we touch upon an application of our main theorem in recent work (together with Hirokawa and Middeldorp) where we study the termination behaviour of Scheme programs. Finally in Section 6.8 we conclude and mention possible future work.

6.2 Preliminaries

We assume familiarity with term rewriting [15, 137]. Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a signature. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We always assume that $\mathcal{F}$ contains at least one constant. The arity of a function symbol $f$ is denoted as $\text{ar}(f)$. Let $>$ be a precedence on the signature $\mathcal{F}$. The rank of a function symbol is defined inductively as follows: $\text{rk}(f) = 1 + \max \{ \text{rk}(g) \mid g \in \mathcal{F} \land f > g \}$ (Here we employ the convention that the maximum of an empty set equals 0.) We write $\leq$ to denote the subterm relation and $\geq$ for its converse. The strict part of $\geq$ is denoted by $\triangleright$. $\text{Var}(t)$ denotes the set of variables occurring in a term $t$. The size (depth) of a term $t$ is denoted as $\text{size}(t)$ ($\text{dp}(t)$). The width of a term $t$ is defined inductively as follows: $\text{wd}(t) = 1$, if $t$ is a variable or a constant, otherwise if $t = f(t_1, \ldots, t_n)$ with $n > 0$, we set $\text{wd}(t) = \max \{ n, \text{wd}(t_1), \ldots, \text{wd}(t_n) \}$. The Buchholz norm of a term $t$ is defined inductively as follows: $\|t\| = 1$, if $t$ is a variable and for $t = f(t_1, \ldots, t_n)$ we set $\|t\| = 1 + \max \{ n, \|t_1\|, \ldots, \|t_n\| \}$. We write $[t_1, \ldots, t_n]$ to denote multisets and $\Psi$ for the summation of multisets.

A term rewrite system (TRS for short) $\mathcal{R}$ over $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a set of rewrite rules $l \rightarrow r$, such that $l \notin \mathcal{V}$ and $\text{Var}(l) \supseteq \text{Var}(r)$. (If not mentioned otherwise, we assume $\mathcal{R}$ is finite.) The root symbols of left-hand sides of rewrite rules are called defined, while all other function symbols are called constructors. For a given signature $\mathcal{F}$ the defined symbols are denoted as $D$, while the constructor symbol are collected in $C$. The smallest rewrite relation that contains $\mathcal{R}$ is denoted by $\rightarrow_{\mathcal{R}}$. We simply write $\rightarrow$ for $\rightarrow_{\mathcal{R}}$ if $\mathcal{R}$ is clear from context. Let $s$ and $t$ be terms. If exactly $n$ steps are preformed to contract $s$ to $t$ we write $s \rightarrow^a t$. A term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is called a normal form if there is no $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $s \rightarrow t$. The innermost rewrite relation $\rightarrow_{\mathcal{R}}$ of a TRS $\mathcal{R}$ is defined on terms as follows: $s \rightarrow_{\mathcal{R}} t$ if there exist a rewrite rule $l \rightarrow r \in \mathcal{R}$, a context $C$, and a substitution $\sigma$
such that \( s = C[l\sigma], t = C[r\sigma] \), and all proper subterms of \( l\sigma \) are normal forms of \( R \). A TRS is called *confluent* if for all \( s, t_1, t_2 \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \) with \( s \rightarrow^* t_1 \) and \( s \rightarrow^* t_2 \) there exists a term \( t_3 \) such that \( t_1 \rightarrow^* t_3 \) and \( t_2 \rightarrow^* t_3 \). A TRS is *non-overlapping* if it has no critical pairs, cf. [15]. A TRS \( R \) is *left-linear* if for all rules \( l \rightarrow r \in R \), all variables in \( l \) occur at most once. If \( R \) is additionally non-overlapping, then \( R \) is called *orthogonal*. Note that every orthogonal TRS is confluent. A *constructor* TRS is a TRS whose signature \( \mathcal{F} \) can be partitioned into the defined symbols \( \mathcal{D} \) and constructor symbols \( \mathcal{C} \) in such a way that the left-hand side of each rule has the form \( f(s_1, \ldots, s_n) \) with \( f \in \mathcal{D} \) and for all \( i: s_i \in \mathcal{T}(\mathcal{C}, \mathcal{V}) \). A defined function symbol is completely defined if it does not occur in any ground term in normal form. A TRS is completely defined if each defined symbol is completely defined. An element of \( \mathcal{T}(\mathcal{C}, \mathcal{V}) \) is called a value; we set \( \text{Val}(R) = \mathcal{T}(\mathcal{C}, \mathcal{V}) \).

We call a TRS terminating if no infinite rewrite sequence exists. The derivation length of a term \( t \) with respect to a terminating TRS \( R \) and rewrite relation \( \rightarrow_R \) is defined as usual: \( \text{DL}(R, t) = \max \{ n \mid \exists t \rightarrow^n t \} \). We call a term \( t = f(t_1, \ldots, t_n) \) constructor-based if all its arguments \( t_i \) are values, i.e., \( t_i \in \mathcal{T}(\mathcal{C}, \mathcal{V}) \) for all \( 1 \leq i \leq n \). The set \( \mathcal{T}_b \) collects all constructor-based terms.

A proper order is a transitive and irreflexive relation. The reflexive closure of a proper order \( \succ \) is denoted as \( \succ^\circ \). A proper order \( \succ \) is well-founded if there is no infinite decreasing sequence \( t_1 \succ t_2 \succ t_3 \cdots \). A well-founded proper order that is also a rewrite relation is called a reduction order. We say a reduction order \( \succ \) and a TRS \( R \) are compatible if \( R \subseteq \succ \). It is well-known that a TRS is terminating if and only if there exists a compatible reduction order.

### 6.3 Main Result

In the sequel \( R \) denotes a constructor TRS over a (possible variadic) signature \( \mathcal{F} \). Let \( \succ \) denote a precedence on \( \mathcal{F} \) such that for all \( f \in \mathcal{D} \) we have for all \( c \in \mathcal{C}: f > c \). (Recall that \( \mathcal{F} \) contains at least one constant.) We assume that \( R \) is completely defined, i.e., ground normal forms and ground values coincide\(^2\).

For each \( n \)-ary function symbol \( f \in \mathcal{D} \) of fixed arity, we suppose the existence of a mapping \( \text{safe} \) that associates with \( f \) a (possibly empty) list \( \{i_1, \ldots, i_m\} \) with \( 1 \leq i_1 < \cdots < i_m \leq n \). For a mapping \( \text{safe} \) and a term \( t = f(t_1, \ldots, t_n) \), \( \text{safe}(f) \) denotes the safe argument positions of \( t \). The argument positions of \( t \) not included in \( \text{safe}(f) \) are called normal and are denoted by \( \text{nrm}(f) \). The mapping \( \text{safe} \) (nrm) is referred to as safe (normal) mapping. We generalise safe (normal) mappings to constructor symbols and variadic function symbols as follows: For each function symbol \( f \in \mathcal{C} \), we fix \( \text{safe}(f) = \{1, \ldots, \text{ar}(f)\} \) and for each variadic function symbol \( f \in \mathcal{D} \) we assert \( \text{safe}(f) = \emptyset \). The normalised signature \( \mathcal{F}^n \) contains a function symbol \( f^n \) for each \( f \in \mathcal{F} \). If \( f \) is of fixed-arity and \( \text{nrm}(f) = \{i_1, \ldots, i_p\} \), then \( \text{ar}(f) = p \). The normalised signature \( \mathcal{C}^n \) is defined accordingly.

\(^2\) The assumption that \( R \) is completely defined arises naturally in the context of implicit characterisation of complexity classes. We follow this convention to some extent, but show that this restriction is not necessary.
6 Complexity Analysis by Rewriting

**Definition 6.3.1.** Let $>$ be a precedence and safe a safe mapping. We define $>_\text{pop}$ inductively as follows: $s = f(s_1, \ldots, s_n) >_\text{pop} t$ if one of the following alternatives holds:

(i) $f$ is a constructor and $s_i >_\text{pop} t$ for some $i \in \{1, \ldots, n\}$,

(ii) $s_i >_\text{pop} t$ for some $i \in \text{nrm}(f)$, or

(iii) $t = g(t_1, \ldots, t_m)$ with $f \in \mathcal{D}$ and $f > g$ and $s >_\text{pop} t_i$ for all $1 \leq i \leq m$.

We write $s >_\text{pop} t$ if $s >_\text{pop} t$ follows by application of clause (i) in Definition 6.3.1.

A similar notation will be used for the orders defined below.

**Definition 6.3.2.** Let $>$ be a precedence and safe a safe mapping. We define the polynomial path order $>_\text{pop*}$ (POP* for short) inductively as follows: $s = f(s_1, \ldots, s_n) >_\text{pop*} t$ if one of the following alternatives holds:

(i) $s >_\text{pop} t$,

(ii) $s_i >_\text{pop*} t$ for some $i \in \{1, \ldots, n\}$,

(iii) $t = g(t_1, \ldots, t_m)$, with $f \in \mathcal{D}$, $f > g$, and the following properties hold:

- $s >_\text{pop*} t_{i_0}$ for some $i_0 \in \text{safe}(g)$ and
- either $s >_\text{pop} t_i$ or $s > t_i$ and $i \in \text{safe}(g)$ for all $i \neq i_0$.

(iv) $t = f(t_1, \ldots, t_m)$ and for $\text{nrm}(f) = \{i_1, \ldots, i_p\}$, $\text{safe}(f) = \{j_1, \ldots, j_q\}$ the following properties hold:

- $[s_{i_1}, \ldots, s_{i_p}] >_\text{pop*} \mathcal{m}ul [t_{i_1}, \ldots, t_{i_p}]$,
- $[s_{j_1}, \ldots, s_{j_q}] >_\text{pop*} \mathcal{m}ul [t_{j_1}, \ldots, t_{j_q}]$.

Here $(>_\text{pop*})_{\mathcal{m}ul}$ denotes the multiset extension of $>_\text{pop*}$ and recall that for variadic function symbols, the set of safe arguments is empty.

**Example 6.3.1.** Consider the following TRS $\mathcal{R}_{\text{insert}}$ (This is a simplification of an example from [?].)

\[
\begin{align*}
\text{if}(\text{true}, x, y) & \rightarrow x & x \geq 0 & \rightarrow \text{true} \\
\text{if}(\text{false}, x, y) & \rightarrow y & 0 \geq s(x) & \rightarrow \text{false} \\
\text{ins}(x, \text{nil}) & \rightarrow \text{cons}(x, \text{nil}) & s(x) \geq s(y) & \rightarrow x \geq y \\
\text{ins}(x, \text{cons}(y, y)) & \rightarrow \text{if}(y \geq x, \text{cons}(x, \text{cons}(y, y)), \text{cons}(y, \text{ins}(x, y)))
\end{align*}
\]

We represent lists with the help of the constructors nil and cons. To show compatibility with POP*, we assume a precedence $>$ that fulfills $\text{ins} > \text{if}$, $\text{ins} > \geq$, $\text{ins} > \text{cons}$, $0 > \text{true}$, and $0 > \text{false}$. Further we define a safe mapping safe as follows:

\[
\begin{align*}
\text{safe}(s) & = \{1\} & \text{safe}(\text{if}) & = \{1, 2, 3\} & \text{safe}(\text{ins}) & = \varnothing \\
\text{safe}(\text{cons}) & = \{1, 2\} & \text{safe}(\geq) & = \{2\}
\end{align*}
\]

It is straightforward to verify that the induced polynomial path order $>_\text{pop*}$ is compatible with $\mathcal{R}_{\text{insert}}$.
An easy inductive argument shows that if \( s \in \text{Val}(\mathcal{R}) \) and \( s \succ_{\text{pop}}^* t \), then \( t \in \text{Val}(\mathcal{R}) \). Note that \( \succ_{\text{pop}}^* \) is not a reduction order. Although \( \succ_{\text{pop}}^* \) is a well-founded proper order that is closed under substitutions, the order is not closed under contexts due to the restrictive definition of clause \([17]\) in the above definition. However we still have the following theorem, which follows as the multiset path order extends \( \succ_{\text{pop}}^* \).

**Theorem 6.3.1.** Every TRS \( \mathcal{R} \) that is compatible with \( \succ_{\text{pop}}^* \) for some well-founded precedence \( \succ \) is terminating.

As normal and safe arguments are distinguishable, we strengthen the notion of runtime complexity as follows:

\[
\mathcal{R}^*_n(m) = \max \{ Dl_{(\mathcal{R}, \cdot \to)}(t) \mid t = f(t_1, \ldots, t_n) \in T_n \text{ and } \sum_{i \in \text{nrm}(f)} \text{size}(t_i) \leq m \}.
\]

This function is called the *normal* runtime complexity.

**Main Theorem.** Let \( \mathcal{R} \) be a finite, completely defined constructor TRS. Assume further \( \mathcal{R} \) is compatible with \( \succ_{\text{pop}}^* \), i.e., \( \mathcal{R} \subseteq \succ_{\text{pop}}^* \). Then the induced (normal) runtime complexity is polynomial.

Assume \( \mathcal{R} \) is a finite, constructor TRS that is not completely defined; i.e., at least one defined function symbol occurs in a ground normal form. To obtain a completely defined TRS it suffices to add suitable rules, thus we arrive at the following corollary, see [12] for the proof.

**Corollary 6.3.1.** Let \( \mathcal{R} \) be a finite, constructor TRS. Assume further \( \mathcal{R} \) is compatible with \( \succ_{\text{pop}}^* \), i.e., \( \mathcal{R} \subseteq \succ_{\text{pop}}^* \). Then the induced (normal) runtime complexity is polynomial.

**Definition 6.3.3.** The *predicative* rewrite relation \( s \not\vdash t \) is defined as follows: \( s \not\vdash t \) if \( s \to t \) by contracting safe argument positions first, i.e., if there exist a rewrite rule \( l \to r \in \mathcal{R} \), a context \( C \), and a substitution \( \sigma \) such that \( s = C[l\sigma], \ t = C[r\sigma] \) and all safe argument position of \( l\sigma \) are in normal form.

Clearly predicative rewriting is a generalisation of innermost rewriting. Essentially following the pattern of the proof of the theorem, we arrive at the following corollary.

**Corollary 6.3.2.** Let \( \mathcal{R} \) be a finite constructor TRS. Assume further \( \mathcal{R} \) is compatible with \( \succ_{\text{pop}}^* \), i.e., \( \mathcal{R} \subseteq \succ_{\text{pop}}^* \). Then for all \( f \in \mathcal{F} \) of arity \( n \), with \( \text{nrm}(f) = \{i_1, \ldots, i_p\} \) and for all values \( s_1, \ldots, s_n \): \( Dl_{(\mathcal{R}, \not\vdash)}(f(s_1, \ldots, s_n)) \) is bounded by a polynomial in the sum of the sizes of the normal argument terms \( s_{i_1}, \ldots, s_{i_p} \).

**Remark 6.3.1.** Beckmann and Weiermann observed in [19] that general rewriting is too powerful to serve as a suitable computation model to characterise the class of polytime computable functions as a TRS. Their notion of a *feasible* rewrite system is reflected adequately in the notion of predicative rewriting.
6.4 Polynomial Path Order on Sequences

In this section we extend definitions and results originally presented in [5] (see also Chapter 4). The main aim is to define a polynomial path order $\preceq_k$ on sequences of terms such that $\preceq_k$ induces polynomial derivation length with respect to a compatible TRS $R$.

Let $\circ \notin \mathcal{F}^n$ be a variadic function symbol. We extend the normalised signature $\mathcal{F}^n$ by $\circ$ and define $Seq(\mathcal{F}^n, \mathcal{V}) = T(\mathcal{F}^n \cup \{\circ\}, \mathcal{V})$. Elements of $Seq(\mathcal{F}^n, \mathcal{V})$ are sometimes referred to as sequences. Instead of $\circ(s_1, \ldots, s_n)$, we usually write $(s_1 \cdots s_n)$ and denote the empty sequence $()$ as $\emptyset$. Let $a = (a_1 \ldots a_n)$ and $b = (b_1 \ldots b_m)$ be elements of $Seq(\mathcal{F}^n, \mathcal{V})$. For $a \neq \emptyset$ and $b \neq \emptyset$ define $a \circ b = (a_1 \ldots a_n b_1 \ldots b_m)$. If $a = \emptyset$ ($b = \emptyset$) we set $a \circ b = b$ ($a \circ b = a$).

Let $\rho$ denote the precedence on $\mathcal{F}^n$ induced by the total precedence $\succ$ on $\mathcal{F}$. Buchholz [30] was the first to observe that finite term rewrite systems compatible with recursive path orders $\succ$ are even compatible to finite approximations of $\succ$. This observation carries over to polynomial path orders. The following definitions generalise the path order on $FP$ (POP for short) as defined in [5]. To keep this exposition short, we only state the definition of approximations of the polynomial path order $\preceq_k$ on sequences. The general definitions for $\succ$ and $\preceq$ is obtained by dropping the restrictions on depth and width, cf. [12]. Note that $\preceq_k$ can be conceived as the limit of the finite approximations $\preceq_k^l$. We use the convention that $f \in \mathcal{F}^n$, i.e., $s = f(s_1, \ldots, s_n)$ implicitly indicates that $f \neq \emptyset$.

**Definition 6.4.1.** Let $k, l \geq 1$ and let $\rho$ be a precedence. We define $\preceq_k^l$ inductively as follows: $s \preceq_k^l t$ for $s = f(s_1, \ldots, s_n)$ or $s = (s_1 \cdots s_n)$ if one of the following alternatives holds:

1. $s_i (\succ^\rho)_k t$ for some $i \in \{1, \ldots, n\}$,
2. $s = f(s_1, \ldots, s_n)$ such that of the following two possibilities holds:
   - $t = g(t_1, \ldots, t_m)$ with $f > g$ or
   - $t = (t_1 \cdots t_m)$,
   and $s \preceq_k^{l-1} t_i$ for all $1 \leq i \leq m$, and $m < k + \text{wd}(s)$, or
3. $s = (s_1 \cdots s_n)$, $t = (t_1 \cdots t_m)$ and the following properties hold:
   - $[t_1, \ldots, t_m] = N_1 \uplus \cdots \uplus N_n$,
   - there exists $i \in \{1, \ldots, n\}$ such that $[s_i] \neq N_i$,
   - for all $1 \leq i \leq n$ such that $[s_i] \neq N_i$ we have $s_i \preceq_k^l r$ for all $r \in N_i$
   - $m < k + \text{wd}(s)$.

We write $\preceq_k$ to abbreviate $\preceq_k^1$.

**Definition 6.4.2.** Let $k, l \geq 1$ and let $\rho$ be a precedence. We define the approximation of the polynomial path order $\preceq_k^l$ on sequences inductively as follows: $s \preceq_k^l t$ for $s = f(s_1, \ldots, s_n)$ or $s = (s_1 \cdots s_n)$ if one of the following alternatives holds:
(i) \( s \succ_k^l t \),

(ii) \( s_i (\succ_k^l) t \) for some \( i \in \{1, \ldots, n\} \),

(iii) \( s = f(s_1, \ldots, s_n), t = (t_1 \cdots t_m) \), and the following properties hold:
- \( s \succ_k^{l-1} t_{i_0} \) for some \( i_0 \in \{1, \ldots, n\} \),
- \( s \succ_k^{l-1} t_i \) for all \( i \neq i_0 \), and
- \( m < k + \text{wd}(s) \),

(iv) \( s = f(s_1, \ldots, s_n), t = f(t_1, \ldots, t_m) \) with \( s_1 \cdots s_n \succ_k^l t_1 \cdots t_m \), or

(v) \( s = (s_1 \cdots s_n), t = (t_1 \cdots t_m) \) and the following properties hold:
- \([t_1, \ldots, t_m] = N_1 \uplus \cdots \uplus N_n \),
- there exists \( i \in \{1, \ldots, n\} \) such that \([s_i] \neq N_i \),
- for all \( 1 \leq i \leq n \) such that \([s_i] \neq N_i \): \( s_i \succ_k^l r \) for all \( r \in N_i \), and
- \( m < k + \text{wd}(s) \).

We write \( \succ_k \) to abbreviate \( \succ_k^k \).

Note that \( \emptyset \) is the minimal element of \( \succ_k \) and \( \succ_k \) and that \( \succ_k \) is a reduction order.
The following lemmas are direct consequences of the definitions.

**Lemma 6.4.1.**

(i) If \( s \succ_k t \) and \( k < l \), then \( s \succ_l t \).

(ii) If \( s \succ_k t \), then \( C[s] \succ_k C[t] \), where \( C[\square] \) denotes a context over \( S_{eq}(F^n, V) \).

**Lemma 6.4.2.** If \( s \succ_k^l t \), then \( \text{dp}(t) \leq \text{dp}(s) + l \) and \( \text{wd}(t) \leq k + \text{wd}(s) \). Moreover, if \( s \succ_k^l t \), then \( \|t\| \leq \|s\| + k + l \).

By Lemma 6.4.2, there exists a (uniform) constant \( c \) such that \( \|t\| \leq \|s\| + c \), whenever \( s \succ_k t \). And thus if we have a \( \succ_k \)-descending sequence \( s = t_0 \succ_k t_1 \succ_k \cdots \succ_k t_\ell \) we conclude that \( \|t_i\| \leq ci + \|s\| \) for all \( i \geq 1 \).

**Definition 6.4.3.** We define

\[
G_k(s) := \max\{\ell \in \mathbb{N} | \exists(t_0, \ldots, t_\ell): s = t_0 \succ_k t_1 \succ_k \cdots \succ_k t_\ell\},
\]

\[
F_{k,p}(m) := \max\{G_k(f(t_1, \ldots, t_n)): \text{rk}(f) = p \land \sum_i G_k(t_i) \leq m\}
\]

In the definition of \( F_{k,p} \), we assume \( f \in F^n \).

A direct consequence of Definition 6.4.3 is that \( G_k((t_1 \cdots t_n)) = n + \sum_{i=1}^n G_k(t_i) \) holds.
The following lemma is generalisation of a similar lemma in [8] and the proof given in [8] can be easily adapted.
Lemma 6.4.3. We define $d_{k,0} := k + 1$ and $d_{k,p+1} := (d_{k,p})^k + 1$. Then for all $k, p$ there exists a constant $c$ (depending only on $k$ and $p$) such that for all $m$: $F_{k,p}(m) \leq c(m + 2)d_{k,p}$.

As a consequence of Lemma 6.4.3 we obtain that $F_{k,p}(m)$ is asymptotically bounded by $m^{d_{k,p}}$ for large enough $m$. The following lemma follows by a standard inductive argument.

Lemma 6.4.4. For all $k$, there exists a constant $c$ such that for $s \in T(C^n \cup \{\circ\}, V)$: $G_k(s) \leq c \cdot \text{size}(s)^2$.

We arrive at the main theorem of this section.

Theorem 6.4.1. For all $f \in F^n$ of arity $n$, for all $s_1, \ldots, s_n \in T(C^n \cup \{\circ\})$, and for all $k$: $G_k(f(s_1, \ldots, s_n))$ is bounded by a polynomial in the sum of the sizes of $s_1, \ldots, s_n$. The polynomial depends only on $k$ and the rank of $f$.

Proof. Let $f \in F^n$ and let $s_1, \ldots, s_n \in T(C^n \cup \{\circ\})$. By Lemma 6.4.3 there exists $c_1 \in \mathbb{N}$ depending on $k$ and $\text{rk}(f)$ such that

$$G_k(f(s_1, \ldots, s_n)) \leq m^{c_1} \quad (6.1)$$

if $\sum_i G_k(s_i) \leq m$ and $m$ is large enough. By Lemma 6.4.4, there exists a constant $c_2$ (depending on the rank of the function symbols in $s_i$) such that $G_k(s_i) \leq c_2 \cdot \text{size}(s_i)^2$. Replacing $m$ in (6.1) by $c_2 \cdot (\sum_i \text{size}(s_i))^2$ and setting $c = c_2^{c_1}$ yields:

$$G_k(f(s_1, \ldots, s_n)) \leq [c_2 \cdot (\sum_i \text{size}(s_i))^2]^{c_1} = c \cdot (\sum_i \text{size}(s_i))^{2c_1}$$

\hfill \Box

### 6.5 Predicative Interpretation

The purpose of this section is to prove our main theorem. Let $R$ denote a completely defined, constructor TRS. We embed the order $\succ_{\text{pop}}$ into $\triangleright_k$ such that $k$ depends only on $R$. This becomes possible if we represent the information on normal and safe arguments underlying the definition of $\succ_{\text{pop}}$ explicitly by interpreting the signature $F$ in the normalised signature $F^n$.

Let $t$ be a term and recall that $\|t\|$ denotes its (Buchholz) norm. We represent the norm unary. Let $s$ denote a fresh nullary function symbol that is minimal in the precedence $>$ on $F^n$. We define $BN(t) = U(\|t\|)$, where $U: \mathbb{N} \to T(\{s, \circ\})$ denotes the representation of $n$ as a sequence $(s \cdots s)$ with $n$ occurrences of the constant $s$. As a direct consequence of the definition, we have: $s \succ t$ implies $BN(s) \triangleright_k BN(t)$ for any $k$. 

78
6.5 Predicative Interpretation

**Definition 6.5.1.** Let safe denote a safe mapping. A *predicative interpretation* (with respect to \(T(F, V)\)) is a pair \((S, N)\) of mappings \(S : T(F, V) \to T(F^n, V)\) and \(N : T(F, V) \to T(F^n, V)\), defined as follows:

\[
S(t) = \begin{cases} 
\emptyset & \text{if } t \in \text{Val}(R) \\
(f^n(N(s_{j_1}), \ldots, N(s_{j_p})), S(s_{i_1}), \ldots, S(s_{i_q})) & \text{if } t \notin \text{Val}(R)
\end{cases}
\]

\[
N(t) = (S(t)) \ominus \text{BN}(t)
\]

In the definition of \(S\), we assume \(t = f(s_1, \ldots, s_n)\), \(\text{nrm}(f) = \{j_1, \ldots, j_p\}\) and \(\text{safe}(f) = \{i_1, \ldots, i_q\}\). (Recall that \(\text{safe}(f) \cup \text{nrm}(f) = \{1, \ldots, n\}\).)

Note that \(N(s) \geq_k S(s)\) (and thus \(N(s) \uparrow_k S(s)\)) holds for any \(k\). Moreover, observe that for any term \(t\), we have \(\text{wd}(N(t)) = 1 + \|t\|\) which follows by a simple inductive argument. We arrive at the two main lemmas of this section.

**Lemma 6.5.1.** Let \(f(l_1, \ldots, l_n) \to r \in R\), let \(\sigma : V \to \text{Val}(R)\) be a substitution and let \(k = 2 \cdot \max\{\text{size}(r) \mid l \to r \in R\}\). If \(f(l_1, \ldots, l_n) >_{\text{pop}} r\) then \(f^n(N(l_1, \sigma), \ldots, N(l_n, \sigma)) \geq_k Q(r \sigma)\) for \(Q \in \{S, N\}\), where \(\text{nrm}(f) = \{i_1, \ldots, i_p\}\).

**Proof.** We sketch the proof plan: Instead of showing the lemma directly, one shows the following stronger property for terms \(s, t \in T(F, V)\) where \(s\) is either a value or of form \(f(s_1, \ldots, s_n)\) such that \(s_i \sigma \in \text{Val}(R)\) for all \(1 \leq i \leq n\).

\[
\text{Let } \ell = \|t\|, \text{ if } f \in D, \text{ then } s >_{\text{pop}} t \text{ implies } Q(s \sigma) >_{2\ell} (f^n(N(s_1, \sigma), \ldots, N(s_p, \sigma)) \geq_k Q(t \sigma); \text{ otherwise } N(s \sigma) >_{2\ell} N(t \sigma) \text{ holds}.
\]

Here we suppose \(\text{safe}(f) = \{p + 1, \ldots, n\}\). To show (i) one proceeds by induction on \(>_{\text{pop}}\). See [12] for the complete proof.

**Lemma 6.5.2.** Let \(l \to r \in R\), let \(\sigma : V \to \text{Val}(R)\) be a substitution, and let \(k = 2 \cdot \max\{\text{size}(r) \mid l \to r \in R\}\). If \(l >_{\text{pop}} r\) then \(Q(l \sigma) \uparrow_k Q(r \sigma)\) for \(Q \in \{S, N\}\).

**Proof.** Similar to the proof of Lemma 6.5.1 one shows the following property for terms \(s, t \in T(F, V)\) where \(s\) is either a value or of form \(f(s_1, \ldots, s_n)\) such that \(s_i \sigma \in \text{Val}(R)\) for all \(1 \leq i \leq n\).

\[
\text{Let } \ell = \|t\|. \text{ If } f \in D, \text{ then } s = f(s_1, \ldots, s_n) >_{\text{pop}} t \text{ implies (i) } f^n(N(s_1, \sigma), \ldots, N(s_p, \sigma)) >_{2\ell} S(t \sigma) \text{ and (ii) } (f^n(N(s_1, \sigma), \ldots, N(s_p, \sigma))) \ominus \text{BN}(s \sigma) >_{2\ell} N(t \sigma). \text{ Otherwise if } f \in C \text{ then } N(s \sigma) >_{2\ell} N(t \sigma) \text{ holds}.
\]

Here we suppose \(\text{safe}(f) = \{p + 1, \ldots, n\}\). To show (i) one proceeds by induction on \(>_{\text{pop}}\). See [12] for the complete proof.

From Lemmata 6.5.1 and 6.5.2 the main lemma of this section follows.

**Lemma 6.5.3.** Let \(s\) and \(t\) be terms such that \(s \uparrow t\) and let \(k = 2 \cdot \max\{\text{size}(r) \mid l \to r \in R\}\). Then \(Q(s) \uparrow_k Q(t)\) for \(Q \in \{S, N\}\).
Main Theorem. Let $\mathcal{R}$ be a finite, completely defined constructor TRS. Assume further $\mathcal{R}$ is compatible with $>_\text{pop}$. Then the induced (normal) runtime complexity is polynomial.

Proof. Let $t = f(t_1, \ldots, t_n)$ be term in $\mathcal{T}_b$ and without loss of generality let $\text{safe}(f) = \{p+1, \ldots, n\}$. We set $k = 2 \cdot \max\{\text{size}(r) \mid l \to r \in \mathcal{R}\}$. By Lemma 6.5.3 any innermost rewrite steps $t \downarrow u$ induces $S(t) \Rightarrow_k S(u)$. Thus we obtain:

$$\text{Di}_{(\mathcal{R}, \Downarrow)}(f(t_1, \ldots, t_n)) = \max\{\ell \mid \exists u \ t \Downarrow^\ell u\}$$

$$\leq \max\{\ell \mid \exists (s_1', \ldots, s'_\ell) : S(t) \Rightarrow_k s_1' \Rightarrow_k \cdots \Rightarrow_k s'_\ell\}$$

$$\leq G_k(S(f(t_1, \ldots, t_n)))$$

Next note that $S(f(t_1, \ldots, t_n)) = (f^n(N(t_1), \ldots, N(t_p)) \varnothing \ldots \varnothing)$. By Theorem 6.4.1 and the observation following Definition 6.4.3 we see that

$$G_k((f^n(N(t_1), \ldots, N(t_p)) \varnothing \ldots \varnothing)) \leq n + 1 + G_k(f^n(N(t_1), \ldots, N(t_p)))$$

Employing Lemma 6.4.3 we see (for a fixed $f$) that $n + 1 + G_k(f^n(N(t_1), \ldots, N(t_p)))$ is asymptotically bounded by a polynomial in the sum of the sizes of the arguments $N(t_1), \ldots, N(t_p)$. By definition $\text{size}(N(t_i)) = \|t_i\| \leq \text{size}(t_i)$ for all $1 \leq i \leq p$.

Hence for each term $t \in \mathcal{T}_b$, $\text{Di}_{(\mathcal{R}, \Downarrow)}(t)$ is bounded by a polynomial in the sum of the sizes of the normal argument terms of $t$. In particular, as the signature $\mathcal{F}$ is finite, the normal runtime complexity function is polynomial.

Remark 6.5.1. In the above theorem we assume a constructor TRS. It is not difficult to see that this restriction is not necessary. (Essentially one replaces the application of Lemmata 6.5.1 and 6.5.2 by the application of the properties $(\dagger)$ and $(\ddagger)$ respectively.) However, the restriction that the arguments of $f$ are in normal form is necessary. Hence we prefer the given formulation of the theorem.

6.6 Experimental Data

To prove compatibility of a given TRS $\mathcal{R}$ with recursive path orders we have to find a precedence $>$ such that the induced order is compatible with $\mathcal{R}$. When we want to orient $\mathcal{R}$ by a polynomial path order $>_\text{pop}$ we additionally require a suitable safe mapping. To automate this search we encode the constraint $s >_\text{pop} t$ into a propositional formula:

$$\tau(s >_\text{pop} t) = \tau_1(s >_\text{pop} t) \lor \tau_2(s >_\text{pop} t) \lor \tau_3(s >_\text{pop} t) \lor \tau_4(s >_\text{pop} t)$$

Here $\tau_1(\cdot)$ is designed to encode clause $(i)$ from Definition 6.3.2. Based on such an encoding, compatibility of a TRS with $>_\text{pop}$ becomes expressible as the satisfiability of the formula $\left(\bigwedge_{l \to r \in \mathcal{R}} \tau(l >_\text{pop} r)\right) \wedge P \wedge S$. Here the subformula $P$ is satisfiable if and only if all the variables $>_f,g$ (defined below) encode a strict precedence, see [147] for a suitable definition of $P$. The subformula $S$ is used to cover the additional conditions imposed on safe mappings defined in the beginning of Section 6.3.
6.6 Experimental Data

We only describe cases (2)–(4), the encoding for case (1)—the comparison using the weaker order $>$ pop—can be easily derived in a similar fashion. If $s = f(s_1, \ldots, s_n)$ we set $\tau_2(s >_{pop} t) = \bigvee_i s_i >_{pops} t$, otherwise $\tau_2(s >_{pops} t) = \bot$. For case (3) we introduce for every function symbol $f$ and argument position $i$ of $f$ the (propositional) variables $\beta_{f,i}$, such that $\beta_{f,i} = true$ represents the assertion $i \in safe(f)$. Moreover, for all function symbols $f, g$ we introduce variables $>_{f,g}$ such that truth of $>_{f,g}$ expresses that $f > g$ holds. If $s = f(s_1, \ldots, s_n)$ and $t = g(t_1, \ldots, t_m)$ for $f \in D$ with $f \neq g$, we define $\tau_3(s >_{pops} t)$ as:

$$>_{f,g} \land \bigvee_{i_0=1}^m (\tau(s >_{pops} t_{i_0}) \land \beta_{g,i_0} \land \bigwedge_{i=1, t \neq i_0} (\tau(s >_{pops} t_{i_0}) \lor (\beta_{g,i} \land (s \triangleright t_{i_0}))))$$

(For $s$, $t$ of different shape, we set $\tau_3(s >_{pops} t) = \bot$.) To deal with case (4) we follow [132]. The main idea is to describe a multiset comparison in terms of multiset covers. Formally, a multiset cover is a pair of mappings $\gamma: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ and $\varepsilon: \{1, \ldots, n\} \rightarrow \{true, false\}$ such that for all $i, j$ ($1 \leq i \leq n$, $1 \leq j \leq m$): if $\varepsilon(i) = true$ then the set $\{j \mid \gamma(j) = i\}$ is a singleton. It is easy to see that $[s_1, \ldots, s_n]$ ($\gamma = \mu$) if there exists a multiset cover $(\gamma, \varepsilon)$ such that for each $j$ there exists an $i$ with $\gamma(j) = i$ and $\varepsilon(i) = true$ implies $s_i = t_j$, while $\varepsilon(i) = false$ implies $s_i > t_j$.

Similarly we obtain $[s_1, \ldots, s_n] >_{mul} [t_1, \ldots, t_m]$ if $[s_1, \ldots, s_n]$ ($\gamma = \mu$) and $\varepsilon(i) = false$ for some $i \in \{1, \ldots, n\}$.

This definition allows an easy encoding of multiset comparisons and based on it, clause (4) of Definition 6.3.2 becomes representable (for terms $s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_m)$) as the conjunction of the following two conditions together with the assumption that there exists a suitable multiset cover $(\gamma, \varepsilon)$:

- whenever $\gamma(j) = i$ then the indicated argument positions $i$ and $j$, are either both normal or both safe,
- at least one cover is strict ($\varepsilon(i) = false$) for some normal argument position $i$ of $f$.

We introduce variables $\gamma_{i,j}$ and $\varepsilon_i$, where $\gamma_{i,j} = true$ represents $\gamma(j) = i$ and $\varepsilon_i = true$ denotes $\varepsilon(i) = true$ ($1 \leq i \leq n$, $1 \leq j \leq m$). Summing up, we set $\tau_4(s >_{pops} \gamma_{mul} t)$ ($s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_m)$) equal to:

$$\bigwedge_{i=1}^n \bigwedge_{j=1}^m \left( \gamma_{i,j} \rightarrow (\varepsilon_i \rightarrow \tau(s_i = t_j)) \land (\neg \varepsilon_i \rightarrow \tau(s_i > t_j)) \land (\beta_{f,i} \leftrightarrow \beta_{f,j}) \right)$$

$$\land \bigwedge_{j=1}^m \left( \gamma_{1,j} \land \cdots \land \gamma_{n,j} \land \bigwedge_{i=1}^n \left( \varepsilon_i \rightarrow one(\gamma_{i,1}, \ldots, \gamma_{i,m}) \right) \land \bigwedge_{i=1}^n \left( \neg \beta_{f,i} \land \neg \varepsilon_i \right) \right)$$

Here $one(\alpha_1, \ldots, \alpha_n)$ is satisfiable if and only if exactly one of the variables $\alpha_1, \ldots, \alpha_n$ is true. And if $s, t$ do not have the assumed form, we set $\tau_4(s >_{pops} \gamma_{mul} t) = \bot.$

81
We compare the polynomial path order \( \text{POP}^* \) to a restricted class of polynomial interpretations (SMC for short) \[26\] and to LMPO \[109\]. SMC refers to simple-mixed polynomial interpretations where constructor symbols are interpreted by a strongly linear (also called additive) polynomial \[26\]. Defined symbols on the other hand are interpreted by simple-mixed polynomials \[43\]. Since \( \text{POP}^* \) and LMPO are in essence syntactic restrictions of MPO we also provide a comparison to MPO. \( \text{POP}^* \) is implemented using the previously described propositional encoding; while the implementation of SMC rests on a propositional encoding of the techniques described in \[43\]. To check satisfiability we employ MiniSat.\(^3\) LMPO and MPO are implemented using an extension of the constraint solving technique described in \[69\], which allows us to compare different implementation techniques at the same time.

As testbed we use those TRSs from the termination problem data base version 4.0 that can be shown terminating with at least one of the tools that participated in the termination competition 2007.\(^4\) We use three different testbeds: \( T \) collects the 957 terminating TRSs from TPDB, \( TC \) collects the 449 TRSs from the TPDB that are also constructor systems, and \( TCO \) collects the 236 TRSs that are terminating, constructor based and orthogonal.\(^5\) The results of our comparisons are given in Table 6.1. The tests presented below were conducted on a small complexity analyser running single-threaded on a 2.1 GHz Intel Core 2 Duo with 1 GB of memory. For each system we used a timeout of 30 seconds.

<table>
<thead>
<tr>
<th></th>
<th>POP*</th>
<th>LMPO</th>
<th>SMC</th>
<th>MPO</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>65</td>
<td>74</td>
<td>156</td>
<td>106</td>
</tr>
<tr>
<td>Maybe</td>
<td>892</td>
<td>812</td>
<td>395</td>
<td>847</td>
</tr>
<tr>
<td>Timeout (30 sec.)</td>
<td>0</td>
<td>71</td>
<td>406</td>
<td>4</td>
</tr>
<tr>
<td><strong>TC</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>41</td>
<td>54</td>
<td>83</td>
<td>65</td>
</tr>
<tr>
<td>Maybe</td>
<td>408</td>
<td>372</td>
<td>271</td>
<td>381</td>
</tr>
<tr>
<td>Timeout (30 sec.)</td>
<td>0</td>
<td>23</td>
<td>95</td>
<td>3</td>
</tr>
<tr>
<td><strong>TCO</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>19</td>
<td>25</td>
<td>38</td>
<td>29</td>
</tr>
<tr>
<td>Maybe</td>
<td>217</td>
<td>201</td>
<td>147</td>
<td>207</td>
</tr>
<tr>
<td>Timeout (30 sec.)</td>
<td>0</td>
<td>10</td>
<td>51</td>
<td>0</td>
</tr>
<tr>
<td><strong>Average yes time (milliseconds)</strong></td>
<td>15</td>
<td>14</td>
<td>1353</td>
<td>10</td>
</tr>
</tbody>
</table>

Some comments: What is noteworthy is the good performance of \( \text{POP}^\ast \) as a direct

\(^3\) Available online at \( \text{http://minisat.se} \).

\(^4\) These 957 systems can be found online: \( \text{http://www.lri.fr/~marche/termination-competition/2007/webform.cgi?command=trs&file=trs-standard.db&timelimit=120} \).

\(^5\) The main reason for this delineation is that in related work \[26\] \[109\] confluent constructor TRS are considered.
6.7 An Application: Complexity of Scheme Programs

**termination** method in comparison to MPO. It is well-known that MPO implies primitive recursive derivation length, cf. [77]. In contrast to this POP* implies polynomial runtime complexity and is thus a much weaker order. Still more than half of the TRSs compatible with MPO are also compatible with POP*. On the other hand the comparison between POP* and LMPO is quite favourable for our approach. Compatibility with LMPO tells us that the given TRS is (in principle) polytime computable, while compatibility with POP* tells additionally that the runtime of a straightforward implementation (using an innermost strategy) is polytime computable. Hence compatibility with POP* provides us with a theoretical stronger result, while the difference on the experimental data appears negligible.

The good performance of SMC in strength is a clear indication that currently (restrictions of) semantic termination techniques (like polynomial interpretations) are of some interest in automatically estimating the runtime complexity of TRSs. This may be surprising, as for additive polynomial interpretations it is (almost) trivial to check that the induced upper bound on the derivation height is polynomial. However, the significant increase in the time necessary to find an additive polynomial interpretation, as indicated in Table 6.1 clearly shows the limits of semantic methods for large examples.

6.7 An Application: Complexity of Scheme Programs

In recent work together with Hirokawa and Middeldorp (see [13]) we study the runtime complexity of (a subset of) Scheme programs by a translation into so-called S-expression rewrite systems (SRS for short). By designing the translation to be complexity preserving, the complexity of the initial Scheme program can be estimated by analysing the complexity of the resulting SRS. Here we indicate how our main theorem is applicable to (a subset of) S-expression rewrite systems, cf. [141].

**Definition 6.7.1.** Let \( K \) be a set of constants, \( V \) be a set of variables such that \( V \cap K = \emptyset \), and \( \circ \notin K \cup V \) a variadic function symbol. We define the set \( S(K, V) \) of S-expressions built from \( K \) and \( V \) as \( T(K \cup \{\circ\}, V) \). We write \((c v_1 \ldots v_n)\) instead of \( \circ(s_1, \ldots, s_n) \). An S-expression rewrite system (SRS for short) is a TRS with the property that the left- and right-hand sides of all rewrite rules are S-expressions.

Let \( S \) be an SRS over \( S(K, V) \) and let \( K = D \cup C \) such that \( D \cap C = \emptyset \). We call the elements of \( C \) constructor constants and the elements of \( D \) defined constants. We momentarily redefine the notion of value in the context of SRSs. The set of values \( \text{Val}(S) \) of \( S \) with respect to \( C \) is inductively defined as follows:

(i) if \( v \in K \) then \( v \in \text{Val}(S) \),

(ii) if \( v_1, \ldots, v_n \in \text{Val}(S) \) and \( c \in C \) then \( (c v_1 \ldots v_n) \in \text{Val}(S) \).

Observe that (defined) constants are values, this reflects that in Scheme procedures are values, cf. [135] and allows for a representation of higher-order programs. Scheme programs are conceivable as SRSs allowing conditional if expressions in conjunction with
an eager, i.e., innermost rewrite strategy. Thus we can delineate a class of SRSs that
easily accommodate a suitably large subset of Scheme programs.

**Definition 6.7.2.** \( S \) is called a **constructor** if, for every \( l \rightarrow r \in S \), \( l = (l_0 \cdots l_n) \) with
\( l_0 \in D \) and \( l_i \in \text{Val}(S) \) for all \( i \in \{1, \ldots, n\} \). (Here the set of values \( \text{Val}(S) \) is defined
with respect to \( C \).)

**Corollary 6.7.1.** Let \( > \) denote a precedence on \( K \) such that for all \( f \in D \) we have
for all \( c \in C \): \( f > c \) and let \( >_{\text{pop}}^* \) denote the induced \( \text{POP}^* \). Let \( S \) be a constructor
SRS compatible with \( >_{\text{pop}}^* \). Then for all \( f \in D \) of arity \( n \) and for all values \( s_1, \ldots, s_n \):
\( D_l(S, i \rightarrow)(\langle f \rangle s_1 \ldots s_n) \) is bounded by a polynomial in the sum of the sizes of the arguments
\( s_1, \ldots, s_n \).

**Proof.** It is important to note that the set of S-expressions \( S(K, V) \) equals \( T(K \cup \{\circ\}, V) \),
i.e., SRSs are **first-order** rewrite systems, whose single defined symbol is the variadic
function symbol \( \circ \).

Hence Theorem 6.7.1 follows almost immediately from Corollary 6.3.1. However the
fact that according to the above definition values may contain defined symbol need to
be taken into account. For that is suffices to redefine Definition 6.5.1 in the natural way.
It is not difficult to argue that suitable adaption of Lemmata 6.5.1 and 6.5.2 to SRSs
are provable.

### 6.8 Conclusion

In this paper we have introduced a restriction of the multiset path order, called **polyno-
mial path order** (\( \text{POP}^* \) for short). Our main result states that \( \text{POP}^* \) induc
polynomial runtime complexity. In Section 6.6 we have provided evidence that our approach
performs well in comparison to related methods. In Section 6.7 the necessary theory to
apply our main theorem in the context of (higher-order) functional languages with eager
evaluations has been developed. In related work (together with Hirokawa and Middeldorpf),
studying the termination behaviour and the runtime complexity of (a subclass of
higher-order) Scheme programs, this basis has proven quite useful, cf. [13].

In concluding we also want to mention that as an easy corollary to our main the-
orem we obtain that \( \text{POP}^* \) also characterises the polytime computable functions. To
be precise the polytime computable functions are exactly the functions computable by
an orthogonal constructor TRS (based on a simple signature) compatible with \( \text{POP}^* \).
(Here **simple** signature means that the size of any constructor term depends linearly on
its depth, an equivalent restriction is necessary in [109].) See [12] for details.

In future work we will strengthen the applicability of our method. The experimental
evidence presented in Section 6.6 shows that compatibility of rewrite systems with \( \text{POP}^* \)
can be easily and quickly tested. However, the strength of the method seems to be impro-
vable. One possible field of future work is to extend \( \text{POP}^* \) to quasi-precedences. The
theoretical changes necessary to accomodate quasi-precedences seem to be manageable.
Another natural extension is to combine \( \text{POP}^* \) with the transformation technique of
semantic labeling, cf. [150]. It is easy to see that semantic labeling (in the basic form)
does not affect the derivation length. Furthermore for finite models the main theorem remains directly applicable.
Proving Quadratic Derivational Complexities using Context Dependent Interpretations

Publication Details

Ranking
The International Conference on Rewriting Techniques and Applications has been ranked A by CORE in 2007.

Abstract
In this paper we study context dependent interpretations, a semantic termination method extending interpretations over the natural numbers, introduced by Hofbauer. We present two subclasses of context dependent interpretations and establish tight upper bounds on the induced derivational complexities. In particular we delineate a class of interpretations that induces quadratic derivational complexity. Furthermore, we present an algorithm for mechanically proving termination of rewrite systems with context dependent interpretations. This algorithm has been implemented and we present ample numerical data for the assessment of the viability of the method.

7.1 Introduction
In order to assess the complexity of a (terminating) term rewrite system (TRS for short) it is natural to look at the maximal length of derivation sequences, as suggested by Hof-

1 This research was partially supported by FWF (Austrian Science Fund) project P20133.
To be precise, let $\mathcal{R}$ denote a finitely branching and terminating TRS over a finite signature. The derivational complexity function with respect to $\mathcal{R}$ (denoted as $d_\mathcal{R}$) relates the length of the longest derivation sequence to the size of the initial term. For direct termination techniques it is often possible to infer an upper bound on $d_\mathcal{R}(n)$ from the termination proof of $\mathcal{R}$, cf. [79, 77, 144, 115, 56]. (Currently it is unknown how to estimate the derivational complexity of a TRS $\mathcal{R}$, if termination of $\mathcal{R}$ has been shown via transformation methods like the dependency pair method or semantic labeling, but see [115, 74] for partial results in this direction.) For example linear derivational complexity can be verified by the use of automata techniques: linear match-bounded TRSs induce linear derivational complexity, see [56]. Unfortunately such a feasible growth rate is not typical. Already termination proofs by polynomial interpretations imply a double-exponential upper bound on the derivational complexity, cf. [79]. In both cases the upper bounds are tight.

However, the tightness of the mentioned bounds does not imply that the upper bounds are always optimal. In particular polynomial interpretations typically overestimate the derivational complexity. In [78] Hofbauer introduced so-called context dependent interpretations as a remedy. These interpretations extend traditional interpretations by introducing an additional parameter. The parameter changes in the course of evaluating a term, which makes the interpretation dependent on the context. The crucial advantage is that context dependent interpretations typically improve the induced bounds on the derivational complexity of TRSs. Furthermore this technique allows the handling of non-simple terminating systems. (See [78] and Section 7.2 for further details.)

In this paper, we establish theoretical and practical extensions of Hofbauer’s approach. As theoretic contributions, we present two subclasses of context dependent interpretations, i.e., we introduce $\Delta$-linear and $\Delta$-restricted interpretations. We show that $\Delta$-linear interpretations induce exponential derivational complexity, while $\Delta$-restricted interpretations induce quadratic derivational complexity. Furthermore, we provide examples showing that these bounds are tight. In [78] it is shown that context dependent interpretations are expressive enough to show termination of TRSs that are not simply terminating. We improve upon this and show that $\Delta$-restricted interpretations suffice here. On the practical side, we design an algorithm that automatically searches for $\Delta$-linear interpretations and $\Delta$-restricted interpretations, which shows that the technique can be mechanised. This answers a question posed by Hofbauer in [78]. The procedure has been implemented and we provide ample numerical data to assess its viability. TRSs with polynomial derivational complexity appear to be of special interest. Thus, we finally compare the applicability of our method to other termination techniques that also induce polynomial derivational complexity.

The remainder of this paper is organised as follows. In the next section we recall basic notions and starting points of this paper. In Section 7.3 we introduce the class of $\Delta$-linear interpretations and describe the algorithm that mechanises the search for $\Delta$-linear and $\Delta$-restricted interpretations. In Section 7.4 we obtain the mentioned results on the derivational complexities induced by either of these interpretations. Furthermore, we show in this section that already $\Delta$-restricted interpretations allow the treatment of
non-simple terminating TRSs. Section 7.5 provides experimental data and finally in Section 7.6 we conclude and mention future work.

7.2 Context Dependent Interpretations

We assume familiarity with the basics of term rewriting, see [15] [137]. Knowledge of context dependent interpretations [78] will be helpful. Below we recall the basic results from the latter paper in a slightly different, but equivalent way, compare [78, 131]. See [78] for the motivation and intuition underlying the introduced concepts.

Let \( \mathcal{F} \) be a finite signature, let \( \mathcal{V} \) be a set of variables and let \( \mathcal{R} \) denote a terminating TRS over \( \mathcal{F} \). The induced relation \( \rightarrow_{\mathcal{R}} \) is assumed to be finitely branching. We simply write \( \rightarrow \) for \( \rightarrow_{\mathcal{R}} \) if \( \mathcal{R} \) is clear from context. The derivation length of a term \( t \) with respect to \( \mathcal{R} \) is defined as follows: \( \text{dl}_{\mathcal{R}}(t) = \max\{n \mid \exists u \ t \rightarrow^n u\} \). The derivational complexity (with respect to \( \mathcal{R} \)) is defined as: \( \text{dc}_{\mathcal{R}}(n) = \max\{\text{dl}_{\mathcal{R}}(t) \mid |t| \leq n\} \), where \(|t|\) denotes the size of \( t \), i.e., the number of symbols of \( t \) as usual. (For example the size of the term \( f(a, x) \) is 3.) We say the derivational complexity of \( \mathcal{R} \) is linear, quadratic, double-exponential, if \( \text{dc}_{\mathcal{R}}(n) \) is bounded by a linear, quadratic, double-exponential function in \( n \), respectively.

A context dependent \( \mathcal{F} \)-algebra (CDA for short) \( \mathcal{C} \) is a family of \( \mathcal{F} \)-algebras over the reals parametrised by a set \( D \subseteq \mathbb{R}^+ \) of positive reals. A CDA \( \mathcal{C} \) associates to each function symbol \( f \in \mathcal{F} \) of arity \( n \), a collection of \( n+1 \) mappings: \( f_{\mathcal{C}} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) and \( f_{\mathcal{C}}^i : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) for all \( 1 \leq i \leq n \). As usual \( f_{\mathcal{C}} \) is called interpretation function, while the mappings \( f_{\mathcal{C}}^i \) are called parameter functions. In addition \( \mathcal{C} \) is equipped with a set \( \{>_{\Delta} \mid \Delta \in D\} \) of proper orders, where we define: \( z >_{\Delta} z' \) if and only if \( z - z' > \Delta \).

Let \( \mathcal{C} \) be a CDA and let a \( \Delta \)-assignment denote a mapping: \( \alpha : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \). We inductively define a mapping \( [\alpha, \Delta]_{\mathcal{C}} \) from the set of terms into the set \( \mathbb{R}_0^+ \) of non-negative reals:

\[
[\alpha, \Delta]_{\mathcal{C}}(t) := \begin{cases} 
\alpha(\Delta, t) & \text{if } t \in \mathcal{V} \\
 f_{\mathcal{C}}(\Delta, [\alpha, f_{\mathcal{C}}^1(\Delta)]_{\mathcal{C}}(t_1), \ldots, [\alpha, f_{\mathcal{C}}^n(\Delta)]_{\mathcal{C}}(t_n)) & \text{if } t = f(t_1, \ldots, t_n). 
\end{cases}
\]

We fix some notational conventions: Due to the special role of the additional variable \( \Delta \), we often write \( f_{\mathcal{C}}[\Delta](z_1, \ldots, z_n) \) instead of \( f_{\mathcal{C}}(\Delta, z_1, \ldots, z_n) \). Furthermore, we usually denote the evaluation of \( t \) as \( [\alpha, \Delta](t) \), if the respective algebra is clear from context.

We say that a CDA \( \mathcal{C} \) is \( \Delta \)-monotone if for all \( \Delta \in D \) and for all \( a_1, \ldots, a_n, b \in \mathbb{R}_0^+ \) with \( a_i > f_{\mathcal{C}}^i(\Delta) b \) for some \( i \in \{1, \ldots, n\} \), we have

\[
f_{\mathcal{C}}[\Delta](a_1, \ldots, a_i, \ldots, a_n) >_{\Delta} f_{\mathcal{C}}[\Delta](a_1, \ldots, b, \ldots, a_n).
\]

Note that if all interpretation functions \( f_{\mathcal{C}}[\Delta] \) are weakly monotone with respect to the standard ordering on \( \mathbb{R}_0^+ \), then validity of the inequalities

\[
f_{\mathcal{C}}[\Delta](z_1, \ldots, z_i + f_{\mathcal{C}}^i(\Delta), \ldots, z_n) - f_{\mathcal{C}}[\Delta](z_1, \ldots, z_i, \ldots, z_n) \geq \Delta,
\]
suffices in order to conclude \( \Delta \)-monotonicity of \( \mathcal{C} \), cf. [78].

A CDA \( \mathcal{C} \) is compatible with a TRS \( \mathcal{R} \) (or \( \mathcal{R} \) is compatible with \( \mathcal{C} \)) if for every rewrite rule \( l \rightarrow r \in \mathcal{R} \), every \( \Delta \in D \), and any assignment \( \alpha : [\alpha, \Delta](l) >_{\Delta} [\alpha, \Delta](r) \) holds.
As running example, we consider the TRS $R_1$ with the single rewrite rule $a(b(x)) \rightarrow b(a(x))$. We assume $D = \mathbb{R}^+$. The following interpretation and parameter functions
\[
\begin{align*}
ac[\Delta](z) &= (1 + \Delta)z \\
ac^1(\Delta) &= \frac{\Delta}{1 + \Delta} \\
b_c[\Delta](z) &= z + 1 \\
b^1_c(\Delta) &= \Delta
\end{align*}
\]
define a CDA $C$ that is $\Delta$-monotone and compatible with $R_1$, compare [78].

Theorem 7.2.1 ([78]). Let $R$ be a TRS and suppose that there exists a $\Delta$-monotone and compatible CDA $C$. Then $R$ is terminating and
\[
\text{dl}_{R_1}(t) \leq \inf_{\Delta \in D} \frac{[\alpha, \Delta](t)}{\Delta}
\]
holds for all terms $t \in T(F, V)$.

The next example clarifies the impact of Theorem 7.2.1, compare [78].

Example 7.2.2. Consider the TRS $R_1$ together with the CDA $C$ in Example 7.2.1. Suppose $c \in F$ is a constant and $c_C[\Delta] = 0$. We assert $D = \mathbb{R}^+$. Then we obtain $[\alpha, \Delta](a^n(b^m(c))) = (1 + \Delta)m$ and hence:
\[
\inf_{\Delta > 0} \frac{[\alpha, \Delta](a^n(b^m(c)))}{\Delta} = \inf_{\Delta > 0} \left( \frac{1}{\Delta} + n \right)m = nm \geq \text{dl}_{R_1}(a^n(b^m(c))).
\]

Furthermore, an easy inductive argument reveals: $\text{dl}_{R_1}(a^n(b^m(c))) = nm$. Hence with respect to the term $a^n(b^m(c))$, compatibility with $C$ entails an optimal upper bound on the derivation length of $R_1$. This is also true for all ground terms. A proof of $\inf_{\Delta > 0} \frac{[\alpha, \Delta](t)}{\Delta} = \text{dl}_{R_1}(t)$ for all $t \in T(F)$ can be found in [78].

Definition 7.2.1. A $\Delta$-quotient is an expression of the form
\[
\frac{\Delta}{a + b\Delta},
\]
where $a, b \in \mathbb{N}$ and either $a > 0$ or $b > 0$. A $\Delta$-quotient $d$ is nontrivial, if $d \neq \Delta$.

Lemma 7.2.1. Let $d_1, d_2$ be $\Delta$-quotients and let $d = d_1[\Delta := d_2]$ denote the result of substituting $d_2$ for $\Delta$ in $d_1$. Then $d$ is a $\Delta$-quotient.

As usual a polynomial $P$ in the variables $z_1, \ldots, z_n$ (over the reals) is a finite sum $\sum_{i=1}^{m} c_i z_1^{n_1} \cdots z_n^{n_n}$. To accommodate $\Delta$-quotients we slightly generalise polynomials.

Definition 7.2.2. An extended monomial $M$ in the variables $\Delta$ and $z_1, \ldots, z_n$ is a finite product $c \cdot \prod v_i$ such that $c$ is an integer and $v_i$ is $x^n$, $x \in \{\Delta, z_1, \ldots, z_n\}$ or $v_i$ is a $\Delta$-quotient. The integer $c$ is called the coefficient and the expression $v_i$ a literal. Finally, an extended polynomial $P$ over $\Delta \in D$ and $z_1, \ldots, z_n \in \mathbb{R}^+_0$ is a finite sum $\sum_i M_i$ of extended monomials $M_i$ (in $\Delta$ and $z_1, \ldots, z_n$).
Note that the coefficients of an extended polynomial are integers. If the context clarifies what is meant, we will drop the qualifier “extended”. Examples 7.2.1 and 7.2.2 as well as the examples studied in [78] suggest a restricted notion of context dependent algebras. This is the subject of the next definition.

**Definition 7.2.3.** A polynomial context dependent interpretation of \( \mathcal{F} \) is a CDA \((\mathcal{C}, \{\Delta\mid \Delta \in D\})\) satisfying the following properties:

- the interpretation function \( f_\mathcal{C} \) is an extended polynomial,
- the parameter set \( D \) equals \( \mathbb{R}^+ \), and
- for each \( f \in \mathcal{F} \) the parameter functions \( f_\mathcal{C}^i \) are \( \Delta \)-quotients.

**Lemma 7.2.2.** Let \( \mathcal{C} \) denote a polynomial context dependent interpretation, let \( \alpha \) be a \( \Delta \)-assignment, and let \( t \) be a term. Then \( [\alpha, \Delta](t) \) is an extended polynomial.

**Proof.** The lemma is a direct consequence of the definitions and Lemma 7.2.1.

**Remark 7.2.1.** Hofbauer showed in [78] that for any monotone polynomial interpretation compatible with a TRS \( \mathcal{R} \), there exists a polynomial context dependent interpretation which is \( \Delta \)-monotone and compatible with \( \mathcal{R} \) and induces at least the same upper bound on the derivational complexity as the polynomial interpretation.

### 7.3 Automated Search for Context Dependent Interpretations

One approach to find context dependent interpretations (semi-)automatically was already mentioned in Hofbauer’s paper [78]. A given polynomial interpretation is suitably lifted to a context dependent interpretation such that monotonicity and compatibility are preserved, but the upper bound on the derivational complexity is often improved. Unfortunately, experimental evidence suggests that the applicability of this heuristics is limited, if one is interested in automatically finding complexity bounds, see Section 7.5 for further details. However, the standard approach for automatically proving termination via polynomial interpretations as stipulated by Contejean et al. [43] can be adapted. The description of this adaption is the topic of this section. We restrict the form of parametric interpretations that we consider.

**Definition 7.3.1.** A (parametric) \( \Delta \)-linear interpretation is a polynomial context dependent interpretation \( \mathcal{C} \) whose interpretation functions and parameter functions have the following form:

\[
 f_\mathcal{C}(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_{(f,i)}z_i + \sum_{i=1}^{n} b_{(f,i)} z_i \Delta + c_f \Delta + d_f \\
 f_\mathcal{C}^i(\Delta) = \frac{\Delta}{a_{(f,i)} + b_{(f,i)} \Delta}
\]
7 Proving Quadratic Derivational Complexities using Context Dependent Interpretations

where the occurring coefficients are supposed to be natural numbers. For a parametric \(\Delta\)-linear interpretation, \(a_{(f,i)}\), \(b_{(f,i)}\), \(c_f\), and \(d_f\) \((f \in \mathcal{F}, 1 \leq i \leq n)\) are called **coefficient variables**.

Note that for any \(\Delta\)-linear interpretation, we have \(a_{(f,i)} > 0\) or \(b_{(f,i)} > 0\) \((f \in \mathcal{F}, 1 \leq i \leq n)\): Any \(\Delta\)-linear interpretation is a polynomial context dependent interpretation by definition. And hence the parameter functions have to be \(\Delta\)-quotients, cf. Definition 7.2.3. Moreover the coefficients \(a_{(f,i)}\), \(b_{(f,i)}\) are used in the interpretation function and the parameter functions. This is necessary for the correctness of Lemma 7.3.1 below.

**Example 7.3.1.** Consider the TRS \(R_1\) from Example 7.2.1. The parametric interpretation and parameter functions have the form:

\[
a_C(\Delta)(z) = az + bz + c\Delta + d \\
b_C(\Delta)(z) = ez + fz + g\Delta + h
\]

\[
a_1(\Delta) = \frac{\Delta}{a + b\Delta} \\
b_1(\Delta) = \frac{\Delta}{e + f\Delta}.
\]

The following lemma is a direct consequence of the definitions.

**Lemma 7.3.1.** Let \(\mathcal{C}\) be an \(\Delta\)-linear interpretation. Then \(\mathcal{C}\) is \(\Delta\)-monotone.

Due to Lemma 7.3.1 in order to prove termination of a given TRS \(R\), it suffices to find a \(\Delta\)-linear interpretation compatible with \(R\). This observation is reflected in the following definition.

**Definition 7.3.2.** Let \(R\) be a TRS and let \(\mathcal{C}\) be a parametric \(\Delta\)-linear interpretation. The **compatibility constraints** of \(R\) with respect to \(\mathcal{C}\) are defined as

\[
CC(R, \mathcal{C}) = \left\{ [\alpha, \Delta](l) - [\alpha, \Delta](r) - \Delta \geq 0 \mid l \rightarrow r \in R \right\} \cup \bigcup \{ a_{(f,i)} + b_{(f,i)} - 1 \geq 0 \mid f \in \mathcal{F}, 1 \leq i \leq \ar(f) \}.
\]

Here \(\ar(f)\) denotes the arity of \(f\) and \(\alpha\) refers to a **symbolic \(\Delta\)-assignment**: Expressions of the form \([\alpha, \Delta](x)\) for \(x \in \mathcal{V}\) remain unevaluated.

While the first half of \(CC(R, \mathcal{C})\) represents compatibility with \(R\), the second set of constraints guarantees that the denominators of the occurring \(\Delta\)-quotients are different from 0. Thus any solution to \(CC(R, \mathcal{C})\), instantiating coefficients with natural numbers, represents a polynomial context dependent interpretation compatible with \(R\).

**Example 7.3.2.** Consider the (parametric) CDA \(\mathcal{C}\) from Example 7.3.1 and set \(\Delta_1 = a_1(\Delta)\) and \(\Delta_2 = b_1(\Delta)\). Let \(\alpha_1 = [\alpha, \Delta_1][\Delta := \Delta_1](x)\) and let \(\alpha_2 = [\alpha, \Delta_2][\Delta := \Delta_2](x)\). Then the constraint \([\alpha, \Delta](a(b(x))) - [\alpha, \Delta](b(a(x))) - \Delta \geq 0\) becomes:

\[
(ace_1 + af\alpha_1\Delta_1 + ag\Delta_1 + be\alpha_1\Delta_1 + bf\alpha_1\Delta_1 + bg\Delta_1 + (bh + c)\Delta + +ah + d) - (ace_2 + be\omega_2\Delta_2 + ce\Delta_2 + af\alpha_2\Delta_2 + bf\alpha_2\Delta_2 + cf\Delta_2 + + (df + g)\Delta + de + h) - \Delta \geq 0.
\]
For all constraints \((P \geq 0) \in CC(R, C)\), \(P\) is an extended polynomial, cf. Lemma 7.2.2. It is easy to see how an extended polynomial (over \(\Delta, z_1, \ldots, z_n\)) is transferable into a (standard) polynomial (over \(\Delta, z_1, \ldots, z_n\)): Multiply (symbolically) with denominators of (nontrivial) \(\Delta\)-quotients till all (nontrivial) \(\Delta\)-quotients are eliminated. This simple procedure is denoted as \(A\). Correctness and termination of the procedure follow trivially.

**Definition 7.3.3.** Let \(R\) be a TRS and let \(C\) be a parametric \(\Delta\)-linear interpretation. The **polynomial compatibility constraints** of \(R\) with respect to \(C\) are defined as follows:

\[
PCC(R, C) := \{P' \geq 0 \mid P \geq 0 \in CC(R, C)\}.
\]

**Example 7.3.3.** Consider the constraint \(P \geq 0\) depicted in Example 7.3.2. To apply the algorithm \(A\) we first have to symbolically multiply with the expression \(a + b\Delta\) and later with \(e + f\Delta\). The resulting constraint \(P' \geq 0\) (with the polynomial \(P'\) in the “variables” \(\Delta, \alpha_1, \text{and } \alpha_2\)) has the form:

\[
\begin{align*}
&((b^2ef + b^2f)\alpha_1\Delta^3 + (2abe f + af^2 + b^2e^2 + bef)\alpha_1\Delta^2 \\
&\quad + (2abef + a^2ef + aef)\alpha_1\Delta + (a^2e^2)\alpha_1) \\
&\quad - ((ab^2)\Delta^3 + (a^2f^2 + 2abef + af + b^2e)\alpha_2\Delta^2 \\
&\quad + (a^2ef + a^2e + abe)\alpha_2\Delta + (a^2e^2)\alpha_2) + ((b^2fh - bdf^2 - bf)\Delta^3 \\
&\quad + (2abfh + b^2ch + bdf - adf^2 - 2bdef - bfh - be - af)\Delta^2 \\
&\quad + (a^2fh + 2abch + adf + bde - 2adef - afh - bde^2 - beh - ae))\Delta \\
&\quad + (a^2eh + ade - ade^2 - ach)) \geq 0.
\end{align*}
\]

We obtain \(PCC(R_1, C) = \{P' \geq 0, a + b - 1 \geq 0, e + f - 1 \geq 0\}\), where the last two constraints reflect that all denominators of \(\Delta\)-quotients are non-zero.

Let \(P \geq 0\) be a constraint in \(PCC(R, C)\) such that \(n\) distinct symbolic assignments \([\alpha, d](x)\) occur in \(P\) (\(x \in V\), \(d\) a \(\Delta\)-quotient). (In Example 7.3.3 two symbolic assignments occur: \(\alpha_1\) and \(\alpha_2\).) Then \(P\) is conceivable as a polynomial in \(\mathbb{Z}[\Delta, z_1, \ldots, z_n]\). It remains to verify that (a suitable instance of) \(P\) is positive, i.e., we have to prove that \(P(\Delta, z_1, \ldots, z_n) \geq 0\) for any values \(\Delta > 0, z_i \geq 0\). This is achieved by testing for absolute positivity instead of positivity, compare [93].

A polynomial \(P\) is **absolutely positive** if \(P\) has non-negative coefficients only. A parametric polynomial \(P\) is called absolutely positive if there exists an instance \(P'\) of \(P\) such that \(P'\) is absolutely positive. Clearly any absolutely positive polynomial is positive. Thus for a given constraint \(P \geq 0 \in PCC(R, C)\) it suffices to find instantiations of the coefficient variables such that all coefficients are natural numbers. This is achieved through the construction of suitable Diophantine inequalities over the coefficients.

**Lemma 7.3.2.** Let \(R\) be a TRS and let \(C\) denote a parametric \(\Delta\)-linear interpretation. If for all \(P \geq 0 \in PCC(R, C)\), \(P\) is absolutely positive then there exists an instantiation of \(C\) compatible with \(R\).

**Proof.** If \(P\) is absolutely positive, there exist natural numbers that can be substituted to the coefficient variables in \(P\) such that the resulting polynomial \(P'\) is absolutely positive
and thus positive. By definition this implies that the constraints in $CC(\mathcal{R}, V)$ are fulfilled. We define an instantiation $\mathcal{C}'$ of $\mathcal{C}$ by applying the same substitution to the coefficient variables in $\mathcal{C}$. Then $\mathcal{C}'$ is compatible with $\mathcal{R}$.  

As an immediate consequence of Lemmata 7.3.1, 7.3.2 and Theorem 7.2.1 we obtain the following theorem.

**Theorem 7.3.1.** Let $\mathcal{R}$ be a TRS and let $\mathcal{C}$ denote a parametric $\Delta$-linear interpretation. Suppose for all $P \geq 0 \in PCC(\mathcal{R}, \mathcal{C})$, $P$ is absolutely positive. Then $\mathcal{C}'$ is compatible with $\mathcal{R}$.

It is easy to see that the Diophantine inequalities induced by Example 7.3.3 cannot be solved, if the symbolic assignments $\alpha_1$ and $\alpha_2$ are treated as different variables. This motivates the next definition.

**Definition 7.3.4.** Given a TRS $\mathcal{R}$ and a $\Delta$-linear interpretation $\mathcal{C}$, the equality constraints of $\mathcal{R}$ with respect to $\mathcal{C}$ are defined as follows:

$$EC(\mathcal{R}, \mathcal{C}) = \{(a + b\Delta) - (c + d\Delta) = 0 \mid \text{Property (\ast) is fulfilled}\}$$

(\ast) There exists $P \geq 0 \in PCC(\mathcal{R}, \mathcal{C})$, $x \in V$ such that $[\alpha, d_1](x)$ and $[\alpha, d_2](x)$ occur in $P$ and $d_1 = \frac{\Delta}{a + b\Delta} \neq \frac{\Delta}{c + d\Delta} = d_2$.

**Example 7.3.4.** Consider Example 7.3.3. Property (\ast) is applicable to the $\Delta$-quotients $d_1$, $d_2$ in the $\Delta$-assignments $\alpha_1 = [\alpha, d_1]$ and $\alpha_2 = [\alpha, d_2]$ as

$$d_1 = \frac{\Delta}{ae + (be + f)\Delta} \neq \frac{\Delta}{ae + (af + b)\Delta} = d_2.$$

Thus the constraint $(ae + (be + f)\Delta) - (ae + (af + b)\Delta) = 0$ occurs in $EC(\mathcal{R}_1, \mathcal{C})$. This is the only constraint in $EC(\mathcal{R}_1, \mathcal{C})$.

Let $P \geq 0 \in PCC(\mathcal{R}, \mathcal{C})$, assume the equality constraints in $EC(\mathcal{R}, \mathcal{C})$ are fulfilled and assume we want to test for absolute positivity of $P$. By assumption distinct symbolic assignments can be treated as equal, which may change the coefficients we need to consider in $P$. This is expressed by writing $P \geq 0 \in PCC(\mathcal{R}, \mathcal{C}) \cup EC(\mathcal{R}, \mathcal{C})$. Furthermore, we call a parametric polynomial a zero polynomial if there exists an instance $P'$ of $P$ such that $P' = 0$.

**Corollary 7.3.1.** Let $\mathcal{R}$ be a TRS and let $\mathcal{C}$ denote a parametric $\Delta$-linear interpretation. Suppose for all $P \geq 0 \in PCC(\mathcal{R}, \mathcal{C}) \cup EC(\mathcal{R}, \mathcal{C})$, $P$ is absolutely positive ($P$ is a zero polynomial). Then $\mathcal{R}$ is terminating and property (7.1) holds for $D = \mathbb{R}^+$.  

Corollary 7.3.1 opens the way to efficiently search for CDAs: Finding a $\Delta$-monotone and compatible CDA $\mathcal{C}$ amounts to solving the Diophantine constraints in $PCC(\mathcal{R}, \mathcal{C}) \cup EC(\mathcal{R}, \mathcal{C})$. Recall that solvability of Diophantine constraints is undecidable [113]. However, there is an easy remedy for this: we restrict the domain of the coefficient variables to a finite one.
Example 7.3.5. Consider the TRS $R_1$ from Example 7.2.1 and the $\Delta$-linear interpretation $C$ from Example 7.3.1. Applying the above described algorithm, the following Diophantine (in)equality need to be solved.

\[
\begin{align*}
    b^2ef + bf^2 - abf^2 - b^2f & \geq 0 \\
    a^2e^2 + a^2ef + aef - 2a^2ef - aeb & \geq 0 \\
    b^2fh - bdf^2 - bf & \geq 0 \\
    a^2eh + ade - ade^2 - aeh & \geq 0 \\
    a + b - 1 & \geq 0 \\
    e + f - 1 & \geq 0 \\
    be + f - af - b = 0 \\
    af^2 + b^2e^2 + be - a^2f^2 - abf - b^2e & \geq 0 \\
    2abfh + b^2eh + bdf - adf^2 - 2bdef - bfh - be - af & \geq 0 \\
    a^2fh + 2abeh + adf + bde - 2ade - afh - bde^2 - beh - ae & \geq 0 .
\end{align*}
\]

Here the constraints $a + b - 1 \geq 0$, $e + f - 1 \geq 0$ guarantee that the denominators of occurring $\Delta$-quotients are positive, and the equality $be + f - af - b = 0$ expresses the equality constraint in $EC(R_1, C)$. Our below discussed implementations of the algorithm presented in this section find the following satisfying assignments for the coefficient variables fully automatically:

\[
a = b = e = h = 1 \quad c = d = f = g = 0 .
\]

7.4 Derivational Complexities Induced by Polynomial Context Dependent Interpretations

In this section we show that the derivational complexity induced by $\Delta$-linear interpretations is exponential and that this bound is tight. Furthermore, we introduce a restricted subclass of $\Delta$-linear interpretations that induces (tight) quadratic derivational complexity.

Recall the TRS $R_1$ considered in Example 7.2.1. This TRS belongs to a family of TRSs $R_k$ for $k > 0$: $a(b(x)) \rightarrow b^k(a(x))$ and it is not difficult to see that for $k \geq 2$ the derivational complexity of $R_k$ is exponential. In [78] $\Delta$-linear interpretations $C_k$ were introduced such that

\[
\inf_{\Delta > 0} \frac{[\alpha, \Delta]_{C_k}(t)}{\Delta} = dl_{R_k}(t) ,
\]

holds for any ground term. I.e., for all $k > 0$ there exist $\Delta$-linear interpretations that optimally bound the derivational complexities of $R_k$. This triggers the question whether we can find such context dependent interpretations automatically. The next example answers this question affirmatively, for $k = 2$.

Example 7.4.1. Consider the TRSs $R_2$: $a(b(x)) \rightarrow b(b(a(x)))$. To find a $\Delta$-linear interpretation, we employ the same parametric interpretation $C$, as in Example 7.3.1 and...
build the set of constraints $CC(R_2, C)$ and consecutively the polynomial compatibility constraints $PCC(R_2, C)$ together with the equality constraints $EC(R_2, C)$. We only state the (automatically) obtained interpretation and parameter functions:

$$a_C(\Delta)(z) = (2 + 2\Delta)z \quad \frac{1}{2 + 2\Delta}$$
$$b_C(\Delta)(z) = z + 1 \quad b_1^C(\Delta) = \Delta .$$

As a consequence of Example 7.4.1 we see the existence of TRSs, compatible with $\Delta$-linear interpretations, whose derivational complexity function is exponential. Moreover, we have the following lemma.

**Lemma 7.4.1.** Let $C$ denote a $\Delta$-linear interpretation and let $K$ denote the maximal coefficient occurring in $C$. Further let $t$ be a ground term, $\alpha$ a $\Delta$-assignment and $\Delta > 0$. Then $[\alpha, \Delta](t) \leq (K + 2)|t|^{\Delta + 1}$.

**Proof.** Straightforward induction on $t$. □

**Theorem 7.4.1.** Let $R$ be a TRS and let $C$ denote a $\Delta$-linear interpretation compatible with $R$. Then $R$ is terminating and $dc_R(n) = 2^{O(n)}$. Moreover there exists a TRS $R$ such that $dc_R(n) = 2^{\Omega(n)}$.

**Proof.** The proof of the upper bound follows the pattern of the proof of Theorem 7.4.2 below. To show that this upper bound is tight, we consider the TRS $R_2$ from Example 7.4.1. It is easy to see that $dc_{R_2}(n) = 2^{\Omega(n)}$ holds. □

In order to establish a termination method that induces polynomial derivational complexity, we restrict the class of $\Delta$-linear interpretations.

**Definition 7.4.1.** A $\Delta$-restricted interpretation is a $\Delta$-linear interpretation. In addition we require that for the interpretation functions and parameter functions

$$f_C(\Delta, z_1, \ldots, z_n) = \sum_{i=1}^{n} a_{(f,i)} z_i + \sum_{i=1}^{n} b_{(f,i)} z_i \Delta + c_f \Delta + d_f$$
$$f_C^I(\Delta) = \frac{\Delta}{a_{(f,i)} + b_{(f,i)} \Delta} ,$$

we have $a_{(f,i)} \in \{0, 1\}$ for all $1 \leq i \leq n$.

**Example 7.4.2.** Consider the TRS $R_1$ from Example 7.2.1. The assignment of coefficient variables as defined in Example 7.3.5 induces a $\Delta$-restricted interpretation.

**Lemma 7.4.2.** Let $C$ denote a $\Delta$-restricted interpretation with coefficients $a_{(f,i)}$, $b_{(f,i)}$, $c_f$, $d_f$ ($f \in \mathcal{F}$, $1 \leq i \leq ar(f)$) and we set

$$M := \max \{ c_f, d_f \mid f \in \mathcal{F} \} \cup \{ 1 \}$$
$$N := \max \{ b_{(f,i)} \mid f \in \mathcal{F}, 1 \leq i \leq ar(f) \} \cup \{ 1 \} .$$

Further let $t$ be a ground term, $\alpha$ a $\Delta$-assignment and let $\Delta > 0$. Then $[\alpha, \Delta](t) \leq M(|t| + N|t|^2 \Delta)$.
7.4 Derivational Complexities Induced by Polynomial Context Dependent Interpretations

**Proof.** We proceed by induction on \( t \). As \( t \in \mathcal{T}(\mathcal{F}) \), the evaluation is independent of the assignment. Hence we write \( [\Delta](t) \) instead of \([\alpha, \Delta](t)\). If \( t = f \in \mathcal{F} \), then

\[
[\Delta](t) = c_f \Delta + d_f \leq M(\Delta + 1) \leq M(|t| + N|t|^2 \Delta).
\]

If on the other hand \( t = f(t_1, \ldots, t_n) \), then

\[
[\Delta](t) = \sum_i (a_{f_i} + b_{f_i} \Delta)[f_i^2(\Delta)](t_i) + c_f \Delta + d_f \quad (7.2)
\]

\[
\leq \sum_i (a_{f_i} + b_{f_i} \Delta)(M(|t_i| + N|t_i|^2 \frac{\Delta}{a_{f_i} + b_{f_i} \Delta}) + c_f \Delta + d_f \quad (7.3)
\]

\[
= \sum_i ((a_{f_i} + b_{f_i} \Delta)M|t_i| + MN|t_i|^2 \Delta) + c_f \Delta + d_f \quad (7.4)
\]

\[
\leq \sum_i ((1 + N\Delta)|t_i| + MN|t_i|^2 \Delta) + M(\Delta + 1) \quad (7.5)
\]

\[
\leq \sum_i |t_i|((1 + N\Delta)M + MN(|t| - 1)\Delta) + M(\Delta + 1) \quad (7.6)
\]

\[
= (|t| - 1)((1 + N\Delta)M + MN(|t| - 1)\Delta) + M(\Delta + 1) \quad (7.7)
\]

\[
= M(|t| - 1)(1 + N\Delta) + N(|t| - 1)^2 \Delta + (\Delta + 1)) \quad (7.8)
\]

\[
\leq M(|t| + N|t|^2 \Delta) \quad (7.9)
\]

In line (7.3) we employ the induction hypothesis, in (7.6) we use \(|t_i| \leq |t| - 1\) and for (7.9) a simple calculation reveals: \((|t| - 1)(1 + N\Delta) + N\Delta(|t| - 1)^2 + (\Delta + 1) = |t| + N|t|^2 \Delta + \Delta - N|t|\Delta \leq |t| + N|t|^2 \Delta\).

\[\square\]

**Theorem 7.4.2.** Let \( \mathcal{R} \) be a TRS and let \( \mathcal{C} \) denote a \( \Delta \)-restricted interpretation compatible with \( \mathcal{R} \). Then \( \mathcal{R} \) is terminating and \( \text{dc}_\mathcal{R}(n) = \mathcal{O}(n^2) \). Moreover there exists a TRS \( \mathcal{R} \) such that \( \text{dc}_\mathcal{R}(n) = \Omega(n^2) \).

**Proof.** By Theorem 7.2.1 \( \mathcal{R} \) is terminating and by Lemma 7.4.2 there exists \( K \in \mathbb{N} \), such that for any ground term \( t \): \([\Delta](t) \leq K(|t| + K|t|^2 \Delta) \leq K^2|t|^2 (\Delta + 1) \) and hence

\[
\text{dl}_\mathcal{R}(t) \leq \inf_{\Delta > 0} \frac{[\Delta](t)}{\Delta} \leq \inf_{\Delta > 0} \frac{K^2|t|^2(\Delta + 1)}{\Delta} = K^2|t|^2.
\]

We obtain \( \text{dl}_\mathcal{R}(t) = \mathcal{O}(|t|^2) \) for any \( t \in \mathcal{T} (\mathcal{F}, \mathcal{V}) \) and thus \( \text{dc}_\mathcal{R}(n) = \mathcal{O}(n^2) \). The tightness of the bound follows by Example 7.2.1.

\[\square\]

By definition the constant employed in Theorem 7.4.2 depends only on the employed interpretation functions. Moreover this dependence is linear. In concluding this section, we want to stress that \( \Delta \)-restricted interpretation are even strong enough to handle non-simple terminating TRSs.
Example 7.4.3 (7.3). Consider the TRS $R$ with the one rule $a(a(x)) \rightarrow a(b(a(x)))$. By applying the algorithm described in Section 7.3, we find the below given $\Delta$-restricted interpretation $C$ automatically:

$$
a_C[\Delta](z) = 2z\Delta + 2 \quad b_C[\Delta](z) = z\Delta \quad a_1^C(\Delta) = \frac{1}{2} \quad b_1^C(\Delta) = 1.
$$

By Theorem 7.3.1, $C$ is compatible with $R$. Hence Theorem 7.4.2 implies that the derivational complexity of $R$ is (at most) quadratic.

### 7.5 Experimental Results

In this section we describe the programs $\text{cdi}_1$, $\text{cdi}_2$, and $\text{cdi}_3$. These programs provide search procedures for context dependent interpretations. The program $\text{cdi}_1$ implements the heuristics of Hofbauer in (7.3), mentioned in Section 7.3 above. On the other hand, programs $\text{cdi}_2$ and $\text{cdi}_3$ implement the algorithm presented in Section 7.3 and incorporate constraint solvers for Diophantine (in)equalities. The program $\text{cdi}_1$ searches for $\Delta$-linear interpretations, while $\text{cdi}_2$ and $\text{cdi}_3$ can search for $\Delta$-linear and $\Delta$-restricted interpretations. We summarise further differences below:

$\text{cdi}_1$ Firstly, the program searches for a polynomial interpretation compatible with a TRS $R$. This interpretation is then lifted to a polynomial context dependent interpretation $C$ as follows: Coefficients of the form $k + 1$ are replaced by $k + \Delta$. Finally Mathematica\(^4\) is invoked to verify that the resulting CDA $C$ is $\Delta$-monotone and compatible with $R$.

$\text{cdi}_2$ This program employs a constraint propagation procedure to solve the Diophantine constraints in $\text{PCC}(R, C) \cup \text{EC}(R, C)$. Essentially the implementation follows the technique suggested in (43).

$\text{cdi}_3$ The Diophantine (in)equalities in $\text{PCC}(R, C) \cup \text{EC}(R, C)$ are translated into propositional logic and suitable assignments are found by employing a SAT solver, in our case MiniSat\(^5\) The implementation follows ideas presented in (51) and employs the plogic library of TTT2\(^6\).

The implementation of the transformation steps as described in Section 7.3, is the same for $\text{cdi}_2$ and $\text{cdi}_3$. The programs $\text{cdi}_1$, $\text{cdi}_2$, and $\text{cdi}_3$ are written in OCaml\(^7\) (and parts of $\text{cdi}_1$ in C). All three programs are fairly small: $\text{cdi}_1$ consists of about 2000 lines of code, while $\text{cdi}_2$ and $\text{cdi}_3$ use roughly 3000 lines of code each. In Table 7.1 we summarise the comparison between the different programs $\text{cdi}_1$, $\text{cdi}_2$, and $\text{cdi}_3$. The numbers in the third line of the table refer to the number of bits maximally used in $\text{cdi}_3$ to encode coefficients. Correspondingly for $\text{cdi}_2$ we used 32 as strict bound on the coefficients. We are interested

\(^4\)http://www.wolfram.com/products/mathematica/
\(^5\)http://minisat.se/
\(^6\)http://colo6-c703.uibk.ac.at/ttt2/
\(^7\)http://www.caml.inria.fr/
7.5 Experimental Results

Table 7.1: 957 terminating TRSs

<table>
<thead>
<tr>
<th></th>
<th>cdi₁</th>
<th>cdi₂</th>
<th>cdi₃</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Δ-lin.</td>
<td>Δ-restr.</td>
<td>Δ-lin.</td>
</tr>
<tr>
<td># success</td>
<td>19</td>
<td>61</td>
<td>62</td>
</tr>
<tr>
<td>average time</td>
<td>-</td>
<td>3132</td>
<td>3652</td>
</tr>
<tr>
<td># timeout</td>
<td>-</td>
<td>276</td>
<td>782</td>
</tr>
</tbody>
</table>

in automatically verifying the complexity of terminating TRSs. Consequentially, as testbed we employ those 957 TRSs from the version 4.0 of the Termination Problem Data Base (TPDB for short) that can be shown terminating with at least one of the tools that participated in the termination competition 2007 [7]. The presented tests were performed single-threaded on a 2.40 GHz Intel® Core™ 2 Duo with 2 GB of memory. For each system we used a timeout of 60 seconds, the times in the tables are given in milliseconds.

Observe that the heuristic proposed in [78] is not suitable as an automatic procedure. (We have not indicated the time spent by cdi₁ as the timing is incomparable to the stand-alone approach of cdi₂ or cdi₃.) With respect to the comparison between cdi₂ and cdi₃, the latter outperforms the former, if at least 2 bits are used. Perhaps surprisingly the performance of cdi₂ and cdi₃ on Δ-restricted and Δ-linear is almost identical. This can be explained by the strong impact of larger bounds for the coefficients $a_{(f,i)}$ ($f \in F$, $1 \leq i \leq ar(f)$) in the complexity of the issuing Diophantine (in)equalities. However, for both programs cdi₂ and cdi₃, the stronger technique gains one crucial system: Example 7.4.1.

Table 7.2 relates existing methods that induce polynomial derivational complexities of TRSs to cdi₃. SL refers to strongly linear interpretations, i.e., only interpretation functions of the form $f_A(x_1, \ldots, x_n) = \sum_i x_i + c$, $c \in \mathbb{N}$ are allowed. Clearly compatibility with strongly linear interpretations induces linear derivational complexity. Secondly, TTTbox refers to the implementation of the match-bound technique as in [98]: Linear TRSs are tested for match-boundedness, non-linear, but non-duplicating TRSs are tested for match-raise-boundedness. This technique again implies linear derivational complexity. (Employing [80] (as in [56]) one sees that any match-raise bounded TRS has linear derivational complexity. Then the claim follows from Lemma 8 in [98].) Note that the restriction to non-duplicating TRS is harmless, as any duplicating TRS induces at least exponential derivational complexity. No further termination methods that induce at most polynomial derivational complexities for TRSs have previously been known. In particular related work on implicit complexity (for example [26, 109, 111, 11, 29]) does not provide methods that induce polynomial derivational complexities, even if sometimes the derivation length can be bounded polynomially, if the set of start terms is suitably restricted. Finally cdi⁺ denotes our standard strategy: First, we search for a strongly

---

*These 957 systems and full experimental evidence can be found at [http://cl-informatik.uibk.ac.at/~aschnabl/experiments/cdi/](http://cl-informatik.uibk.ac.at/~aschnabl/experiments/cdi/).
linear interpretation. If such an interpretation cannot be found, then a $\Delta$-restricted interpretation is sought (with 5 bits as bound).

Some comments on the results reported in Table 7.2. By definition the set of TRSs compatible with a strongly linear interpretation is a (strict) subset of those treatable with $\text{cdi}^+$. On the other hand the comparison between $\text{T}T\text{Tbobx}$ and $\text{cdi}^+$ (or $\text{cdi}_3$) may appear not very favourable for our approach. However, $\text{cdi}^+$ (and $\text{cdi}_3$) can handle TRSs that cannot be handled by $\text{T}T\text{Tbobx}$. More precisely with respect to $\Delta$-restricted interpretations $\text{cdi}^+$ (and $\text{cdi}_3$) can handle 38 (37) TRSs that cannot be handled with $\text{T}T\text{Tbobx}$. For instance the following example can only be handled with $\text{cdi}^+$ (and $\text{cdi}_3$).

**Example 7.5.1.** Consider the following rewrite system $R_+$. (This is Example 2.11 in Steinbach and Kübler’s collection [136].)

\begin{align*}
0 + y &\rightarrow y \\
0 - y &\rightarrow y \\
s(x) + y &\rightarrow s(x+y) \\
s(x) &\rightarrow x
\end{align*}

It is easy to see that $R_+$ is compatible with the following (automatically generated) $\Delta$-restricted interpretation $C$.

\begin{align*}
-C[\Delta](x,y) &= x + y + 3y\Delta + 2\Delta \\
-C[\Delta](x) &= x + 2 \\
C[\Delta](x) &= x + \frac{\Delta}{1+\Delta} \\
C[\Delta](y) &= y + \frac{\Delta}{1+\Delta} \\
C[\Delta](0) &= \frac{\Delta}{1+\Delta}
\end{align*}

with parameter functions: $-\frac{1}{C}(\Delta) = +\frac{2}{C}(\Delta) = s_1(\Delta) = \Delta$, $-\frac{2}{C}(\Delta) = +\frac{2}{C}(\Delta) = \Delta$, $-\frac{2}{C}(\Delta) = +\frac{2}{C}(\Delta) = \Delta$, and $+\frac{1}{C}(\Delta) = \frac{\Delta}{1+\Delta}$. Due to Theorem 7.4.2 we conclude quadratic derivational complexity, while the standard polynomial interpretation would only allow to conclude an exponential upper bound. Note that the deduced quadratic derivational complexity provides an optimal upper bound.

Another issue is the high average yes time (and the higher number of timeouts) of $\text{cdi}_3$ and $\text{cdi}^+$ in relation to existing techniques. Although a closer look reveals that the total times spent by $\text{T}T\text{Tbobx}$ and $\text{cdi}^+$ (or $\text{cdi}_3$) is relatively equal, an improvement of the efficiency of the introduced tools seems worthwhile.

**Remark 7.5.1.** Note that $\text{cdi}^+$ in conjunction with $\text{T}T\text{Tbobx}$ can automatically verify that 163 TRSs in the testbed are of at most quadratic derivational complexity. Put differently more than 10\% of all 1381 TRSs (and more than a third of the 445 non-duplicating TRSs) in version 4.0 of the TPDB are of quadratic derivational complexity.
7.6 Conclusion

In this paper we have presented two subclasses of context dependent interpretations, and established tight upper bounds on the induced derivational complexities. More precisely, we have delineated two subclasses: $\Delta$-linear and $\Delta$-restricted context dependent interpretations that induce exponential and quadratic derivational complexity, respectively. Further, we introduced an algorithm for mechanically proving termination of rewrite systems with context dependent interpretations. As a consequence we established a technique to automatically verify quadratic derivational complexity of TRSs. Finally, we reported on different implementations of this algorithm and presented numerical data to compare these implementations with existing methods that allow to automatically verify polynomial derivational complexity of TRSs.

We believe the here presented approach can be extended further. A starting point for future work would be to decide whether it is possible to define additional subclasses of context dependent interpretations inducing polynomial derivational complexities that grow faster than quadratic. One possible approach is to drop the restriction to integer coefficients and thus generalise the notion of polynomial context dependent interpretations. By Tarski’s quantifier elimination method, such an extension turns the undecidable positivity problem for Diophantine (in)equalities into a decidable problem. Further research will clarify the impact of this extension. A crucial problem in practical considerations is the known ineffectivity of quantifier elimination, see for example [34].
8

Automated Complexity Analysis Based on the Dependency Pair Method

Publication Details

I am indebted to Dieter Hofbauer, who spotted an unfortunate mistake in Lemma 22 of the published version. This mistake has been rectified below. The mistake influenced the given experimental evidence and I would like to thank Andreas Schnabl and Martin Avanzini for providing me with adjusted experimental data.

Ranking
The International Joint Conference on Automated Reasoning has been ranked A+ by CORE in 2007.

Abstract
In this paper, we present a variant of the dependency pair method for analysing runtime complexities of term rewrite systems automatically. This method is easy to implement, but significantly extends the analytic power of existing direct methods. Our findings extend the class of TRSs whose linear or quadratic runtime complexity can be detected automatically. We provide ample numerical data for assessing the viability of the method.

8.1 Introduction

Term rewriting is a conceptually simple but powerful abstract model of computation that underlies much of declarative programming. In order to assess the complexity of a

---

1 This research was partially supported by FWF (Austrian Science Fund) project P20133.
(terminating) term rewrite system (TRS for short) it is natural to look at the maximal length of derivation sequences, as suggested by Hofbauer and Lautemann in \cite{79}. More precisely, the derivational complexity function with respect to a (terminating and finitely-branching) TRS $R$ relates the length of the longest derivation sequence to the size of the initial term. For direct termination techniques it is often possible to establish upper-bounds on the growth rate of the derivational complexity function from the termination proof of $R$, see for example \cite{79, 77, 144, 78, 115, 56}.

However, if one is interested in methods that induce feasible (i.e., polynomial) complexity, the existing body of research is not directly applicable. On one hand this is due to the fact that for standard techniques the derivational complexity cannot be contained by polynomial growth rates. (See \cite{56} for the exception to the rule.) Already termination proofs by polynomial interpretations induce a double-exponential upper-bound on the derivational complexity, cf. \cite{79}. On the other hand this is—to some extent—the consequence of the definition of derivational complexity as this measure does not discriminate between different types of initial terms, while in modelling declarative programs the type of the initial term is usually quite restrictive. The following example clarifies the situation.

**Example 8.1.1.** Consider the TRS $R$

1: $\text{is\_empty}(\text{nil}) \rightarrow \top$
2: $\text{is\_empty}(x :: y) \rightarrow \bot$
3: $\text{hd}(x :: y) \rightarrow x$
4: $\text{tl}(x :: y) \rightarrow y$
5: $\text{append}(x, y) \rightarrow \text{ifappend}(x, y, x)$
6: $\text{ifappend}(x, y, \text{nil}) \rightarrow y$
7: $\text{ifappend}(x, y, u :: v) \rightarrow u :: \text{append}(v, y)$

Although the functions computed by $R$ are obviously feasible this is not reflected in the derivational complexity of $R$. Consider rule 5, which we abbreviate as $C[x] \rightarrow D[x, x]$. Since the maximal derivation length starting with $C^n[x]$ equals $2^n - 1$ for all $n > 0$, $R$ admits (at least) exponential derivational complexity.

After a moment one sees that this behaviour is forced upon us, as the TRS $R$ may duplicate variables, i.e., $R$ is duplicating. Furthermore, in general the applicability of the above results is typically limited to simple termination. (But see \cite{78, 115, 56} for exceptions to this rule.) To overcome the first mentioned restriction we propose to study runtime complexities of rewrite systems. The runtime complexity function with respect to a TRS $R$ relates the length of the longest derivation sequence to the size of the arguments of the initial term, where the arguments are supposed to be in normal form. In order to overcome the second restriction, we base our study on a fresh analysis of the dependency pair method. The dependency pair method \cite{9} is a powerful (and easily automatable) method for proving termination of term rewrite systems. In contrast to the above cited direct termination methods, this technique is a transformation technique, allowing for applicability beyond simple termination.

Studying (runtime) complexities induced by the dependency pair method is challenging. Below we give an (easy) example showing that the direct translations of original
theorems formulated in the context of termination analysis is destined to failure in the context of runtime complexity analysis. If one recalls that the dependency pair method is based on the observation that from an arbitrary non-terminating term one can extract a minimal non-terminating subterm, this is not surprising. Through a very careful investi-
gation of the original formulation of the dependency pair method (see [9, 57], but also [72]), we establish a runtime complexity analysis based on the dependency pair method. In doing so, we introduce weak dependency pairs and weak innermost dependency pairs as a general adaption of dependency pairs to (innermost) runtime complexity analysis. Here the innermost runtime complexity function with respect to a TRS \( R \) relates the length of the longest innermost derivation sequence to the size of the arguments of the initial term, where again the arguments are supposed to be in normal form.

Our main result shows how natural improvements of the dependency pair method, like usable rules, reduction pairs, and argument filterings become applicable in this context. Moreover, for innermost rewriting, we establish an easy criterion to decide when weak innermost dependency pairs can be replaced by “standard” dependency pairs without introducing fallacies. Thus we establish (for the first time) a method to analyse the derivation length induced by the (standard) dependency pair method for innermost rewriting. We have implemented the technique and experimental evidence shows that the use of weak dependency pairs significantly increases the applicability of the body of existing results on the estimation of derivation length via termination techniques. In particular, our findings extend the class of TRSs whose linear or quadratic runtime complexity can be detected automatically.

The remainder of this paper is organised as follows. In the next section we recall basic notions and starting points of this paper. Sections 8.3 and 8.4 introduce weak dependency pairs and discuss the employability of the usable rule criterion. In Section 8.5 we show how to estimate runtime complexities through relative rewriting and in Section 8.6 we state our Main Theorem. The presented technique has been implemented and we provide ample numerical data for assessing the viability of the method. This evidence can be found in Section 8.7. Finally in Section 8.8 we conclude and mention possible future work.

8.2 Preliminaries

We assume familiarity with term rewriting [15, 137] but briefly review basic concepts and notations. Let \( V \) denote a countably infinite set of variables and \( F \) a signature. The set of terms over \( F \) and \( V \) is denoted by \( T(F, V) \). The root symbol of a term \( t \) is either \( t \) itself, if \( t \in V \), or the symbol \( f \), if \( t = f(t_1, \ldots, t_n) \). The set of position \( \text{Pos}(t) \) of a term \( t \) is defined as usual. We write \( \text{Pos}_G(t) \subseteq \text{Pos}(t) \) for the set of positions of subterms, whose root symbol is contained in \( G \subseteq F \). The subterm relation is denoted as \( \sqsubseteq \). \( \text{Var}(t) \) denotes the set of variables occurring in a term \( t \) and the size \( |t| \) of a term is defined as the number of symbols in \( t \).

A term rewrite system (TRS for short) \( R \) over \( T(F, V) \) is a finite set of rewrite rules \( l \rightarrow r \), such that \( l \notin V \) and \( \text{Var}(l) \supseteq \text{Var}(r) \). The smallest rewrite relation that contains
\( \mathcal{R} \) is denoted by \( \rightarrow_{\mathcal{R}} \). The transitive closure of \( \rightarrow_{\mathcal{R}} \) is denoted by \( \mathcal{R}^* \), and its transitive and reflexive closure by \( \rightarrow_{\mathcal{R}}^* \). We simply write \( \rightarrow \) for \( \rightarrow_{\mathcal{R}} \) if \( \mathcal{R} \) is clear from context. A term \( s \in T(F, V) \) is called a normal form if there is no \( t \in T(F, V) \) such that \( s \rightarrow t \). With \( \mathcal{NF}(\mathcal{R}) \) we denote the set of all normal forms of a term rewrite system \( \mathcal{R} \). The innermost rewrite relation \( \mathcal{R} \rightarrow \mathcal{R} \) of a TRS \( \mathcal{R} \) is defined on terms as follows: \( s \mathcal{R} \rightarrow t \) if there exist a rewrite rule \( l \rightarrow r \in \mathcal{R} \), a context \( C \), and a substitution \( \sigma \) such that \( s = C[l\sigma], t = C[r\sigma] \), and all proper subterms of \( l\sigma \) are normal forms of \( \mathcal{R} \). The set of defined function symbols is denoted as \( D \), while the constructor symbols are collected in \( C \). We call a term \( t = f(t_1, \ldots, t_n) \) basic if \( f \in D \) and \( t_i \in T(C, V) \) for all \( 1 \leq i \leq n \). A TRS \( \mathcal{R} \) is called duplicating if there exists a rule \( l \rightarrow r \in \mathcal{R} \) such that a variable occurs more often in \( r \) than in \( t \). We call a TRS terminating if no infinite rewrite sequence exists. Let \( s \) and \( t \) be terms. If exactly \( n \) steps are performed to rewrite \( s \) to \( t \) we write \( s \mathcal{R} \rightarrow^n t \). The derivation length of a terminating term \( t \) with respect to a TRS \( \mathcal{R} \) and rewrite relation \( \mathcal{R} \rightarrow \) is defined as: \( \mathsf{dl}(s, \mathcal{R} \rightarrow) = \max\{n \mid \exists t \mathcal{R} \rightarrow^n t \} \). Let \( \mathcal{R} \) be a TRS and \( T \) be a set of terms. The runtime complexity function with respect to a relation \( \mathcal{R} \rightarrow \) on \( T \) is defined as follows:

\[
\mathsf{rc}(n, T, \mathcal{R} \rightarrow) = \max\{\mathsf{dl}(t, \mathcal{R} \rightarrow) \mid t \in T \text{ and } |t| \leq n\}.
\]

In particular we are interested in the (innermost) runtime complexity with respect to \( \rightarrow_{\mathcal{R}} \) (\( \mathcal{R} \rightarrow_{\mathcal{R}} \)) on the set \( T_\mathcal{B} \) of all basic terms, as defined above\(^2\). More precisely, the runtime complexity function (with respect to \( \mathcal{R} \)) is defined as \( \mathsf{rc}_{\mathcal{R}}(n) := \mathsf{rc}(n, T_\mathcal{B}, \mathcal{R} \rightarrow_{\mathcal{R}}) \) and we define the innermost runtime complexity function as \( \mathsf{rc}_{\mathcal{R}}^1(n) := \mathsf{rc}(n, T_\mathcal{B}, \mathcal{R} \rightarrow) \). Finally, the derivational complexity function (with respect to \( \mathcal{R} \)) becomes definable as follows: \( \mathsf{dc}_{\mathcal{R}}(n) = \mathsf{rc}(n, T, \mathcal{R} \rightarrow) \), where \( T \) denotes the set of all terms \( T(F, V) \). We sometimes say the (innermost) runtime complexity of \( \mathcal{R} \) is linear, quadratic, or polynomial if \( \mathsf{rc}_{\mathcal{R}}^1(n) \) is bounded linearly, quadratically, or polynomially in \( n \), respectively. Note that the derivational complexity and the runtime complexity of a TRS \( \mathcal{R} \) may be quite different: In general it is not possible to bound \( \mathsf{dc}_{\mathcal{R}} \) polynomially in \( \mathsf{rc}_{\mathcal{R}} \), as witnessed by Example 8.1.1 and the observation that the runtime complexity of \( \mathcal{R} \) is linear (see Example 8.4.2 below).

A proper order is a transitive and irreflexive relation and a preorder is a transitive and reflexive relation. A proper order \( \succ \) is well-founded if there is no infinite decreasing sequence \( t_1 \succ t_2 \succ t_3 \cdots \). A well-founded proper order that also is a rewrite relation is called a reduction order. We say a reduction order \( \succ \) and a TRS \( \mathcal{R} \) are compatible if \( \mathcal{R} \subseteq \succ \). It is well-known that a TRS is terminating if and only if there exists a compatible reduction order. An \( F \)-algebra \( A \) consists of a carrier set \( A \) and a collection of interpretations \( f_A \) for each function symbol in \( F \). A well-founded and monotone algebra (WMA for short) is a pair \( (A, \succ) \), where \( A \) is an algebra and \( \succ \) is a well-founded proper order on \( A \) such that every \( f_A \) is monotone in all arguments. An assignment \( \alpha : V \rightarrow A \) is a function mapping variables to elements in the carrier. A WMA naturally

---

\(^2\) We can replace \( T_\mathcal{B} \) by the set of terms \( f(t_1, \ldots, t_n) \) with \( f \in D \), whose arguments \( t_i \) are in normal form, while keeping all results in this paper.
induces a proper order $>_{\mathcal{A}}$ on terms: $s >_{\mathcal{A}} t$ if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments $\alpha: \mathcal{V} \to \mathcal{A}$.

8.3 The Dependency Pair Method

The purpose of this section is to take a fresh look at the dependency pair method from the point of complexity analysis. Familiarity with \[9, 72\] will be helpful. The dependency pair method for termination analysis is based on the observation that from an arbitrary non-terminating term one can extract a minimal non-terminating subterm. For complexity analysis we employ a similar observation: From a given term $t$ one can extract a list of subterms whose sum of the derivation lengths is equal to the derivational length of $t$.

Let $X$ be a set of symbols. We write $C(t_1, \ldots, t_n)_X$ to denote $C[t_1, \ldots, t_n]$, whenever $\text{root}(t_i) \in X$ for all $1 \leq i \leq n$ and $C$ is an $n$-hole context containing no $X$-symbols. (Note that the context $C$ may be degenerate and does not contain a hole $\Box$ or it may be that $C$ is a hole.) Then, every term $t$ can be uniquely written in the form $C(t_1, \ldots, t_n)_X$.

**Lemma 8.3.1.** Let $t$ be a terminating term, and let $\sigma$ be a substitution. Then we have $\text{dl}(t \sigma, \rightarrow_{\mathcal{R}}) = \sum_{1 \leq i \leq n} \text{dl}(t_i \sigma, \rightarrow_{\mathcal{R}})$, whenever $t = C(t_1, \ldots, t_n)_{D \cup V}$.

We define the function $\text{com}$ as a mapping from tuples of terms to terms as follows: $\text{com}(t_1, \ldots, t_n)$ is $t_1$ if $n = 1$, and $c(t_1, \ldots, t_n)$ otherwise. Here $c$ is a fresh $n$-ary function symbol called *compound symbol*. The above lemma motivates the next definition of *weak dependency pairs*.

**Definition 8.3.1.** Let $t$ be a term. We set $t^\sharp := t$ if $t \in \mathcal{V}$, and $t^\sharp := f^\sharp(t_1, \ldots, t_n)$ if $t = f(t_1, \ldots, t_n)$. Here $f^\sharp$ is a new $n$-ary function symbol called *dependency pair symbol*. For a signature $\mathcal{F}$, we define $\mathcal{F}^\sharp = \mathcal{F} \cup \{ f^\sharp \mid f \in \mathcal{F} \}$. Let $\mathcal{R}$ be a TRS. If $l \rightarrow r \in \mathcal{R}$ and $r = C(u_1, \ldots, u_n)_{D \cup V}$, then the rewrite rule $l^\sharp \rightarrow \text{com}(u_1^\sharp, \ldots, u_n^\sharp)$ is called a *weak dependency pair* of $\mathcal{R}$. The set of all weak dependency pairs is denoted by $\text{WDP}(\mathcal{R})$.

While dependency pair symbols are defined with respect to $\text{WDP}(\mathcal{R})$, these symbols are not defined with respect to the original system $\mathcal{R}$. In the sequel defined symbols, refer to the defined function symbols of $\mathcal{R}$.

**Example 8.3.1 (continued from Example 8.1.1).** The set $\text{WDP}(\mathcal{R})$ consists of the next seven weak dependency pairs:

$$
\begin{align*}
5: \quad & \text{is_empty}^\sharp(\text{nil}) \rightarrow c \\
6: \quad & \text{is_empty}^\sharp(x :: y) \rightarrow d \\
7: \quad & \text{hd}^\sharp(x :: y) \rightarrow x \\
8: \quad & \text{tl}^\sharp(x :: y) \rightarrow y \\
9: \quad & \text{append}^\sharp(x, y) \rightarrow \text{ifappend}^\sharp(x, y, x) \\
10: \quad & \text{ifappend}^\sharp(x, y, \text{nil}) \rightarrow y \\
11: \quad & \text{ifappend}^\sharp(x, y, u :: v) \rightarrow e(u, \text{append}^\sharp(v, y))
\end{align*}
$$

**Lemma 8.3.2.** Let $t \in T(\mathcal{F}, \mathcal{V})$ be a terminating term with $\text{root}(t) \in \mathcal{D}$. We have $\text{dl}(t, \rightarrow_{\mathcal{R}}) = \text{dl}(t^\sharp, \rightarrow_{\text{WDP}(\mathcal{R}) \cup \mathcal{R}})$.
Proof. We show $\text{dl}(t, \rightarrow_R) \leq \text{dl}(t^\ell, \rightarrow_{\text{WIDP}(R) \cup R})$ by induction on $\ell = \text{dl}(t, \rightarrow_R)$. If $\ell = 0$, the inequality is trivial. Suppose $\ell > 0$. Then there exists a term $u$ such that $t \rightarrow_R u$ and $\text{dl}(u, \rightarrow_R) = \ell - 1$. We distinguish two cases depending on the rewrite position $p$.

- If $p$ is a position below the root, then clearly $\text{root}(u) = \text{root}(t) \in D$ and $t^\ell \rightarrow_R u^\ell$. The induction hypothesis yields $\text{dl}(u, \rightarrow_R) \leq \text{dl}(u^\ell, \rightarrow_{\text{WIDP}(R) \cup R})$, and we obtain $\ell \leq \text{dl}(t^\ell, \rightarrow_{\text{WIDP}(R) \cup R})$.

- If $p$ is a root position, then there exist a rewrite rule $l \rightarrow r \in R$ and a substitution $\sigma$ such that $l = t \sigma$ and $u = r \sigma$. We have $r = C\langle u_1, \ldots, u_n \rangle_D \psi_V$ and thus by definition $l^\ell \rightarrow \text{com}(u_1^\ell, \ldots, u_n^\ell) \in \text{WIDP}(R)$ such that $t^\ell = l^\ell \sigma$. Now, either $u_i \in V$ or $\text{root}(u_i) \in D$ for every $1 \leq i \leq n$. Suppose $u_i \in V$. Then $u_i^\ell \sigma = u_i \sigma$ and clearly no dependency pair symbol can occur and thus,

$$
\text{dl}(u_i \sigma, \rightarrow_R) = \text{dl}(u_i^\ell \sigma, \rightarrow_R) = \text{dl}(u_i \sigma)^\ell, \rightarrow_{\text{WIDP}(R) \cup R}) .
$$

Otherwise, if $\text{root}(u_i) \in D$ then $u_i^\ell \sigma = (u_i \sigma)^\ell$. Hence $\text{dl}(u_i \sigma, \rightarrow_R) \leq \text{dl}(u, \rightarrow_R) < l$, and we conclude $\text{dl}(u_i \sigma, \rightarrow_R) \leq \text{dl}(u_i^\ell \sigma, \rightarrow_{\text{WIDP}(R) \cup R})$ from the induction hypothesis. Therefore,

$$
\ell = \text{dl}(u, \rightarrow_R) + 1 = \sum_{1 \leq i \leq n} \text{dl}(u_i \sigma, \rightarrow_R) + 1 \leq \sum_{1 \leq i \leq n} \text{dl}(u_i^\ell \sigma, \rightarrow_{\text{WIDP}(R) \cup R}) + 1
$$

$$
\leq \text{dl}(\text{com}(u_1^\ell, \ldots, u_n^\ell) \sigma, \rightarrow_{\text{WIDP}(R) \cup R}) + 1 = \text{dl}(t^\ell, \rightarrow_{\text{WIDP}(R) \cup R}) .
$$

Here we used Lemma 8.3.1 for the second equality.

Note that $t$ is $R$-reducible if and only if $t^\ell$ is $\text{WIDP}(R) \cup R$-reducible. Hence as $t$ is terminating, $t^\ell$ is terminating on $\rightarrow_{\text{WIDP}(R) \cup R}$. Thus, similarly, $\text{dl}(t, \rightarrow_R) \geq \text{dl}(t^\ell, \rightarrow_{\text{WIDP}(R) \cup R})$ is shown by induction on $\text{dl}(t^\ell, \rightarrow_{\text{WIDP}(R) \cup R})$.

Lemma 8.3.3. Let $t$ be a terminating term and $\sigma$ a substitution such that $x \sigma$ is a normal form of $R$ for all $x \in \text{Var}(t)$. Then $\text{dl}(t \sigma, \rightarrow_R) = \sum_{1 \leq i \leq n} \text{dl}(t_i \sigma, \rightarrow_R)$, whenever $t = C\langle t_1, \ldots, t_n \rangle_D$.

Definition 8.3.2. Let $R$ be a TRS. If $l \rightarrow r \in R$ and $r = C\langle u_1, \ldots, u_n \rangle_D$ then the rewrite rule $l^\ell \rightarrow \text{com}(u_1^\ell, \ldots, u_n^\ell)$ is called a weak innermost dependency pair of $R$. The set of all weak innermost dependency pairs is denoted by $\text{WIDP}(R)$.

Example 8.3.2 (continued from Example 8.1.1). The set $\text{WIDP}(R)$ consists of the following seven weak dependency pairs (with respect to $\rightarrow$):

$$
is\_empty^\ell(\text{nil}) \rightarrow c \quad \text{append}^\ell(x, y) \rightarrow \text{ifappend}^\ell(x, y, x)$$
$$
is\_empty^\ell(x :: y) \rightarrow d \quad \text{ifappend}^\ell(x, y, \text{nil}) \rightarrow g$$
$$
\text{hd}^\ell(x :: y) \rightarrow e \quad \text{ifappend}^\ell(x, y, u :: v) \rightarrow \text{append}^\ell(v, y)$$
$$
\text{tl}^\ell(x :: y) \rightarrow f .
$$
The next lemma adapts Lemma 8.3.2 to innermost rewriting.

**Lemma 8.3.4.** Let $t$ be an innermost terminating term in $T(\mathcal{F}, \mathcal{V})$ with $\text{root}(t) \in \mathcal{D}$. We have $\text{dl}(t, \rightarrow_R) = \text{dl}(t, \rightarrow_{\text{WIDP}(\mathcal{R}) \cup \mathcal{R}})$.

We conclude this section by discussing the applicability of standard dependency pairs (9) in complexity analysis. For that we recall the standard definition of dependency pairs.

**Definition 8.3.3 (9).** The set $\text{DP}(\mathcal{R})$ of (standard) dependency pairs of a TRS $\mathcal{R}$ is defined as $\{l^\sharp \rightarrow u^\sharp \mid l \rightarrow r \in \mathcal{R}, u \triangleright r, \text{root}(u) \in \mathcal{D}\}$.

The following example shows that Lemma 8.3.2 (Lemma 8.3.4) does not hold if we replace weak (innermost) dependency pairs with standard dependency pairs.

**Example 8.3.3.** Consider the one-rule TRS $\mathcal{R}$: $f(s(x)) \rightarrow g(f(x), f(x))$. $\text{DP}(\mathcal{R})$ is the singleton of $f^\sharp(s(x)) \rightarrow f^\sharp(x)$. Let $t_n = f(s^n(x))$ for each $n \geq 0$. Since $t_{n+1} \rightarrow_R g(t_n, t_n)$ holds for all $n \geq 0$, it is easy to see $\text{dl}(t_{n+1}, \rightarrow_R) \geq 2^n$, while $\text{dl}(t_{n+1}, \rightarrow_{\text{DP}(\mathcal{R}) \cup \mathcal{R}}) = n$.

Hence, in general we cannot replace weak dependency pairs with (standard) dependency pairs. However, if we restrict our attention to innermost rewriting, we can employ dependency pairs in complexity analysis without introducing fallacies, when specific conditions are met.

**Lemma 8.3.5.** Let $t$ be an innermost terminating term with $\text{root}(t) \in \mathcal{D}$. If all compound symbols in $\text{WIDP}(\mathcal{R})$ are nullary, $\text{dl}(t, \rightarrow_R) \leq \text{dl}(t, \rightarrow_{\text{DP}(\mathcal{R}) \cup \mathcal{R}}) + 1$ holds.

**Example 8.3.4 (continued from Example 8.3.2).** The occurring compound symbols are nullary. $\text{DP}(\mathcal{R})$ consists of the following two dependency pairs:

\[
\text{append}^\sharp(x, y) \rightarrow \text{ifappend}^\sharp(x, y, x) \quad \text{ifappend}^\sharp(x, y, u :: v) \rightarrow \text{append}^\sharp(v, y)
\]

### 8.4 Usable Rules

In the previous section, we studied the dependency pair method in the light of complexity analysis. Let $\mathcal{R}$ be a TRS and $\mathcal{P}$ a set of weak dependency pairs, weak innermost dependency pairs, or standard dependency pairs of $\mathcal{R}$. Lemmata 8.3.2, 8.3.4, and 8.3.5 describe a strong connection between the length of derivations in the original TRSs $\mathcal{R}$ and the transformed (and extended) system $\mathcal{P} \cup \mathcal{R}$. In this section we show how we can simplify the new TRS $\mathcal{P} \cup \mathcal{R}$ by employing usable rules.

**Definition 8.4.1.** We write $f \succ_d g$ if there exists a rewrite rule $l \rightarrow r \in \mathcal{R}$ such that $f = \text{root}(l)$ and $g$ is a defined function symbol in $\text{Fun}(r)$. For a set $\mathcal{G}$ of defined function symbols we denote by $\mathcal{R} \mid \mathcal{G}$ the set of rewrite rules $l \rightarrow r \in \mathcal{R}$ with $\text{root}(l) \in \mathcal{G}$. The set $\mathcal{U}(t)$ of usable rules of a term $t$ is defined as $\mathcal{R} \mid \{g \mid f \succ_d^* g \text{ for some } f \in \text{Fun}(t)\}$. Finally, if $\mathcal{P}$ is a set of (weak) dependency pairs then $\mathcal{U}(\mathcal{P}) = \bigcup_{t \in \mathcal{P}} \mathcal{U}(t)$. 

109
Example 8.4.1 (continued from Examples 8.3.1 and 8.3.2). The sets of usable rules are empty (and thus equal) for the weak dependency pairs and for the weak innermost dependency pairs, i.e., we have \( \mathcal{U}(\text{WDP}(\mathcal{R})) = \mathcal{U}(\text{WDP}(\mathcal{R})) = \emptyset \).

The usable rule criterion in termination analysis (cf. [62, 72]) asserts that a non-terminating rewrite sequence of \( \mathcal{R} \cup \text{DP}(\mathcal{R}) \) can be transformed into a non-terminating rewrite sequence of \( \mathcal{U}(\text{DP}(\mathcal{R})) \cup \text{DP}(\mathcal{R}) \cup \{ g(x, y) \rightarrow x, g(x, y) \rightarrow y \} \), where \( g \) is a fresh function symbol. Because \( \mathcal{U}(\text{DP}(\mathcal{R})) \) is a (small) subset of \( \mathcal{R} \) and most termination methods can handle \( g(x, y) \rightarrow x \) and \( g(x, y) \rightarrow y \) easily, the termination analysis often becomes easy by switching the target of analysis from the former TRS to the latter TRS. Unfortunately the transformation used in [62, 72] increases the size of starting terms, therefore we cannot adopt this transformation approach. Note, however that the usable rule criteria for innermost termination [57] can be directly applied in the context of complexity analysis. Nevertheless, one may show a new type of usable rule criterion by exploiting the basic property of a starting term. Recall that \( \mathcal{T}_b \) denotes the set of basic terms; we set \( \mathcal{T}_b^\sharp = \{ t^\sharp \mid t \in \mathcal{T}_b \} \).

**Lemma 8.4.1.** Let \( \mathcal{P} \) be a set of (weak) dependency pairs and let \( (t_i)_{i=0,1,...} \) be a (finite or infinite) derivation of \( \mathcal{R} \cup \mathcal{P} \). If \( t_0 \in \mathcal{T}_b^\sharp \) then \( (t_i)_{i=0,1,...} \) is a derivation of \( \mathcal{U}(\mathcal{P}) \cup \mathcal{P} \).

**Proof.** Let \( \mathcal{G} \) be the set of all non-usable symbols with respect to \( \mathcal{P} \). We write \( P(t) \) if \( t_q \in \mathcal{N}(\mathcal{R}) \) for all \( q \in \mathcal{Pos}_G (t) \). Since \( t_i \rightarrow_{\mathcal{U}(\mathcal{P}) \cup \mathcal{P}} t_{i+1} \) holds whenever \( P(t_i) \) and \( t_i \rightarrow_{\mathcal{R} \cup \mathcal{P}} t_{i+1} \), it is sufficient to show \( P(t_i) \) for all \( i \). We perform induction on \( i \).

(i) Assume \( i = 0 \). Since \( t_0 \in \mathcal{T}_b^\sharp \), we have \( t_0 \in \mathcal{N}(\mathcal{R}) \) and thus \( t_0 |_p \in \mathcal{N}(\mathcal{R}) \) for all positions \( p \). The assertion \( P \) follows trivially.

(ii) Suppose \( i > 0 \). By induction hypothesis, there exist \( l \rightarrow r \in \mathcal{U}(\mathcal{P}) \cup \mathcal{P} \), \( p \in \mathcal{Pos}(t_{i-1}) \), and a substitution \( \sigma \) such that \( t_{i-1} |_p = l \sigma \) and \( t_i |_p = r \sigma \). In order to show property \( P \) for \( t_i \), we fix a position \( q \in \mathcal{G} \). We have to show \( t_i |_q \in \mathcal{N}(\mathcal{R}) \).

We distinguish three cases:

- Suppose that \( q \) is above \( p \). Then \( t_{i-1} |_q \) is reducible, but this contradicts the induction hypothesis \( P(t_{i-1}) \).
- Suppose \( p \) and \( q \) are parallel but distinct. Since \( t_{i-1} |_q = t_i |_q \in \mathcal{N}(\mathcal{R}) \) holds, we obtain \( P(t_i) \).
- Otherwise, \( q \) is below \( p \). Then, \( t_i |_q \) is a subterm of \( r \sigma \). Because \( r \) contains no \( \mathcal{G} \)-symbols by the definition of usable symbols, \( t_i |_q \) is a subterm of \( x \sigma \) for some \( x \in \text{Var}(r) \subseteq \text{Var}(l) \). Therefore, \( t_i |_q \) is also a subterm of \( t_{i-1} \), from which \( t_i |_q \in \mathcal{N}(\mathcal{R}) \) follows. We obtain \( P(t_i) \).

\[ \square \]

The following theorem follows from Lemmata 8.3.1, 8.3.4, and 8.3.5 in conjunction with the above Lemma 8.4.1. It adapts the usable rule criteria to complexity analysis.

\[ ^3 \text{Note that Theorem 8.4.1 only holds for basic terms } t \in \mathcal{T}_b^\sharp. \text{ In order to show this, we need some additional technical lemmas, which are the subject of the next section.} \]
Theorem 8.4.1. Let $\mathcal{R}$ be a TRS and let $t \in \mathcal{T}_B$. If $t$ is terminating with respect to $\rightarrow$ then $\text{dl}(t, \rightarrow) \leq \text{dl}(t^2, \rightarrow_{U(\mathcal{P}) \cup \mathcal{P}})$, where $\rightarrow$ denotes $\rightarrow_{\mathcal{R}}$ or $\rightarrow_{\mathcal{R}}$ depending on whether $\mathcal{P} = \text{WDP}(\mathcal{R})$ or $\mathcal{P} = \text{WIDP}(\mathcal{R})$. Moreover, suppose all compound symbols in $\text{WIDP}(\mathcal{R})$ are nullary then $\text{dl}(t, \rightarrow_{\mathcal{R}}) \leq \text{dl}(t^2, \rightarrow_{U(\text{DP}(\mathcal{R})) \cup \text{DP}(\mathcal{R})}) + 1$.

It is worth stressing that it is (often) easier to analyse the complexity of $U(\mathcal{P}) \cup \mathcal{P}$ than the complexity of $\mathcal{R}$. To clarify the applicability of the theorem in complexity analysis, we consider two restrictive classes of polynomial interpretations, whose definitions are motivated by [26].

A polynomial $P(x_1, \ldots, x_n)$ (over the natural numbers) is called strongly linear if $P(x_1, \ldots, x_n) = x_1 + \cdots + x_n + c$ where $c \in \mathbb{N}$. A polynomial interpretation is called linear restricted if all constructor symbols are interpreted by strongly linear polynomials and all other function symbols by a linear polynomial (monotone with respect to the standard order $>$ on $\mathbb{N}$). If on the other hand the non-constructor symbols are interpreted by quadratic polynomials, the polynomial interpretation is called quadratic restricted. Here a polynomial is quadratic if it is a sum of monomials of degree at most 2. It is easy to see that if a TRS $\mathcal{R}$ is compatible with a linear or quadratic restricted interpretation, the runtime complexity of $\mathcal{R}$ is linear or quadratic, respectively (see also [26]).

Corollary 8.4.1. Let $\mathcal{R}$ be a TRS and let $\mathcal{P} = \text{WDP}(\mathcal{R})$ or $\mathcal{P} = \text{WIDP}(\mathcal{R})$. If $U(\mathcal{P}) \cup \mathcal{P}$ is compatible with a linear or quadratic restricted interpretation, the (innermost) runtime complexity function $r_{\mathcal{R}}^{(i)}$ with respect to $\mathcal{R}$ is linear or quadratic, respectively. Moreover, suppose all compound symbols in $\text{WIDP}(\mathcal{R})$ are nullary and $U(\text{DP}(\mathcal{R})) \cup \text{DP}(\mathcal{R})$ is compatible with a linear (quadratic) restricted interpretation, then $\mathcal{R}$ admits at most linear (quadratic) innermost runtime complexity.

Proof. Let $\mathcal{R}$ be a TRS. For simplicity we suppose $\mathcal{P} = \text{WDP}(\mathcal{R})$ and assume the existence of a linear restricted interpretation $\mathcal{A}$, compatible with $U(\mathcal{P}) \cup \mathcal{P}$. Clearly this implies the well-foundedness of the relation $\rightarrow_{U(\mathcal{P}) \cup \mathcal{P}}$, which in turn implies the well-foundedness of $\rightarrow_{\mathcal{R}}$, cf. Lemma 8.4.1. Hence Theorem 8.4.1 is applicable and we conclude $\text{dl}(t, \rightarrow_{\mathcal{R}}) \leq \text{dl}(t^2, \rightarrow_{U(\text{DP}(\mathcal{R})) \cup \text{DP}(\mathcal{R})})$. On the other hand, compatibility with $\mathcal{A}$ implies that $\text{dl}(t^2, \rightarrow_{U(\text{DP}(\mathcal{R})) \cup \text{DP}(\mathcal{R})}) = O(|t^2|)$. As $|t^2| = |t|$, we can combine these equalities to conclude linear runtime complexity of $\mathcal{R}$. \hfill $\Box$

The below given example applies Corollary 8.4.1 to the motivating Example 8.1.1 introduced in Section 8.1.

Example 8.4.2 (continued from Example 8.3.1). We take a quadratic restricted interpretation $\mathcal{B}$ into $\mathbb{N} \setminus \{0\}$ with $c_B = d_B = n_B = 1$, $e_B(x, y) = x + y$, $x : B = x + y$, $hd_B^i(x) = x + 1$, $tl_B^i(x) = x + 1$, $\text{is\_empty}_B^i(x) = x + 1$, $\text{append}_B^i(x, y) = x^2 + 3x + y + 1$, and $\text{ifappend}_B^i(x, y, z) = 2x + y + z^2 + z$. Then $\mathcal{B}$ interprets $\text{WDP}(\mathcal{R})$ as follows:

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2 &gt; 1 &amp; 8</td>
<td>$x + y + 1 &gt; y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$x + y + 1 &gt; 1$</td>
<td>9</td>
<td>$x^2 + 3x + y + 1 &gt; 3x + y + x^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$x + y + 1 &gt; x$</td>
<td>10</td>
<td>$2x + y + 2 &gt; y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$2x + y + u^2 + 2uv + v^2 + u + v &gt; u + v^2 + 3v + y + 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Therefore, $\text{WDP}(\mathcal{R}) \subseteq \mathbb{R}$ holds. Hence, the runtime complexity of $\mathcal{R}$ for full rewriting is quadratic. (Recall that $U(\text{WDP}(\mathcal{R})) = \emptyset$.)

### 8.5 The Weight Gap Principle

We recall the notion of relative rewriting \cite{[53],[137]}.

**Definition 8.5.1.** Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs. We write $\rightarrow_{\mathcal{R}/\mathcal{S}}$ for $\rightarrow_{\mathcal{S}} \cdot \rightarrow_{\mathcal{R}} \cdot \rightarrow_{\mathcal{S}}$ and we call $\rightarrow_{\mathcal{R}/\mathcal{S}}$ the relative rewrite relation of $\mathcal{R}$ over $\mathcal{S}$.

Since $dl(t, \rightarrow_{\mathcal{R}/\mathcal{S}})$ corresponds to the number of $\rightarrow_{\mathcal{R}}$-steps in a maximal derivation of $\rightarrow_{\mathcal{R} \cup \mathcal{S}}$ from $t$, we easily see the bound $dl(t, \rightarrow_{\mathcal{R}/\mathcal{S}}) \leq dl(t, \rightarrow_{\mathcal{R} \cup \mathcal{S}})$. In this section we study the opposite, i.e., we figure out a way to give an upper-bound of $dl(t, \rightarrow_{\mathcal{R} \cup \mathcal{S}})$ by a function of $dl(t, \rightarrow_{\mathcal{R}/\mathcal{S}})$.

First we introduce the key ingredient, strongly linear interpretations, a very restrictive form of polynomial interpretations. Let $F$ denote a signature. A strongly linear interpretation (SLI for short) is a WMA $(\mathcal{A}, \succ)$ that satisfies the following properties:

(i) the carrier of $\mathcal{A}$ is the set of natural numbers $\mathbb{N}$, (ii) all interpretation functions $f_{\mathcal{A}}$ are strongly linear, (iii) the proper order $\succ$ is the standard order $>$ on $\mathbb{N}$. Note that an SLI $\mathcal{A}$ is conceivable as a weight function. We define the maximum weight $M_{\mathcal{A}}$ of $\mathcal{A}$ as $\max\{f_{\mathcal{A}}(0, \ldots, 0) \mid f \in F\}$. Let $\mathcal{A}$ denote an SLI, let $\alpha_0$ denote the assignment mapping any variable to 0, i.e., $\alpha_0(x) = 0$ for all $x \in \mathcal{V}$, and let $t$ be a term. We write $[t]$ as an abbreviation for $[\alpha_0]_{\mathcal{A}}(t)$.

**Lemma 8.5.1.** Let $\mathcal{A}$ be an SLI and let $t$ be a term. Then $[t] \leq M_{\mathcal{A}} \cdot |t|$ holds.

**Proof.** By induction on $t$. If $t \in \mathcal{V}$ then $[t] = 0 \leq M_{\mathcal{A}} \cdot |t|$. Otherwise, suppose $t = f(t_1, \ldots, t_n)$, where $f_{\mathcal{A}}(x_1, \ldots, x_n) = x_1 + \ldots + x_n + c$. By the induction hypothesis and $c \leq M_{\mathcal{A}}$ we obtain the following inequalities:

$$[t] = f_{\mathcal{A}}([t_1], \ldots, [t_n]) \leq [t_1] + \cdots + [t_n] + c$$
$$\leq M_{\mathcal{A}} \cdot |t_1| + \cdots + M_{\mathcal{A}} \cdot |t_n| = M_{\mathcal{A}} \cdot |t|.$$

\[\square\]

The conception of strongly linear interpretations as weight functions allows us to study (possible) weight increase throughout a rewrite derivation. This observation is reflected in the next definition.

**Definition 8.5.2.** Let $\mathcal{A}$ be an algebra and let $\mathcal{R}$ be a TRS. The weight gap $\Delta(\mathcal{A}, \mathcal{R})$ of $\mathcal{A}$ with respect to $\mathcal{R}$ is defined on $\mathbb{N}$ as follows: $\Delta(\mathcal{A}, \mathcal{R}) = \max\{r \div [t] \mid l \rightarrow r \in \mathcal{R}\}$, where $\div$ is defined as usual: $m \div n := \max\{m - n, 0\}$.

The following weight gap principle is a direct consequence of the definitions.

\[\text{Note that } \rightarrow_{\mathcal{R}/\mathcal{S}} = \rightarrow_{\mathcal{R}}, \text{ if } \mathcal{S} = \emptyset.\]
Lemma 8.5.2. Let \( R \) be a non-duplicating TRS and \( A \) an SLI. If \( s \to_R t \) then \([s] + \Delta(A, R) \geq [t]\).

We stress that the lemma does only require the non-duplicating condition. Indeed, the implication in the lemma holds even if the TRS \( R \) is not compatible with a strongly linear interpretation. This principle brings us to the next theorem.

Theorem 8.5.1. Let \( R \) and \( S \) be TRSs, \( A \) an SLI compatible with \( S \) and \( R \) non-duplicating. Then we have
\[
\text{dl}(t, \to_{R \cup S}) \leq (1 + \Delta(A, R)) \cdot \text{dl}(t, \to_{R/S}) + M_A \cdot |t|,
\]
whenever \( t \) is terminating on \( R \cup S \).

Proof. Let \( m = \text{dl}(t, \to_{R/S}), \) let \( n = |t|, \) and set \( \Delta = \Delta(A, R). \) Any derivation of \( \to_{R \cup S} \) is representable as follows
\[
s_0 \to_S^k t_0 \to_R s_1 \to_S^i t_1 \to_R \cdots \to_S^m t_m,
\]
and without loss of generality we may assume that the derivation is maximal. We observe the next two facts.

(a) \( k_i \leq [s_i] - [t_i] \) holds for all \( 0 \leq i \leq m. \) This is because \([s] \geq [t] + 1\) whenever \( s \to_R t \) by the assumption \( S \subseteq \succ A, \) and we have \( s_i \to_S^k t_i. \)

(b) \([s_{i+1}] - [t_i] \leq \Delta \) holds for all \( 0 \leq i < m \) as due to Lemma 8.5.2 we have \([t_i] + \Delta \geq [s_{i+1}]. \)

We obtain the following inequalities:
\[
\text{dl}(s_0, \to_{R \cup S}) = m + k_0 + \cdots + k_m
\]
\[
\leq m + ([s_0] - [t_0]) + \cdots + ([s_m] - [t_m])
\]
\[
= m + [s_0] + ([s_1] - [t_0]) + \cdots + ([s_m] - [t_{m-1}]) - [t_m]
\]
\[
\leq m + [s_0] + m\Delta - [t_m]
\]
\[
\leq m + [s_0] + m\Delta
\]
\[
\leq m + M_A \cdot n + m\Delta = (1 + \Delta)m + M_A \cdot n.
\]

Here we used (a) \( m \)-times in the second line, (b) \( m - 1 \)-times in the fourth line, and Lemma 8.5.1 in the last line. \( \square \)

The next example clarifies that the conditions expressed in Theorem 8.5.1 are essentially optimal: We cannot replace the assumption that the algebra \( A \) is strongly linear with a weaker assumption: Already if \( A \) is a linear polynomial interpretation, the derivation height of \( R \cup S \) cannot be bounded polynomially in \( \text{dl}(t, \to_{R/S}) \) and \( |t| \) alone.

Example 8.5.1. Consider the TRSs \( R \)
\[
\exp(0) \to s(0) \quad \text{d}(0) \to 0
\]
\[
\exp(r(x)) \to \text{d}(\exp(x)) \quad \text{d}(s(x)) \to s(s(d(x)))
\]
This TRS formalises the exponentiation function. Setting \( t_n = \exp(r^n(0)) \) we obtain \( \text{dl}(t_n, \rightarrow_R) \geq 2^n \) for each \( n \geq 0 \). Thus the runtime complexity of \( R \) is (at least) exponential. In order to show the claim, we split \( R \) into two TRSs \( R_1 = \{ \exp(0) \rightarrow s(0), \exp(r(x)) \rightarrow d(\exp(x)) \} \) and \( R_2 = \{ d(0) \rightarrow 0, d(s(x)) \rightarrow s(d(x)) \} \). Then it is easy to verify that the next linear polynomial interpretation \( A \) is compatible with \( R_2: 0_A = 0, d_A(x) = 3x, \) and \( s_A(x) = x + 1 \). Moreover an upper-bound of \( \text{dl}(t_n, \rightarrow_{R_1/R_2}) \) can be estimated by using the following polynomial interpretation \( B: 0_B = 0, d_B(x) = s_B(x) = x, \) and \( \exp_B(x) = r_B(x) = x + 1 \). Since \( \rightarrow_{R_1} \subseteq \rightarrow_B \) and \( \rightarrow_{R_2} \subseteq \rightarrow_B \) hold, we have \( \rightarrow_{R_1/R_2} \subseteq \rightarrow_B \). Hence, \( \text{dl}(t_n, \rightarrow_{R_1/R_2}) \leq |a_0|_B(t_n) = n + 2 \). But clearly from this we cannot conclude a polynomial bound on the derivation length of \( R_1 \cup R_2 = R \), as the runtime complexity of \( R \) is exponential, at least.

To conclude this section, we show that Theorem 8.4.1 can only hold for basic terms \( t \in T_b^\circ \).

**Example 8.5.2.** Consider the one-rule TRS \( R = \{ a(b(x)) \rightarrow b(a(x)) \} \) from Example 2.50. It is not difficult to see that \( \text{dl}(a^n(b(x)), \rightarrow_R) = 2^n - 1 \), see [78]. The set WDP\((R)\) consists of just one dependency pair \( a^*_R(b(x)) \rightarrow a^*_R(x) \). In particular the set of usable rules is empty. The following SLI \( A \) is compatible with WDP\((R)\): \( a^*_A(x) = a_A(x) = x \) and \( b_A(x) = x + 1 \). Hence, due to Lemma 8.5.1 we can conclude the existence of a constant \( K \) such that \( \text{dl}(t^2, \rightarrow_{\text{WDP}(R)}) \leq K \cdot |t| \). Due to Theorem 8.4.1 we conclude linear runtime complexity of \( R \).

### 8.6 Reduction Pairs and Argument Filterings

In this section we study the consequences of combining Theorem 8.4.1 and Theorem 8.5.1. In doing so, we adapt reduction pairs and argument filterings ([9]) to runtime complexity analysis. Let \( R \) be a TRS, and let \( A \) be a strongly linear interpretation and suppose we consider weak, weak innermost, or (standard) dependency pairs \( P \), such that \( P \) is non-duplicating. If \( U(P) \subseteq \rightarrow_A \) then there exist constants \( K, L \geq 0 \) (depending on \( P \) and \( A \) only) such that

\[
\text{dl}(t, \rightarrow_R) \leq K \cdot \text{dl}(t^2, \rightarrow_{P/\text{WDP}(P)}) + L \cdot |t^2|,
\]

for all terminating basic terms \( t \in T_b^\circ \). This follows from the combination of Theorems 8.4.1 and 8.5.1. Thus, in order to estimate the derivation length of \( t \) with respect to \( R \) it suffices to estimate the maximal \( P \) steps, i.e., we have to estimate \( \text{dl}(t^2, \rightarrow_{P/\text{WDP}(P)}) \) suitably. Consider a maximal derivation \( (t_i)_{i=0,\ldots,n} \) of \( \rightarrow_{P/\text{WDP}(P)} \) with \( t_0 = t^2 \). For every \( 0 \leq i < n \) there exist terms \( u_i \) and \( v_i \) such that

\[
t_i \rightarrow^*_U(P) u_i \rightarrow_P v_i \rightarrow^*_U(P) t_{i+1}.
\]

(8.1)

Let \( \succsim \) be a pair of orders with \( \succsim \circ \succsim \subseteq \succsim \). If \( u_i \succsim v_i \succsim t_{i+1} \), then \( t^2 = t_0 \succsim t_1 \succsim \cdots \succsim t_n \). Therefore, \( \text{dl}(t^2, \rightarrow_{P/\text{WDP}(P)}) \) can be bounded in the maximal length of \( \succsim \)-descending steps. We formalise these observations through the use of *reduction pairs* and *collapsible orders*.
6.6 Reduction Pairs and Argument Filterings

Definition 8.6.1. Let \( \mathcal{R} \) be a TRS, let \( \mathcal{P} \) be a set of weak dependency pairs of \( \mathcal{R} \) and let \( G \) denote a mapping associating a term (over \( \mathcal{F} \) and \( \mathcal{V} \)) and a proper order \( \succ \) with a natural number. An order \( \succ \) on terms is \( G \)-collapsible for a TRS \( \mathcal{R} \) if \( s \rightarrow_{\mathcal{P} \cup \mathcal{U}(\mathcal{P})} t \) and \( s \succ t \) implies \( G(s, \succ) > G(t, \succ) \). An order \( \succ \) is collapsible for a TRS \( \mathcal{R} \), if there is a mapping \( G \) such that \( \succ \) is \( G \)-collapsible for \( \mathcal{R} \).

Note that most reduction orders are collapsible. For instance, if \( \mathcal{A} \) is a polynomial interpretation then \( \succ_{\mathcal{A}} \) is collapsible, as witnessed by the evaluation function \([\alpha_0]_{\mathcal{A}}\). Furthermore, simplification orders like MPO, LPO and KBO are collapsible (cf. \([14], [15])\).

Definition 8.6.2. A rewrite preorder is a preorder on terms which is closed under contexts and substitutions. A reduction pair \( (\succ\succ, \succ) \) consists of a rewrite preorder \( \succ\succ \) and a compatible well-founded order \( \succ \) which is closed under substitutions. Here compatibility means the inclusion \( \succ\succ \cdot \succ \cdot \succ \subseteq \succ \). A reduction pair \( (\succ\succ, \succ) \) is called collapsible for a TRS \( \mathcal{R} \) if \( \succ \) is collapsible for \( \mathcal{R} \).

Recall the derivation in (8.1): Due to compound symbols the rewrite step \( u_i \rightarrow_{\mathcal{P}} v_i \) may take place below the root. Hence \( \mathcal{P} \subseteq \succ \) does not ensure \( u_i \succ v_i \). To address this problem we introduce a notion of safety that is based on the next definitions.

Definition 8.6.3. The set \( T^c_\mathcal{P} \) is inductively defined as follows (i) \( T^c_\mathcal{P} = T \cup T \subseteq T^c_\mathcal{P} \), where \( T^c = \{ t^c \mid t \in T \} \) and (ii) \( c(t_1, \ldots, t_n) \in T^c_\mathcal{P} \), whenever \( t_1, \ldots, t_n \in T^c_\mathcal{P} \) and \( c \) is a compound symbol.

Definition 8.6.4. A proper order \( \succ \) on \( T^c_\mathcal{P} \) is called safe if \( c(s_1, \ldots, s_i, \ldots, s_n) \succ c(s_1, \ldots, t, \ldots, s_n) \) for all \( n \)-ary compound symbols \( c \) and all terms \( s_1, \ldots, s_n, t \) with \( s_i \succ t \). A reduction pair \((\succ\succ, \succ)\) is called safe if \( \succ \) is safe.

Lemma 8.6.1. Let \( \mathcal{P} \) be a set of weak, weak innermost, or standard dependency pairs, and let \( (\succ\succ, \succ) \) be a safe reduction pair such that \( \mathcal{U}(\mathcal{P}) \subseteq \succ\succ \) and \( \mathcal{P} \subseteq \succ \). If \( s \in T^c_\mathcal{P} \) and \( s \rightarrow_{\mathcal{P} \cup \mathcal{U}(\mathcal{P})} t \) then \( s \succ t \) and \( t \in T^c_\mathcal{P} \).

Employing Theorem 8.4.1, Theorem 8.5.1 and Lemma 8.6.1 we arrive at our Main Theorem.

Theorem 8.6.1. Let \( \mathcal{R} \) be a TRS, let \( \mathcal{A} \) be an SLI, let \( \mathcal{P} \) be the set of weak, weak innermost, or (standard) dependency pairs, such that \( \mathcal{P} \) is non-duplicating, and let \( (\succ\succ, \succ) \) be a safe and \( G \)-collapsible reduction pair such that \( \mathcal{U}(\mathcal{P}) \subseteq \succ\succ \) and \( \mathcal{P} \subseteq \succ \). If in addition \( \mathcal{U}(\mathcal{P}) \subseteq \succ_{\mathcal{A}} \) then for any \( t \in T_b \), we have \( \text{dl}(t, \rightarrow) \leq p(G(t^c, \succ), |t|) \), where \( p(m, n) := (1 + \Delta(\mathcal{A}, \mathcal{P})) \cdot m + M_\mathcal{A} \cdot n \) and \( \rightarrow \) denotes \( \rightarrow_{\mathcal{R}} \) or \( \rightarrow_{\mathcal{R}} \) depending on whether \( \mathcal{P} = \text{WDP}(\mathcal{R}) \) or \( \mathcal{P} = \text{WIDP}(\mathcal{R}) \). Moreover if all compound symbols in \( \text{WIDP}(\mathcal{R}) \) are nullary we have \( \text{dl}(t, \rightarrow_{\mathcal{R}}) \leq p(G(t^c, \succ), |t|) + 1 \).

\(^5\) On the other hand it is easy to construct non-collapsible orders: Suppose we extend the natural numbers \( \mathbb{N} \) by a non-standard element \( \infty \) such that for any \( n \in \mathbb{N} \) we set \( \infty > n \). Clearly we cannot collapse \( \infty \) to a natural number.
Proof. First, observe that the assumptions imply that any basic term \( t \in \mathcal{T}_b \) is terminating with respect to \( \mathcal{R} \). This is a direct consequence of Lemma 8.4.1 and Lemma 8.6.1 in conjunction with the assumptions of the theorem. Without loss of generality, we assume \( \mathcal{P} = \text{WDP}(\mathcal{P}) \). By Theorem 8.4.1 and 8.5.1 we obtain:

\[
dl(t, \rightarrow) \leq dl(t^2, \rightarrow_{U(\mathcal{P})} \cup \mathcal{P}) \leq p(dl(t^2, \rightarrow_{P/\mathcal{U}(\mathcal{P})}), |t^2|) \\
\leq p(G(t^2, \succ), |t^2|) = p(G(t^2, \succ), |t|)
\]

In the last line we exploit that \( |t^2| = |t| \). 

Note that there exist two subtle disadvantages of Theorem 8.6.1 in comparison to Theorem 8.4.1. First the Main Theorem requires that the set of weak, weak innermost, or (standard) dependency pairs \( \mathcal{P} \) is non-duplicating. Second, the requirement that the usable rules are compatible with some SLI, implies that all usable rules must be non-duplicating. Hence the set \( \mathcal{U}(\mathcal{P}) \cup \mathcal{P} \) must not contain duplicating rules. This is not necessary to meet the requirements of Theorem 8.4.1.

In order to construct safe reduction pairs one may use safe algebras, i.e., weakly monotone well-founded algebras \((\mathcal{A}, \succ)\) such that the interpretations of compound symbols are strictly monotone with respect to \( \succ \). Another way is to apply an argument filtering to a reduction pair.

Definition 8.6.5. An argument filtering for a signature \( \mathcal{F} \) is a mapping \( \pi \) that assigns to every \( n \)-ary function symbol \( f \in \mathcal{F} \) an argument position \( i \in \{1, \ldots, n\} \) or a (possibly empty) list \([i_1, \ldots, i_m]\) of argument positions with \( 1 \leq i_1 < \cdots < i_m \leq n \). The signature \( \mathcal{F}_\pi \) consists of all function symbols \( f \) such that \( \pi(f) \) is some list \([i_1, \ldots, i_m]\), where in \( \mathcal{F}_\pi \) the arity of \( f \) is \( m \). Every argument filtering \( \pi \) induces a mapping from \( T(\mathcal{F}, \mathcal{V}) \) to \( T(\mathcal{F}_\pi, \mathcal{V}) \), also denoted by \( \pi \):

\[
\pi(t) = \begin{cases} 
  t & \text{if } t \text{ is a variable} \\
  \pi(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } \pi(f) = i \\
  f(\pi(t_{i_1}), \ldots, \pi(t_{i_m})) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } \pi(f) = [i_1, \ldots, i_m] 
\end{cases}
\]

An argument filtering \( \pi \) is called safe if \( \pi(c) = [1, \ldots, n] \) for all \( n \)-ary compound symbols.

For a relation \( R \) on \( T(\mathcal{F}, \mathcal{V}) \) we define \( R^\pi \) on \( T(\mathcal{F}_\pi, \mathcal{V}) \) as follows: \( s R^\pi t \) if and only if \( \pi(s) R \pi(t) \).

Lemma 8.6.2. If \((\mathcal{A}, \succ)\) is a safe algebra then \((\succ_{\mathcal{A}}, \succ_{\mathcal{A}})\) is a safe reduction pair, where \( \succ \) denotes the reflexive closure of \( \succ \). Furthermore, \((\succ_{\pi}, \succ_{\pi})\) is a safe reduction pair if \((\succ, \succ)\) is a safe reduction pair and \( \pi \) is a safe argument filtering.

Following the pattern of the proof of Corollary 8.4.1 it is an easy exercise to extend Theorem 8.6.1 to a method for complexity analysis.

Corollary 8.6.1. Let \( \mathcal{R} \) be a TRS, let \( \mathcal{A} \) be an SLI, let \( \mathcal{P} \) be the set of weak, weak innermost, or standard dependency pairs, such that \( \mathcal{P} \) is non-duplicating, where the compound symbols in \( \text{WIDP}(\mathcal{R}) \) are nullary, if \( \mathcal{P} = \text{DP}(\mathcal{R}) \). Moreover let \( \mathcal{B} \) be a linear
or quadratic restricted interpretation such that $(\geq_B, >_B)$ forms a safe reduction pair with $\mathcal{U}(\mathcal{P}) \subseteq \geq_B$ and $\mathcal{P} \subseteq >_B$. If $\mathcal{U}(\mathcal{P}) \subseteq >_A$ then the (innermost) runtime complexity function $rc^i_R$ with respect to $R$ is linear or quadratic, respectively.

Note that if $\mathcal{U}(\mathcal{P}) = \emptyset$, the compatibility of $\mathcal{U}(\mathcal{P})$ with an SLI is trivially satisfiable. In this special case by taking the SLI $A$ that interprets all symbols with the identity function, we obtain $dl(t, \rightarrow) \leq G(t^{\sharp}, >_B) + |t|$ because $\Delta(A, \emptyset) = 0$ and $M_A = 1$.

As a consequence of Theorem 8.6.1 and Lemma 8.3.5 we obtain the following corollary.

**Corollary 8.6.2.** Let $R$ be a TRS, let $A$ be an SLI, let all compound symbols in $\text{WIDP}(R)$ be nullary, let $\text{WIDP}(R)$ be non-duplicating, and let $B$ be a linear or quadratic restricted interpretation such that $(\geq_B, >_B)$ forms a reduction pair with $\mathcal{U}(\text{DP}(R)) \subseteq \geq_B$ and $\text{DP}(R) \subseteq >_B$. If in addition $\mathcal{U}(\text{DP}(R)) \subseteq >_A$ then the innermost runtime complexity function $rc^i_R$ with respect to $R$ is linear or quadratic, respectively.

Corollary 8.6.2 establishes (for the first time) a method to analyse the derivation length induced by the standard dependency pair method for innermost rewriting. More general, if all compound symbols in $\text{WIDP}(R)$ are nullary and there exists a collapsible reduction pair $(\geq, >)$ such that $\mathcal{U}(\mathcal{P}) \subseteq \geq$ and $\mathcal{P} \subseteq >$, then the innermost runtime complexity of $R$ is linear in the maximal length of $>-$descending steps. Clearly for string rewriting (cf. [137]) the compound symbols in $\text{WIDP}(R)$ are always nullary and all rules in $\text{WIDP}(R)$ are non-duplicating. Hence the syntactic requirements are always met.

### 8.7 Experiments

In order to test the practical feasibility of the here established methods, we implemented a complexity analyser based on syntactical transformations for dependency pairs and usable rules together with polynomial orders (based on [139]). To deal efficiently with polynomial interpretations, the issuing constraints are encoded in propositional logic in a similar spirit as in [51]. Assignments are found by employing a state-of-the-art SAT solver, in our case MiniSat\(^6\). Furthermore, strongly linear interpretations are handled by a decision procedure for Presburger arithmetic.

In a similar way, the new techniques have also been incorporated into the Tyrolean Complexity Tool (TCT for short) that incorporates the most powerful techniques to analyse the complexity of rewrite systems that are currently at hand.\(^7\) For compilation of the here presented experimental data we used the latter implementation.

As suitable test bed we used the rewrite systems in the Termination Problem Data Base version 4.0.\(^8\) This test bed comprises 1739 TRSs. The presented tests were performed on a server with 8 Dual-Core 2.6 GHz AMD\(^\circledR\) Opteron\(^\text{TM}\) Processor 8220 CPUs, for a total of 16 cores. 64 GB of RAM are available. For each system we used a timeout of 60 seconds, the times in the tables are given in seconds. Tables 8.1 and 8.2 summarise

---

\(^6\) http://minisat.se/
\(^7\) http://cl-informatik.uibk.ac.at/software/tct/
\(^8\) http://colo5-c703.uibk.ac.at:8080/termcomp/
Table 8.1: Results for Linear Runtime Complexities

<table>
<thead>
<tr>
<th></th>
<th>full rewriting</th>
<th>innermost rewriting</th>
</tr>
</thead>
<tbody>
<tr>
<td>LR</td>
<td>Cor. 8.4.1</td>
<td>Cor. 8.6.1</td>
</tr>
<tr>
<td></td>
<td>both</td>
<td>Cor. 8.4.1 (DP)</td>
</tr>
<tr>
<td></td>
<td>Cor. 8.6.1 (DP)</td>
<td>both</td>
</tr>
<tr>
<td>S</td>
<td>139</td>
<td>144</td>
</tr>
<tr>
<td></td>
<td>93</td>
<td>(136)</td>
</tr>
<tr>
<td>F</td>
<td>1591</td>
<td>1577</td>
</tr>
<tr>
<td></td>
<td>1646</td>
<td>(1581)</td>
</tr>
<tr>
<td>T</td>
<td>8</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(22)</td>
</tr>
</tbody>
</table>

Table 8.2: Results for Quadratic Runtime Complexities

<table>
<thead>
<tr>
<th></th>
<th>full rewriting</th>
<th>innermost rewriting</th>
</tr>
</thead>
<tbody>
<tr>
<td>QR</td>
<td>Cor. 8.4.1</td>
<td>Cor. 8.6.1</td>
</tr>
<tr>
<td></td>
<td>both</td>
<td>Cor. 8.4.1 (DP)</td>
</tr>
<tr>
<td></td>
<td>Cor. 8.6.1 (DP)</td>
<td>both</td>
</tr>
<tr>
<td>S</td>
<td>182</td>
<td>183</td>
</tr>
<tr>
<td></td>
<td>93</td>
<td>(166)</td>
</tr>
<tr>
<td>F</td>
<td>524</td>
<td>492</td>
</tr>
<tr>
<td></td>
<td>473</td>
<td>(855)</td>
</tr>
<tr>
<td>T</td>
<td>864</td>
<td>924</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>(884)</td>
</tr>
</tbody>
</table>

Text written in *italics* below the number of successes or failures indicates total time of success cases or failure cases, respectively. The columns marked “Cor. 8.4.1” and “Cor. 8.6.1” refer to the applicability of the respective corollaries. In the column marked “both” we indicate the results, we obtain when we first try to apply Corollary 8.6.1 and if this fails Corollary 8.4.1. Table 8.1 shows the experimental results for linear runtime complexities based on LR. The columns marked “Cor. 8.4.1” and “Cor. 8.6.1” refer to the applicability of the respective corollaries. In the column marked “both” we indicate the results, we obtain when we first try to apply Corollary 8.6.1 and if this fails Corollary 8.4.1. Table 8.2 summarises experimental results for quadratic runtime complexities based on QR. On the studied test bed there are 1567 TRSs such that one may switch from WIDP(R) to DP(R). For the individual tests, we indicated the results in parentheses for these versions of Corollary 8.4.1 and Corollary 8.6.1.

8.8 Conclusion

In this paper we studied the runtime complexity of rewrite systems. We have established a variant of the dependency pair method that is applicable in this context and is...
easily mechanisable. In particular our findings extend the class of TRSs whose linear or quadratic runtime complexity can be detected automatically. We provided ample numerical data for assessing the viability of the method. To conclude, we mention possible future work. In the experiments presented, we have restricted our attention to interpretation based methods inducing linear or quadratic (innermost) runtime complexity. Recently in [11] (see Chapter 6) a restriction of the multiset path order, called polynomial path order has been introduced that induces polynomial runtime complexity. In future work we will test to what extent this is effectively combinable with our Main Theorem. Furthermore, we strive to extend the approach presented here to handle dependency graphs [9].
9

Complexity, Graphs, and the Dependency Pair Method

Publication Details


The mistake mentioned in Chapter 8 resulted in a flawed Proposition 9 in the published version. This mistake has been rectified below. This influenced the given experimental evidence and I would like to thank Andreas Schnabl and Martin Avanzini for providing me with adjusted experimental data.

Ranking

The Conference on Logic Programming and Automated Reasoning has been ranked A by CORE in 2007.

Abstract

This paper builds on recent efforts (see Chapter 8) to exploit the dependency pair method for verifying feasible, i.e., polynomial runtime complexities of term rewrite systems automatically. We extend our earlier results by revisiting dependency graphs in the context of complexity analysis. The obtained new results are easy to implement and considerably extend the analytic power of our existing methods. The gain in power is even more significant when compared to existing methods that directly, i.e., without the use of transformations, induce feasible runtime complexities. We provide ample numerical data for assessing the viability of the method.

1 This research was partially supported by FWF (Austrian Science Fund) project P20133.
9 Complexity, Graphs, and the Dependency Pair Method

9.1 Introduction

Term rewriting is a conceptually simple but powerful abstract model of computation that underlies much of declarative programming. Runtime complexity is a notion for capturing time complexities of functions defined by a term rewriting system (TRS for short) introduced in [74] (see Chapter 8 but also [106, 26, 11]). In recent research we revisited the basic dependency pair method [9] in order to make it applicable for complexity analysis, cf. [74]. The dependency pair method introduced by Arts and Giesl [9] is one of the most powerful methods in termination analysis. The method enables us to use several powerful techniques including, usable rules, reduction pairs, argument filterings, and dependency graphs. Our main results in [74] show how natural improvements of the dependency pair method, like usable rules, reduction pairs, and argument filterings become applicable in the context of complexity analysis. In this paper, we will extend these recent results further.

The dependency pair method for termination analysis is based on the observation that from an arbitrary non-terminating term one can extract a minimal non-terminating subterm. For that one considers dependency pairs that essentially encode recursive calls in a TRS. Note that with respect to the TRS defined in Example 9.1.1 below, one finds 5 such pairs (see Section 9.4 for further details).

Example 9.1.1. Consider the following TRS \( \mathcal{R} \) which computes a permutation of lists:

\[
\begin{align*}
1: & \quad \text{app}(\text{nil}, y) \rightarrow y \\
2: & \quad \text{app}(n :: x, y) \rightarrow n :: \text{app}(x, y) \\
3: & \quad \text{reverse}(\text{nil}) \rightarrow \text{nil} \\
4: & \quad \text{reverse}(n :: x) \rightarrow \text{app}(\text{reverse}(x), n :: \text{nil}) \\
5: & \quad \text{shuffle}(\text{nil}) \rightarrow \text{nil} \\
6: & \quad \text{shuffle}(n :: x) \rightarrow n :: \text{shuffle}(\text{reverse}(x))
\end{align*}
\]

A very well-studied refinement of the dependency pair method are dependency graphs. To show termination of a TRS, it suffices to guarantee that none of the cycles in \( \text{DG}(\mathcal{R}) \) can give rise to an infinite rewrite sequence. (Here a cycle \( \mathcal{C} \) is a nonempty set of dependency pairs of \( \mathcal{R} \) such that for every two pairs \( s \rightarrow t \) and \( u \rightarrow v \) in \( \mathcal{C} \) there exists a nonempty path in \( \mathcal{C} \) from \( s \rightarrow t \) to \( u \rightarrow v \).) More precisely it suffices to prove for every cycle \( \mathcal{C} \) in the dependency graph \( \text{DG}(\mathcal{R}) \), that there are no \( \mathcal{C} \)-minimal rewrite sequences (see [57], but also [72, 62]). To achieve this one may consider each cycle independently, i.e., for each cycle it suffices to find a reduction pair \( (\succsim, \succ) \) (cf. Section 9.2) such that \( \mathcal{R} \subseteq \succsim, \mathcal{C} \subseteq \succsim \) and \( \mathcal{C} \cap \succ \neq \emptyset \), i.e., at least one dependency pair in \( \mathcal{C} \) is strictly decreasing.

Example 9.1.2 (continued from Example 9.1.1). The dependency graph \( \text{DG}(\mathcal{R}) \), whose nodes are the mentioned 5 dependency pairs, has the following form:

\[
\begin{align*}
10' \rightarrow 11' \rightarrow 8' \rightarrow 7' \\
\rightarrow 9'
\end{align*}
\]

This is Example 3.12 in Arts and Giesl’s collection of TRSs [10].
9.2 Preliminaries

This graph contains the (maximal) cycles \{7', 9', 10\}. As already mentioned, it suffices to consider each of these three cycles individually.

The main contribution of this paper is to extend the dependency graph refinement of the dependency pair method to complexity analysis. This is a challenging task, and we face a couple of difficulties, documented via suitable examples below. To overcome these obstacles we adapt the standard notion of dependency graph suitably and introduce weak (innermost) dependency graphs, based on weak dependency pairs, which have been studied in [74] (see also Chapter 8). Moreover, we observe that in the context of complexity analysis, it is not enough to focus on the (maximal) cycles of a (weak) dependency graph. Instead, we show how cycle detection is to be replaced by path detection, in order to salvage the (standard) technique of dependency graphs for runtime complexity considerations.

The remainder of the paper is organised as follows. After recalling basic notions in Section 9.2, we recall in Section 9.3 main results from [74] that will be extended in the sequel. In Section 9.4 we establish our dependency graph analysis for complexity analysis. Finally, we conclude in Section 9.5, where we assess the applicability of our method.

9.2 Preliminaries

We assume familiarity with term rewriting [15, 137], but briefly review basic concepts and notations. Moreover, we assume familiarity with standard notions in graph theory (see for example [68, Chapter 1]).

Let \( V \) denote a countably infinite set of variables and \( F \) a signature. The set of terms over \( F \) and \( V \) is denoted by \( T(F, V) \) (\( T \) for short). The root symbol of a term \( t \) is either \( t \) itself, if \( t \in V \), or the symbol \( f \), if \( t = f(t_1, \ldots, t_n) \). The set of positions \( \text{Pos}(t) \) of a term \( t \) is defined as usual. We write \( \text{Pos}_G(t) \subseteq \text{Pos}(t) \) for the set of positions of subterms whose root symbol is contained in \( G \subseteq F \). The descendants of a position with respect to a rewrite sequence are defined as usual, cf. [137]. The subterm relation is denoted as \( \sqsubseteq \).

\( \text{Var}(t) (\text{Fun}(t)) \) denotes the set of variables (functions) occurring in a term \( t \). The size \( |t| \) of a term is defined as the number of symbols in \( t \). A term rewrite system \( R \) over \( T(F, V) \) is a finite set of rewrite rules \( l \rightarrow r \), such that \( l \notin V \) and \( \text{Var}(l) \supseteq \text{Var}(r) \). The smallest rewrite relation that contains \( R \) is denoted by \( \rightarrow^* \), and its transitive and reflexive closure by \( \rightarrow_{\text{ref}} \). We simply write \( \rightarrow \) for \( \rightarrow_{\text{ref}} \) if \( R \) is clear from context. A term \( s \in T(F, V) \) is called a normal form if there is no \( t \in T(F, V) \) such that \( s \rightarrow t \). The innermost rewrite relation \( \rightarrow_{\text{inn}} \) of a TRS \( R \) is defined on terms as follows: \( s \rightarrow_{\text{inn}} t \) if there exists a rewrite rule \( l \rightarrow r \in R \), a context \( C \), and a substitution \( \sigma \) such that \( s = C[l\sigma] \), \( t = C[r\sigma] \), and all proper subterms of \( l\sigma \) are normal forms of \( R \). The set of defined symbols is denoted as \( D \), while the constructor symbols are collected in \( C \). We call a term \( t = f(t_1, \ldots, t_n) \) basic if \( f \in D \) and \( t_i \in T(C, V) \) for all \( 1 \leq i \leq n \).

We call a TRS terminating if no infinite rewrite sequence exists. The \( n \)-fold composition of \( \rightarrow \) is denoted as \( \rightarrow^n \) and the derivation length of a terminating term \( t \) with respect \( ^3 \) Recall that a cycle \( C \) is maximal, if there is no longer cycle containing \( C \).
to a TRS \( \mathcal{R} \) and rewrite relation \( \rightarrow_{\mathcal{R}} \) is defined as: \( \text{dl}(s, \rightarrow_{\mathcal{R}}) := \max\{n \mid \exists t \rightarrow^{n} t\} \).

Let \( \mathcal{R} \) be a TRS and \( T \) be a set of terms. The runtime complexity function with respect to a relation \( \rightarrow \) on \( T \) is defined as follows:

\[
\text{rc}(n, T, \rightarrow) := \max\{\text{dl}(t, \rightarrow) \mid t \in T \text{ and } |t| \leq n\}.
\]

In particular we are interested in the (innermost) runtime complexity with respect to \( \rightarrow_{\mathcal{R}} \) on the set \( T_{b} \) of all basic terms. More precisely, the runtime complexity function (with respect to \( \mathcal{R} \)) is defined as \( \text{rc}_{\mathcal{R}}(n) := \text{rc}(n, T_{b}, \rightarrow_{\mathcal{R}}) \) and we define the innermost runtime complexity function as \( \text{rc}_{i\mathcal{R}}(n) := \text{rc}(n, T_{b}, \overset{\rightarrow}{\sim}_{\mathcal{R}}) \). Note that the derivational complexity function (with respect to \( \mathcal{R} \)) becomes definable as follows:

\[
\text{dc}_{\mathcal{R}}(n) := \text{rc}(n, T, \rightarrow_{\mathcal{R}}),
\]

where \( T \) denotes the set of all terms \( T(\mathcal{F}, \mathcal{V}) \), compare [79]. We sometimes say the (innermost) runtime complexity of \( \mathcal{R} \) is linear, quadratic, or polynomial if \( \text{rc}_{i\mathcal{R}}(i) \) is bounded by a linear, quadratic, or polynomial function in \( n \), respectively.

A proper order is a transitive and irreflexive relation and a preorder is a transitive and reflexive relation. A proper order \( > \) is well-founded if there is no infinite decreasing sequence \( t_{1} \succ t_{2} \succ t_{3} \cdots \). An \( \mathcal{F} \)-algebra \( \mathcal{A} \) consists of a carrier set \( A \) and an interpretation \( f_{\mathcal{A}} \) for each function symbol in \( \mathcal{F} \). A well-founded and monotone algebra (WMA for short) is a pair \((\mathcal{A}, >)\), where \( \mathcal{A} \) is an algebra and \( > \) is a well-founded proper order on \( A \) such that every \( f_{\mathcal{A}} \) is monotone (with respect to \( > \)) in all arguments. An assignment \( \alpha : \mathcal{V} \rightarrow A \) is a function mapping variables to elements in the carrier, and \([\alpha]_{\mathcal{A}}(\cdot)\) denotes the usual evaluation function associated with \( \mathcal{A} \). A WMA naturally induces a proper order \( >_{\mathcal{A}} \) on terms: \( s >_{\mathcal{A}} t \) if \([\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)\) for all assignments \( \alpha : \mathcal{V} \rightarrow A \). For the reflexive closure \( \succeq \) of \( > \), the preorder \( \succeq_{\mathcal{A}} \) is similarly defined. Clearly the proper order \( >_{\mathcal{A}} \) is a reduction order, i.e., if \( \mathcal{R} \subseteq >_{\mathcal{A}} \), for a TRS \( \mathcal{R} \), then we can conclude termination of \( \mathcal{R} \). A rewrite preorder is a preorder on terms which is closed under contexts and substitutions. A reduction pair \((\succeq, >)\) consists of a rewrite preorder \( \succeq \) and a compatible well-founded order \( > \) which is closed under substitutions. Here compatibility means the inclusion \( \succeq \cdot > \cdot \succeq \subseteq > \). Note that for any WMA \( \mathcal{A} \) the pair \((\succeq_{\mathcal{A}}, >_{\mathcal{A}})\) constitutes a reduction pair.

We call a WMA \( \mathcal{A} \) based on the natural numbers \( \mathbb{N} \) a polynomial interpretation, if all functions \( f_{\mathcal{A}} \) are polynomials. A polynomial \( P(x_{1}, \ldots, x_{n}) \) (over the natural numbers) is called strongly linear if \( P(x_{1}, \ldots, x_{n}) = x_{1} + \cdots + x_{n} + c \) where \( c \in \mathbb{N} \). A polynomial interpretation is called linear restricted if all constructor symbols are interpreted by strongly linear polynomials and all other function symbols by linear polynomials. If on the other hand the non-constructor symbols are interpreted by quadratic polynomials, the polynomial interpretation is called quadratic restricted. Here a polynomial is quadratic if it is a sum of monomials of degree at most 2 (see [43]). It is easy to see that if a TRS \( \mathcal{R} \) is compatible with a linear or quadratic restricted interpretation, the runtime complexity of \( \mathcal{R} \) is linear or quadratic, respectively (see [74] but also [26]).

Finally, we introduce a very restrictive class of polynomial interpretations: strongly linear interpretations (SLI for short). A polynomial interpretation is called strongly

---

4 We can replace \( T_{b} \) by the set of terms \( f(t_{1}, \ldots, t_{n}) \) with \( f \in \mathcal{D} \), whose arguments \( t_{i} \) are in normal form, while keeping all results in this paper.
9.3 Complexity Analysis Based on the Dependency Pair Method

In this section, we recall central definitions and results established in [74] (see also Chapter [8]). We kindly refer the reader to [74] for additional examples and underlying intuitions.

We write \( C(t_1, \ldots, t_n)_X \) to denote \( C[t_1, \ldots, t_n] \), whenever root\((t_i) \subseteq X \) for all \( 1 \leq i \leq n \) and \( C \) is an \( n \)-hole context containing no \( X \)-symbols. Let \( t \) be a term. We set \( t^l := t \) if \( t \subseteq \mathcal{V} \), and \( t^l := f^l(t_1, \ldots, t_n) \) if \( t = f(t_1, \ldots, t_n) \). Here \( f^l \) is a new \( n \)-ary function symbol called dependency pair symbol. For a signature \( \mathcal{F} \), we define \( \mathcal{F}^\perp = \mathcal{F} \cup \{ f^l \mid f \in \mathcal{F} \} \).

**Definition 9.3.1.** Let \( \mathcal{R} \) be a TRS. If \( l \rightarrow r \in \mathcal{R} \) and \( r = C(u_1, \ldots, u_n)_D \) then the rewrite rule \( l^l \rightarrow \text{COM}(u_1^l, \ldots, u_n^l) \) is called a weak dependency pair of \( \mathcal{R} \). Here \( \text{COM} \) is defined with a fresh \( n \)-ary function symbol \( c \) (corresponding to \( l \rightarrow r \)) as follows: \( \text{COM}(t_1, \ldots, t_n) = t_1 \) if \( n = 1 \), and \( c(t_1, \ldots, t_n) \) otherwise. The symbol \( c \) is called compound symbol. The set of all weak dependency pairs is denoted by WDP(\( \mathcal{R} \)).

**Example 9.3.1** (continued from Example [9.1.1]). The set WDP(\( \mathcal{R} \)) consists of the next 6 weak dependency pairs.

- 7: \( \text{app}^l(nil, y) \rightarrow y \)
- 8: \( \text{app}^l(n : x, y) \rightarrow c(n, \text{app}^l(x, y)) \)
- 9: \( \text{reverse}^l(nil) \rightarrow d \)
- 10: \( \text{reverse}^l(n :: x) \rightarrow \text{app}^l(\text{reverse}(x), n :: nil) \)
- 11: \( \text{shuffle}^l(nil) \rightarrow e \)
- 12: \( \text{shuffle}^l(n :: x) \rightarrow f(n, \text{shuffle}^l(\text{reverse}(x))) \)

**Definition 9.3.2.** Let \( \mathcal{R} \) be a TRS. If \( l \rightarrow r \in \mathcal{R} \) and \( r = C(u_1, \ldots, u_n)_D \) then the rewrite rule \( l^l \rightarrow \text{COM}(u_1^l, \ldots, u_n^l) \) is called a weak innermost dependency pair of \( \mathcal{R} \). The set of all weak innermost dependency pairs is denoted by WIDP(\( \mathcal{R} \)).

Definitions [9.3.1] and [9.3.2] should be compared to the definition of “standard” dependency pairs.

**Definition 9.3.3** ([8]). The set DP(\( \mathcal{R} \)) of (standard) dependency pairs of a TRS \( \mathcal{R} \) is defined as \( \{ l^l \rightarrow u^l \mid l \rightarrow r \in \mathcal{R}, u \subseteq r, \text{root}(u) \subseteq \mathcal{D} \} \).

**Example 9.3.2** (continued from Example [9.3.1]). As already mentioned in the introduction, the TRS \( \mathcal{R} \) admits 5 (standard) dependency pairs. Note that the sets DP(\( \mathcal{R} \)) and WDP(\( \mathcal{R} \)) are incomparable. For example \( \text{app}^l(nil, y) \rightarrow y \in \text{WDP}(\mathcal{R}) \setminus \text{DP}(\mathcal{R}) \) while \( \text{shuffle}^l(x) \rightarrow \text{reverse}^l(x) \in \text{DP}(\mathcal{R}) \setminus \text{WDP}(\mathcal{R}) \).

We write \( f \triangleright_d g \) if there exists a rewrite rule \( l \rightarrow r \in \mathcal{R} \) such that \( f = \text{root}(l) \) and \( g \) is a defined symbol in \( \text{Fun}(r) \). For a set \( \mathcal{G} \) of defined symbols we denote by \( \mathcal{R}[\mathcal{G}] \) the set of rewrite rules \( l \rightarrow r \in \mathcal{R} \) with \( \text{root}(l) \in \mathcal{G} \). The set \( U(t) \) of usable rules of a term \( t \) is defined as \( \mathcal{R}[\{ g \mid f \triangleright_d g \text{ for some } f \in \text{Fun}(t) \}] \). Finally, if \( \mathcal{P} \) is a set of (weak or weak innermost) dependency pairs then \( U(\mathcal{P}) = \bigcup_{l \rightarrow r \in \mathcal{P}} U(r) \).
Proposition 9.3.1 (Chapter 8 Theorem 8.4.1). Let $\mathcal{R}$ be a TRS and let $t \in T_b$. If $t$ is terminating with respect to $\rightarrow$ then $d(t, \rightarrow) \leq d(t, \rightarrow_{U(\mathcal{P})})$, where $\rightarrow$ denotes $\rightarrow_\mathcal{R}$ or $\rightarrow_\mathcal{R}$ depending on whether $\mathcal{P} = WDP(\mathcal{R})$ or $\mathcal{P} = WIDP(\mathcal{R})$.

We recall the notion of relative rewriting [137]. Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs. We write $\rightarrow_{\mathcal{R}/\mathcal{S}}$ for $\rightarrow_\mathcal{R} \cdot \rightarrow_\mathcal{R} \cdot \rightarrow_\mathcal{S}$ and we call $\rightarrow_{\mathcal{R}/\mathcal{S}}$ the relative rewrite relation of $\mathcal{R}$ over $\mathcal{S}$. (Note that $\rightarrow_{\mathcal{R}/\mathcal{S}} = \rightarrow_\mathcal{R}$, if $\mathcal{S} = \emptyset$.) Let $\mathcal{A}$ denote a strongly linear interpretation.

Proposition 9.3.2 (Chapter 8 Theorem 8.5.1). Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs, $\mathcal{A}$ an SLI compatible with $\mathcal{S}$, and $\mathcal{R}$ non-duplicating. There exist constants $K$ and $L$, depending only on $\mathcal{R}$ and $\mathcal{A}$, such that $d\left(t, \rightarrow_{\mathcal{R}/\mathcal{S}}\right) \leq K \cdot d\left(t, \rightarrow_{\mathcal{R}}\right) + L \cdot |t|$ for all terminating terms $t$ on $\mathcal{R} \cup \mathcal{S}$.

We need some further definitions. Let $\mathcal{R}$ be a TRS, let $\mathcal{P}$ be a set of weak or weak innermost dependency pairs of $\mathcal{R}$ and let $G$ denote a mapping associating a term (over $\mathcal{T}^2$ and $\mathcal{Y}$) and a proper order $\succ$ with a natural number. An order $\succ$ on terms is $G$-collapsible for a TRS $\mathcal{R}$ if $s \rightarrow_{\mathcal{R},U(\mathcal{P})} t$ and $s \succ t$ implies $G(s, \succ) > G(t, \succ)$. An order $\succ$ is collapsible for a TRS $\mathcal{R}$, if there is a mapping $G$ such that $\succ$ is $G$-collapsible for $\mathcal{R}$.

We write $\mathcal{T}_0^\mathcal{R}$ for \{t | t \in T_b\}. The set $\mathcal{T}_0^\mathcal{R}$ is inductively defined as follows (i) $\mathcal{T}_0^\mathcal{R} \subseteq \mathcal{T}_0^\mathcal{R}$, where $\mathcal{T}_0^\mathcal{R} = \{t \mid t \in \mathcal{T}\}$ and (ii) $c(t_1, \ldots, t_n) \in \mathcal{T}_0^\mathcal{R}$, whenever $t_1, \ldots, t_n \in \mathcal{T}_0^\mathcal{R}$ and $c$ a compound symbol. A proper order $\succ$ on $\mathcal{T}_0^\mathcal{R}$ is called safe if $c(s_1, \ldots, s_i, \ldots, s_n) \succ c(s_1, \ldots, t, \ldots, s_n)$ for all $n$-ary compound symbols $c$ and all terms $s_1, \ldots, s_n, t$ with $s_i \succ t$. A reduction pair $(\succ, \prec)$ is called collapsible for a TRS $\mathcal{R}$ if $\succ$ is collapsible for $\mathcal{R}$. It is called safe if the well-founded order $\succ$ is safe. In order to construct a non-duplicating reduction pairs one may use safe algebras, i.e., weakly monotone well-founded algebras $(\mathcal{A}, \succ)$ such that the interpretations of compound symbols are strictly monotone with respect to $\succ$.

It is easy to see that if $(\mathcal{A}, \succ)$ is a safe algebra then $(\succ_{\mathcal{A}}, \succ_{\mathcal{A}})$ is a safe reduction pair.

Proposition 9.3.3 (Chapter 8 Theorem 8.6.1). Let $\mathcal{R}$ be a TRS, let $\mathcal{A}$ be an SLI, let $\mathcal{P}$ be a set of weak or weak innermost dependency pairs, such that $\mathcal{P}$ is non-duplicating, and let $(\succ_{\mathcal{A}})$ be a safe and $G$-collapsible reduction pair such that $U(\mathcal{P}) \subseteq \succ_{\mathcal{A}}$ and $\mathcal{P} \subseteq \succ_{\mathcal{A}}$.

If in addition $U(\mathcal{P}) \subseteq \succ_{\mathcal{A}}$ then for any $t \in T_b$, there exist constants $K$ and $L$ (depending only on $\mathcal{R}$ and $\mathcal{A}$) such that $d\left(t, \rightarrow\right) \leq K \cdot G(t, \succ_{\mathcal{A}}) + L \cdot |t|$. Here $\rightarrow$ denotes $\rightarrow_\mathcal{R}$ or $\rightarrow_\mathcal{R}$ depending on whether $\mathcal{P} = WDP(\mathcal{R})$ or $\mathcal{P} = WIDP(\mathcal{R})$.

Suppose the assertions of the proposition are met and there exists a polynomial $p$ such that $G(t, \succ_{\mathcal{A}}) \leq p(|t|)$ holds. Then, as an easy corollary to Proposition 9.3.3 we observe that the runtime complexity induced by $\mathcal{R}$ is majorised by $p$.

### 9.4 Dependency Graphs

In this section, we study a natural refinement of the dependency pair method, namely dependency graphs (see [9, 57, 59, 72]) in the context of complexity analysis. We start

---

Note that most reduction orders are collapsible. E.g. if $\mathcal{A}$ is a polynomial interpretation then $\succ_{\mathcal{A}}$ is collapsible, as one may take any $\alpha$ and set $G(t, \succ_{\mathcal{A}}) := [\alpha]_{\mathcal{A}}(t)$.  

126
with a brief motivation. Let $\mathcal{R}$ be a TRS, let $\mathcal{P}$ denote a set of weak or weak innermost dependency pairs and let $(s_i)_{i=0, \ldots, n}$ denote a maximal derivation $D$ with respect to $\mathcal{R}$ with $s_0 \in T_b$. In order to estimate the length $\ell$ of this derivation it suffices to estimate the length of the derivation $t_0 \rightarrow^{*}_{t(U(\mathcal{P})) U(\mathcal{P}) \cup \mathcal{P}} t_n$, where $t_0 = s_0^\sharp \in T^\sharp_b$, cf. Proposition 9.3.1. If we suppose that $\mathcal{P}$ is non-duplication and that there exists an SLI such that $U(\mathcal{P}) \subseteq A$, we may estimate the derivation length $\ell$ by finding one (safe and collapsible) reduction pair $(\succeq, \succ)$ such that $U(\mathcal{P}) \subseteq \succeq$ and $\mathcal{P} \subseteq \succ$ holds, cf. Proposition 9.3.3. On the other hand in termination analysis—as already mentioned in the introduction—it suffices to guarantee that for any cycle $\mathcal{C}$ in the dependency graph $DG(\mathcal{R})$, there are no $\mathcal{C}$-minimal rewrite sequences, cf. [57]. Hence, we strive to extend this idea to complexity analysis.

### 9.4.1 From Cycle Analysis to Path Detection

Let us recall the definition of a dependency graph and extend it suitably to weak and weak innermost dependency pairs.

**Definition 9.4.1.** Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ and let $\mathcal{P}$ be the set of weak, weak innermost, or (standard) dependency pairs. The nodes of the weak dependency graph $WDG(\mathcal{R})$, weak innermost dependency graph $WIDG(\mathcal{R})$, or dependency graph $DG(\mathcal{R})$ are the elements of $\mathcal{P}$ and there is an arrow from $s \rightarrow t$ to $u \rightarrow v$ if and only if there exist a context $C$ and substitutions $\sigma, \tau : V \rightarrow T(F, V)$ such that $t\sigma \rightarrow^* C[u\tau]$, where $\rightarrow$ denotes $\rightarrow_R$ or $i \rightarrow_R$ depending on whether $\mathcal{P} = WDP(\mathcal{R}), \mathcal{P} = DP(\mathcal{R})$ or $\mathcal{P} = WIDP(\mathcal{R})$, respectively.

**Example 9.4.1 (continued from Example 9.3.1).** The weak dependency graph $WDG(\mathcal{R})$ has the following form.

```
10 → 8 → 7     9 → 12 → 11
```

We recall a theorem on the dependency graph refinement in conjunction with usable rules and innermost rewriting (see [57], but also [70]). Similar results hold in the context of full rewriting, see [62, 72].

**Theorem 9.4.1 ([57]).** A TRS $\mathcal{R}$ is innermost terminating if for every maximal cycle $\mathcal{C}$ in the dependency graph $DG(\mathcal{R})$ there exists a reduction pair $(\succeq, \succ)$ such that $U(\mathcal{C}) \subseteq \succeq$ and $\mathcal{C} \subseteq \succ$.

The following example shows that we cannot directly employ Theorem 9.4.1 in the realm of complexity analysis. Even though in this setting we can restrict our attention to a specific strategy: innermost rewriting.

**Example 9.4.2.** Consider the TRS $\mathcal{R}_{exp}$

- $exp(0) \rightarrow s(0)$
- $d(0) \rightarrow 0$
- $exp(r(x)) \rightarrow d(exp(x))$
- $d(s(x)) \rightarrow s(s(d(x)))$
DP(\(R_{\text{exp}}\)) consists of three pairs: 1: \(\exp^\sharp(r(x)) \to d^\sharp(\exp(x))\), 2: \(\exp^\sharp(r(x)) \to \exp^\sharp(x)\), and 3: \(d^\sharp(s(x)) \to d^\sharp(x)\). Hence the dependency graph DG(\(R_{\text{exp}}\)) contains two maximal cycles: \{2\} and \{3\}. It is easy to see how to define two reduction pairs \(+,>_{A}\) and \(+,>_{B}\) such that the conditions of the theorem are fulfilled. For that it suffices to define interpretations \(A\) and \(B\), respectively. Because one can find suitable linear restricted ones for \(A\) and \(B\), compatibility with these interpretations apparently induces linear runtime complexity of \(R_{\text{exp}}\), cf. [26, 74] (even for full rewriting). However, we must not conclude linear innermost runtime complexity for \(R_{\text{exp}}\) in this setting, as \(R_{\text{exp}}\) formalises the exponentiation function and setting \(t_n = \exp(r^n(0))\) we obtain \(dl(t_n, \to_R) > 2^n\) for each \(n \geq 0\). Thus the innermost runtime complexity of \(R_{\text{exp}}\) is exponential.

Note that the problem exemplified by Example 9.4.2 cannot be circumvented by replacing the dependency graph employed in Theorem 9.4.1 with the weak (innermost) dependency graph. Furthermore, observe that while Proposition 9.3.1 allows us to replace in Example 9.4.2 the innermost rewrite relation \(\sim_R\) by the (sometimes simpler) rewrite relation \(\sim_{DP(R)} \cup DP(R)\), this is of no help: The exponential length of \(t^\sharp_n\) in Example 9.4.2 with respect to \(U(DP(R)) \cup DP(R)\) is not due to the cycles \{2\} or \{3\}, but achieved through the non-cyclic pair 1 and its usable rules. These observations are cast into Definition 9.4.2. below.

A graph is called \textit{strongly connected} if any node is connected with every other node by a path. A \textit{strongly connected component} (SCC for short) is a maximal strongly connected subgraph.

\textbf{Definition 9.4.2.} Let \(G\) be a graph, let \(\equiv\) denote the equivalence relation induced by SCCs, and let \(P\) be an SCC in \(G\). The set of all source nodes in \(G/\equiv\) is denoted by \(\text{Src}\). Let \(l \to r\) be a dependency pair in \(G\), let \(K \in G/\equiv\) and let \(C\) denote the SCC represented by \(K\). Then we write \(l \to r \in K\) if \(l \to r \in C\).

\textbf{Example 9.4.3} (Continued from Example 9.4.1). There are 6 (trivial) SCCs in WDG(\(R\)), all being trivial. Hence the graph WDG(\(R\))/\(\equiv\) has the following form:

\[10 \to 8 \to 7 \quad 9 \quad 12 \to 11\]

Here \(\text{Src} = \{\{9\}, \{10\}, \{12\}\}\).

\section*{9.4.2 Refinement Based on Path Detection}

We re-consider the motivating derivation \(D\):

\[t_0 \to_{U(\mathcal{P}) \cup \mathcal{P}} t_1 \to_{U(\mathcal{P}) \cup \mathcal{P}} \cdots \to_{U(\mathcal{P}) \cup \mathcal{P}} t_n ,\]

where \(t_0 \in T^\sharp_{R}\). To simplify the exposition, we set \(\mathcal{P} = WDP(\mathcal{R})\) and \(G = WDG(\mathcal{R})\). Momentarily we assume that all compound symbol are of arity 0, as is for instance

\footnotetext[6]{Note that in the literature SCCs are sometimes defined as \textit{maximal cycles}. This alternative definition is of limited use in our context as we must not ignore trivial SCCs.}
the case in Example 9.3.1. Above we asserted that there exists an SLI \( \mathcal{A} \) such that \( \mathcal{U}(\mathcal{P}) \subseteq \succ \mathcal{A} \). Hence Proposition 9.3.2 is applicable. Thus, to estimate the length of the derivation (9.1) it suffices to consider the following relative rewriting derivation:

\[
t_0 \rightarrow_{p/\mathcal{U}(\mathcal{P})} t_1 \rightarrow_{p/\mathcal{U}(\mathcal{P})} \cdots \rightarrow_{p/\mathcal{U}(\mathcal{P})} t_n.
\]

(9.2)

Exploiting the given assumptions, it is not difficult to see that derivation (9.2) is representable as follows:

\[
t_0 \rightarrow_{\ell_1} t_1 \rightarrow_{\ell_2} t_2 \rightarrow_{\ell_3} t_3 \rightarrow \cdots \rightarrow_{\ell_m} t_m,
\]

(9.3)

where, \((\mathcal{P}_1, \ldots, \mathcal{P}_m)\) is a path in \( G/\equiv \) with \( \mathcal{P}_1 \in \text{Src} \) and \( \ell_i \geq 0 \) (\( i = 1, \ldots, n \)). Since the length \( \ell \) of the pictured \( \rightarrow_{p/\mathcal{U}(\mathcal{P})} \)-rewrite sequence equals \( \ell_1 + \cdots + \ell_m \), this suggests that we can estimate each \( \ell_j \) (\( j \in \{1, \ldots, m\} \)) independently. We assume the existence of a family of SLIs \( \mathcal{B}_j \) (\( j \in \{1, \ldots, m\} \)) such that \( \mathcal{U}(\mathcal{P}_1) \cup \cdots \cup \mathcal{U}(\mathcal{P}_j) \subseteq \succ \mathcal{B}_j \) and \( \mathcal{P}_j \subseteq \succ \mathcal{B}_j \) holds for every \( j \). From this we can conclude \( \ell_j = \Omega(|t_{\ell_j}|) \) for all \( j \in \{1, \ldots, m\} \). The next step is to estimate each \( \ell_j \) by a function (preferable a polynomial) in \( |t_0| \). As each of the WMAs \( \mathcal{B}_j \) is assumed to be strongly linear, we can even conclude \( [a_0]_{\mathcal{B}_j}(t_{\ell_j}) = \Omega(|t_{\ell_j}|) \). (Here \( a_0 \) denotes the assignment mapping any variable to 0.) In sum, we obtain for each \( j \in \{1, \ldots, m\} \), the existence of a constant \( c_j \) such that \( |t_{\ell_j}| \leq c_j \cdot |t_0| \) and thus there exists a linear polynomial \( p(x) \) such that \( \ell_j \leq p(|t_0|) \). However, some care is necessary in assessing this observation: Note that the given argument cannot be used to deduce polynomial runtime complexity, if we weaken the assumption that the algebras \( \mathcal{B}_j \) are strongly linear only slightly. Hence, we replace the direct application of Proposition 9.3.2 as follows.

**Lemma 9.4.1.** \( n \leq \text{dl}(s, \rightarrow_{\mathcal{R}_2/(S_1 \cup S_2)}) \) whenever \( s \rightarrow_{S_1^*} \rightarrow_{\mathcal{R}_2/S_2}^{n} u. \)

**Proof.** Straightforward. \( \square \)

We lift the assumption that all compound symbols are of arity at most 0. Perhaps surprisingly this generalisation complicates the matter considerably. First a maximal derivation need no longer be of the form given in (9.3) which is exemplified by Example 9.4.4 below.

**Example 9.4.4.** Consider the TRS \( \mathcal{R} = \{ f(0) \rightarrow \text{leaf}, f(s(x)) \rightarrow \text{branch}(f(x), f(x)) \} \).

The set \( \text{WDP}(\mathcal{R}) \) consists of the two weak dependency pairs: 1: \( f^l(0) \rightarrow c_1 \) and 2: \( f^l(s(x)) \rightarrow c_2(f^l(x), f^l(x)) \). Hence the weak dependency graph \( \text{WDG}(\mathcal{R}) \) contains 2 SCCs: \( \{2\} \) and \( \{1\} \). Clearly \( \text{Src} = \{\{2\}\} \). Let \( t_n = f^l(s^n(0)) \). Consider the following sequence:

\[
t_2 \rightarrow_{\{2\}^2} c_2(c_2(t_0, t_0), t_1) \rightarrow_{\{1\}^2} c_2(c_2(c_1, t_0), t_1) \rightarrow_{\{2\}^3} c_2(c_2(c_1, t_0), c_2(t_0, t_0)) \rightarrow_{\{1\}^3} c_2(c_2(c_1, c_1), c_2(c_1, c_1)).
\]

This derivation does not have the form \( (9.3) \), because it is based on the sequence \( (\{2\}, \{1\}, \{2\}, \{1\}) \), which is not a path in \( \text{WDG}(\mathcal{R})/\equiv \).
Note that the derivation in Example 9.4.4 can be reordered (without affecting its length) such that the derivation becomes based on a path. Still, not every derivation can be abstracted to a path. Consider a maximal (with respect to subset inclusion) component of WDG(\(R\))/\(\equiv\). Clearly this component forms a directed acyclic graph \(G\), and without loss of generality we assume in the following that \(G\) is a tree \(T\) with root in \(S_{rc}\). Otherwise, observe that any directed acyclic graph \(G\) can be unfolded to a forest \(F\) and that the size of \(G\) is bounded in the size of \(F\). Moreover if \(T \in F\) is of maximal depth, then the size of \(F\) is linearly bounded in the size of \(T\) as the number of trees in \(F\) depends only on \(R\). Suppose further that \(T\) is not degenerated to a branch.

Then a given derivation may only be abstractable by different paths in \(T\), as exemplified by Example 9.4.5.

**Example 9.4.5.** Consider the TRS \(R = \{ f \rightarrow c(g,h), g \rightarrow a, h \rightarrow a \}\). Thus WDP(\(R\)) consists of three dependency pairs: 1: \(f^\sharp \rightarrow c_1(g^\sharp, h^\sharp)\), 2: \(g^\sharp \rightarrow c_2\), and 3: \(h^\sharp \rightarrow c_3\). Let \(P := \text{WDP}(R)\), then clearly \(P = \text{WDG}(R) = \text{WDG}(R)/\(\equiv\). Consider the following derivation
\[
f^\sharp \rightarrow_P c_1(g^\sharp, h^\sharp) \rightarrow_P c_1(c_2, h^\sharp) \rightarrow_P c_1(c_2, c_3)
\]
This derivation is composed from the paths \((\{1\}, \{2\})\) and \((\{1\}, \{3\})\).

Fortunately, we can circumvent these obstacles. Let \(P\) denote the set of weak or weak innermost dependency pairs of a TRS \(R\). We make the following easy observation.

**Lemma 9.4.2.** Let \(G\) denote a weak or weak innermost dependency graph. Let \(C \subseteq G\) and let \(D: s \rightarrow_{\text{c}C/\text{U}(P)}^* t\) denote a derivation based on \(C\) with \(s \in T^C_e\). Then \(D\) has the following form: \(s = s_0 \rightarrow_{\text{c}C/\text{U}(P)} s_1 \rightarrow_{\text{c}C/\text{U}(P)} \cdots \rightarrow_{\text{c}C/\text{U}(P)} s_n = t\) where each \(s_i \in T^C_e\).

**Proof.** It is easy to see that \(D\) has the presented form and that for each \(i \in \{0, \ldots, n\}\) there exists a context \(C\) such that \(s_i = C[u_{i1}^\sharp, \ldots, u_{ir}^\sharp]\) and \(C\) consists of compound symbols only. This establishes the lemma.

Motivated by Example 9.4.4 we observe that a weak (innermost) dependency pair containing an \(m\)-ary \((m > 1)\) compound symbol can only induces \(m\) independent derivations. Hence, we can reorder derivations to achieve the structure of derivation (9.3). This is formally proven via the next two lemmas.

**Lemma 9.4.3.** Let \(G\) denote a weak or weak innermost dependency graph and let \(K\) and \(L\) denote two different nodes in \(G/\(\equiv\)\) such that there is no edge from \(K\) to \(L\). Let \(s_0 \in T^K_{\mathcal{E}}\) and suppose the existence of a derivation \(D\) of the following form: \(s_0 \rightarrow_{K/\text{U}(P)}^n s_n \rightarrow_{L/\text{U}(P)} t_0 \rightarrow_{L/\text{U}(P)}^m t_m\). Then there exists a derivation \(D'\) which has the form \(t'_0 \rightarrow_{L/\text{U}(P)}^m t'_m \rightarrow_{L/\text{U}(P)} s'_0 \rightarrow_{K/\text{U}(P)} s'_n\) with \(t'_0 \in T^L_{\mathcal{E}}\).

**Proof.** Consider the following two dependency pairs: 1: \(u_k^\sharp \rightarrow \text{COM}(v_{k1}^\sharp, \ldots, v_{kr}^\sharp)\) and 2: \(u_l^\sharp \rightarrow \text{COM}(v_{l1}^\sharp, \ldots, v_{lr}^\sharp)\). Here the dependency pair 1 belongs to \(K\) and denotes the last dependency pair employed in \(D\) before the path leaves \(K\) into \(L\), while 2 denotes the first pair in \(L\). The assumption that there is no edge from \(K\) to \(L\) can be reformulated as follows:
(†) No context \( C \) and no substitutions \( \sigma, \tau: \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V}) \) exist such that

\[
\text{COM}(v^\sigma_{k1}, \ldots, v^\sigma_{kr}) \rightarrow^*_{\mathcal{U}(\mathcal{P})} C[u^\tau_1 t],
\]

holds.

To prove the lemma, we proceed by induction on \( n \). It suffices to consider the step case \( n > 1 \). By assumption the last rewrite step in the subderivation \( D_0: s_0 \rightarrow^*_c \mathcal{U}(\mathcal{P}) \) \( s_n \) employs dependency pair \( 1 \). Let \( p \in \mathcal{P}\text{os}(s_n) \) denote the position of the reduct \( \text{COM}(v^\sigma_{k1}, \ldots, v^\sigma_{kr}) \) in \( s_n \). By assumption there exists a derivation \( s_n \rightarrow^*_{\mathcal{U}(\mathcal{P})} t_0 \). Let \( q \in \mathcal{P}\text{os}(s_n) \) denote the position of the redex in \( s_n \) that is contracted as first step in this reduction. Without loss of generality we can assume that both positions are parallel to each other. Otherwise one of the following cases applies. Either \( p < q \) or \( p \geq q \). But clearly the first case contradicts the assumption (†). Hence, assume the second. But this is also impossible. Lemma 9.4.2 yields that \( s_{n_0} \in \mathcal{T}_c^\delta \), which contradicts that \( q \) is redex with respect to \( \mathcal{U}(\mathcal{P}) \). Repeating this argument we see that position \( p \) has exactly one descendant in \( t_0 \). A similar argument shows that all redex positions in the subderivation \( D_1: t_0 \rightarrow^*_{\mathcal{U}(\mathcal{P})} t_m \) are parallel to (descendants of) \( p \). Hence, we can move the last rewrite step \( s_{n-1} \rightarrow_c s_n \) in the derivation \( D_0 \) after the derivation \( D_1 \). Note that in each of the terms \( (t_i)_{i=1, \ldots, m} \) the position \( p \) exists and denotes the term \( \text{COM}(v^\sigma_{k1}, \ldots, v^\sigma_{kr}) \).

Hence, the replacement of \( \text{COM}(v^\sigma_{k1}, \ldots, v^\sigma_{kr}) \) everywhere by \( u^\sigma_k \sigma \) does not affect the validity of the rewrite sequence. Furthermore the set \( \mathcal{T}_c^\delta \) is closed under this operation. Now, the induction hypothesis becomes applicable to derive the existence of the sought derivation \( D' \).

Let \( \mathcal{G} \) denote a weak or weak innermost dependency graph and let \( D: s \rightarrow^\ell t \) denote a derivation, such that \( s \in \mathcal{T}_c^\ell \). Here \( \rightarrow \) denotes either \( \rightarrow_{\mathcal{P}/\mathcal{U}(\mathcal{P})} \) or \( \rightarrow_{\mathcal{P}/\mathcal{U}(\mathcal{P})} \). We say that \( D \) is based on \( (\mathcal{P}_1, \ldots, \mathcal{P}_m) \) in \( \mathcal{G}_\equiv \) if \( D \) is of the form

\[
s \overset{(i)}{\rightarrow} \mathcal{P}_{i_1}/\mathcal{U}(\mathcal{P}) \cdots \overset{(i)}{\rightarrow} \mathcal{P}_{i_m}/\mathcal{U}(\mathcal{P}) \ t,
\]

with \( i_1, \ldots, i_m \geq 0 \). We arrive at the main lemma of this section.

**Lemma 9.4.4.** Let \( \mathcal{P} \) denote a set of weak or weak innermost dependency pairs, let \( s \in \mathcal{T}_c^\ell \) and let \( D: s \rightarrow^\ell t \) denote a maximal derivation, where \( \rightarrow \) denotes \( \rightarrow_{\mathcal{P}/\mathcal{U}(\mathcal{P})} \) or \( \rightarrow_{\mathcal{P}/\mathcal{U}(\mathcal{P})} \) respectively. Suppose that \( D \) is based on \( (\mathcal{P}_1, \ldots, \mathcal{P}_m) \) and \( \mathcal{P}_1 \in \text{Src} \). Then there exists a derivation \( D': s \rightarrow^\ell t \) based on \( (\mathcal{P}'_1, \ldots, \mathcal{P}'_{m'}) \), with \( \mathcal{P}'_1 \in \text{Src} \) such that all \( \mathcal{P}'_i \) (\( i \in \{1, \ldots, m'\} \)) are pairwise distinct.

**Proof.** Without loss of generality, we restrict our attention to weak dependency pairs. To prove the lemma, we consider a sequence \( (\mathcal{P}_1, \ldots, \mathcal{P}_m) \), where there exist indices \( i, j \) and \( k \) with \( i < j < k \) and \( \mathcal{P}_i = \mathcal{P}_k \). By induction on \( j - i \) we show that this path is transformable into a sequence \( (\mathcal{P}'_1, \ldots, \mathcal{P}'_{m'}) \) of the required form. It suffices to prove the step case. Moreover, we can assume without loss of generality that \( k = j + 1 \). Consider the two dependency pairs: 1: \( i^\ell_j \rightarrow \text{COM}(u^\tau_{j1}, \ldots, u^\tau_{jr}) \) and 2: \( i^\ell_k \rightarrow \text{COM}(u^\tau_{k1}, \ldots , u^\tau_{kr}) \).
Dependency pair 1 belongs to \( \mathcal{P}_j \) and denotes the last dependency pair employed in \( D \) before the sequence leaves \( \mathcal{P}_j \) into \( \mathcal{P}_k \), while 2 denotes the first pair in \( \mathcal{P}_k \). We consider two cases:

(i) Assume there exist a context \( C \) and substitutions \( \sigma, \tau : \mathcal{V} \to T(\mathcal{F}, \mathcal{V}) \) such that the following holds: \( \text{com}(u^j_1 \sigma, \ldots, u^j_p \sigma) \rightarrow^* C[l^j_k \tau] \). Thus by definition of weak dependency graphs the node in WDG(\( \mathcal{R} \)) representing dependency pair 1 is connected to the node representing dependency pair 2. In particular every node in the SCCs represented by \( \mathcal{P}_i = \mathcal{P}_k \) is connected to every node in the SCC represented by \( \mathcal{P}_j \). This implies that \( \mathcal{P}_i = \mathcal{P}_j = \mathcal{P}_k \) contradicting the assumption.

(ii) Otherwise, there is no edge between \( \mathcal{P}_j \) and \( \mathcal{P}_k \) in the graph \( \mathcal{G}_{/\equiv} \) and by the assumptions on \( (\mathcal{P}_1, \ldots, \mathcal{P}_\ell) \) we find a derivation of the following form: \( D_0 : s_{j_1} \rightarrow^p_{\mathcal{P}_j/\mathcal{U}(\mathcal{P})} s_{j_p}^{-u(\mathcal{P})} s_{k_1} \rightarrow^q_{\mathcal{P}_k/\mathcal{U}(\mathcal{P})} s_{k_q} \). Due to Lemma 9.4.3 there exists a derivation \( D_1 : s'_{k_1} \rightarrow^q_{\mathcal{P}_k/\mathcal{U}(\mathcal{P})} s'_{k_q} \rightarrow^p_{\mathcal{P}_j/\mathcal{U}(\mathcal{P})} s'_{j_p} \) so that the number of (weak) dependency pair steps is unchanged. The sequence \( (\mathcal{P}_1, \ldots, \mathcal{P}_j, \mathcal{P}_k, \ldots, \mathcal{P}_m) \) is reorderable into \( (\mathcal{P}_1, \ldots, \mathcal{P}_k, \mathcal{P}_j, \ldots, \mathcal{P}_m) \) without affecting the length \( \ell \) of the \( \rightarrow_{\mathcal{P}/\mathcal{U}(\mathcal{P})} \)-rewrite sequence. By assumption \( k = j + 1 \), hence the induction hypothesis becomes applicable and we conclude the existence of a path \( (\mathcal{P}_1', \ldots, \mathcal{P}_m') \) fulfilling the assertions of the lemma.

Finally, we arrive at the main contribution of this paper.

**Theorem 9.4.2.** Let \( \mathcal{R} \) be a TRS, let \( \mathcal{P} \) be the set of weak or weak innermost dependency pairs, let \( A \) denote the maximum arity of compound symbols and let \( K \) denote the number of SCCs in the weak (innermost) dependency graph \( \mathcal{G} \). Suppose \( t \in \mathcal{T}_b^\ell \) is (innermost) terminating and define

\[
L(t) := \max\{\text{dl}(t, (i)_{\mathcal{P}_m/\mathcal{S}}) \mid (\mathcal{P}_1, \ldots, \mathcal{P}_m) \text{ is a path in } \mathcal{G}_{/\equiv} \text{ such that } \mathcal{P}_1 \in \text{Src} \},
\]

where \( \mathcal{S} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{m-1} \cup \mathcal{U}(\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_m) \). Then \( \text{dl}(t, (i)_{\mathcal{P}/\mathcal{U}(\mathcal{P})}) \leq K^2 \cdot L(t) \).

**Proof.** Let \( (\mathcal{P}_1, \ldots, \mathcal{P}_m) \) be a path in \( \mathcal{P}_{/\equiv} \) such that \( \mathcal{P}_1 \in \text{Src} \) and let \( D : t \rightarrow^\ell u \), denote a maximal derivation based on this path. (Here \( \rightarrow \) denotes \( \rightarrow_{\mathcal{P}/\mathcal{U}(\mathcal{P})} \) or \( \rightarrow_{\mathcal{P}/\mathcal{U}(\mathcal{P})} \).) Lemma 9.4.4 yields that \( D \) has the following form:

\[
t = t_0 \rightarrow^\ell_{\mathcal{P}_1/\mathcal{U}(\mathcal{P}_1)} t_1 \rightarrow^\ell_{\mathcal{P}_2/\mathcal{U}(\mathcal{P}_1) \cup \mathcal{U}(\mathcal{P}_2)} \cdots \rightarrow^\ell_{\mathcal{P}_m/\mathcal{U}(\mathcal{P}_1) \cup \cdots \cup \mathcal{U}(\mathcal{P}_m)} t_m = u , \tag{9.4}
\]

where \( t_0 \in \mathcal{T}_b^\ell \) and \( t_i \in \mathcal{T}_c^\ell \) for all \( i \geq 1 \). It suffices to estimate \( \ell_j \) for all \( j = 1, \ldots, m \) suitably. Let \( j \) be arbitrary, but fixed. Consider the subderivation \( D' \) of (9.4) where \( m \) is replaced by \( j \). Clearly \( D' \) is contained in the following derivation:

\[
t \rightarrow^*_{\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{j-1} \cup \mathcal{U}(\mathcal{P}_1) \cup \cdots \cup \mathcal{U}(\mathcal{P}_{j-1})} \rightarrow^\ell_{\mathcal{P}_j/\mathcal{U}(\mathcal{P}_1) \cup \cdots \cup \mathcal{U}(\mathcal{P}_j)} t_j
\]

132
Hence Lemma 9.4.1 is applicable, thus $\ell_j \leq d_l(t, \rightarrow_{P_j \cup \ldots \cup P_{j-1} \cup \mathcal{U}(P_j) \cup \ldots \cup \mathcal{U}(P_1)})$. As $\mathcal{U}(P_1) \cup \ldots \cup \mathcal{U}(P_j) \subseteq \mathcal{U}(P_1 \cup \ldots \cup P_j)$ we conclude $\ell_j \leq L(t)$ and obtain $\ell = \ell_1 + \ell_2 + \ldots + \ell_m \leq K \cdot L(t)$.

Above we argued that any connected component in $\mathcal{P}/\equiv$ is a tree. Clearly the number of nodes in this tree is less than $K$. Thus an arbitrary derivation can at most be based on $K$-many different paths. As the length of a derivation $D$ based on a specific path can be estimated by $K \cdot L(t)$, we conclude that the length of an arbitrary derivation is less than $K^2 \cdot L(t)$. This completes the proof of the theorem.

Theorem 9.4.2 together with Proposition 9.3.2 form a suitable analog of Theorem 9.4.1:

Let $\mathcal{P}$ be the set of weak or weak innermost dependency pairs. Suppose for every path $\mathcal{W}_i := (P_{i1}, \ldots, P_{im})$ in $\mathcal{P}$ there exist an SLI $A_j$ compatible with the usable rules of $\bigcup_{j \in \{i_1, \ldots, i_m\}} P_j$. Assume the existence of a safe and $G$-collapsible reduction pairs $(\succeq, >_i)$ such that $\mathcal{U}(\bigcup_{j \in \{i_1, \ldots, i_m\}} P_j) \cup \bigcup_{j \in \{i_1, \ldots, i_m-1\}} P_j$ is compatible with $\succeq$, and $P_{im}$ is compatible with $>_i$, such that $P_{im}$ is non-duplicating. Then for any $t \in T_B$ the derivation height $d_l(t, (\circ)^i)$ with respect to (innermost) rewriting is majorised by $G(t, >_i)$ and $|t|$.

**Corollary 9.4.1.** Let $\mathcal{R}$ be a TRS, let $\mathcal{P}$ be the set of weak (innermost) dependency pairs, and let $G$ denote the weak (innermost) dependency graph. Suppose for every path $\mathcal{W}_i := (P_{i1}, \ldots, P_{im})$ in $G/\equiv$ there exist an SLI $A_j$ and linear (quadratic) restricted interpretations $B_j$ such that $(\succeq, >_j)$ is a path in $G/\equiv$. Assume the existence of a safe and $G$-collapsible reduction pairs $(\succeq, >_i)$ such that $\mathcal{U}(\bigcup_{j \in \{i_1, \ldots, i_m\}} P_j) \cup \bigcup_{j \in \{i_1, \ldots, i_m-1\}} P_j$ is compatible with $\succeq$, and $P_{im}$ is compatible with $>_i$, such that $P_{im}$ is non-duplicating. Then the runtime complexity of a TRS $\mathcal{R}$ is linear or quadratic, respectively.

**Proof.** Observe that the assumptions imply that any basic term $t \in T_B$ is terminating with respect to $\mathcal{R}$: Any infinite derivation with respect to $\mathcal{R}$ starting in $t$ can be translated into an infinite derivation with respect to $\mathcal{U}(\mathcal{R}) \cup \mathcal{P}$ (see [74, Lemma 16]). Moreover, as the number of paths in $G/\equiv$ is finite, there exists a component $P_j$ that represents an infinite rewrite sequence. This is a contradiction. Without loss of generality, we assume $\mathcal{P} = WDP(\mathcal{P})$ and $G = WDG(\mathcal{P})$. Note that the reduction pair $(\succeq, >_B)$ is safe and collapsible. Hence for all $i$, the length of any $\rightarrow_{P_{im}/S}$-rewrite sequence is less than $p_i(|t|)$, where $p_n$ denotes a linear (or quadratic) polynomial, depending on $|t|$ only. (Here $S = P_{i1} \cup \ldots \cup P_{i-m-1} \cup \mathcal{U}(P_{i1} \cup \ldots \cup P_{im})$.) In analogy to the operator $L$, we define $M(t) := \max\{d_l(t, \rightarrow_{P_{im}/S}) \mid (P_{i1}, \ldots, P_{im}) \text{ is a path in } G/\equiv \text{ such that } P_{im} \in \text{Src}\}$. An application of Proposition 9.3.2 yields $M(t) = O(p_i(|t|))$. Following the pattern of the proof of the theorem, we establish the existence of a polynomial $p$ such that $d_l(t, \rightarrow_{\mathcal{P},\mathcal{U}(\mathcal{P})}) \leq p(|t|)$ holds for any basic term $t$. Finally, the corollary follows by an application of Proposition 9.3.1.

As mentioned above, in the dependency graph refinement for termination analysis it suffices to guarantee for each cycle $\mathcal{C}$ that there exist no $\mathcal{C}$-minimal rewrite sequences. For that one only needs to find a reduction pair $(\succeq, >)$ such that $\mathcal{R} \subseteq \succeq$, $\mathcal{C} \subseteq \succeq$, and $\mathcal{C} \cap > \neq \emptyset$. Thus, considering Theorem 9.4.2, it is tempting to think that it should suffice to replace strongly connected components by cycles and the stronger conditions should apply. However this intuition is deceiving as shown by the next example.

133
Example 9.4.6. Consider the TRS \( \mathcal{R} \) of \( f(s(x), 0) \to f(x, s(0)) \) and \( f(x, s(y)) \to f(x, y) \). WDP(\( \mathcal{R} \)) consists of 1: \( f^2(s(x), 0) \to f^2(x, s(x)) \) and 2: \( f^2(x, s(y)) \to f^2(x, y) \), and the weak dependency graph WDG(\( \mathcal{R} \)) contains two cycles \( \{1, 2\} \) and \( \{2\} \). There are two linear restricted interpretations \( A \) and \( B \) such that \( \{1, 2\} \subseteq \geq_A \cup \geq_A \), \( \{1\} \subseteq \geq_A \), and \( \{2\} \subseteq \geq_B \). Here, however, we must not conclude linear runtime complexity, because the runtime complexity of \( \mathcal{R} \) is at least quadratic.

9.5 Conclusion

In this section we provide (experimental) evidence on the applicability of the technique for complexity analysis established in this paper. We briefly consider the efficient implementation of the techniques provided by Theorem 9.4.2 and Corollary 9.4.1. Firstly, in order to approximate (weak) dependency graphs, we adapted (innermost) dependency graph estimations using the functions TCAP (ICAP) \[^59]\). Secondly, note that a graph including \( n \) nodes may contain an exponential number of paths. However, to apply Corollary 9.4.1 it is sufficient to handle only paths in the following set. Note that if we can assume that \( G/\equiv \) is a tree, then this set contains at most \( n^2 \) paths.

\[
\{(P_1, \ldots, P_k) \mid (P_1, \ldots, P_m) \text{ is a maximal path and } k \leq m\}, \quad \text{(9.5)}
\]

Example 9.5.1. Consider the following non-total terminating \( \mathcal{R} \).

13: \( p(f(f(x))) \to q(f(g(x))) \)  \quad 15: \( p(g(g(x))) \to q(g(f(x))) \)
14: \( q(f(f(x))) \to p(f(g(x))) \)  \quad 16: \( q(g(g(x))) \to p(g(f(x))) \)

The weak dependency pairs WDP(\( \mathcal{R} \)) are given as follows:

17: \( p^2(f(f(x))) \to q^2(f(f(x))) \)  \quad 19: \( p^2(g(g(x))) \to q^2(g(f(x))) \)
18: \( q^2(f(f(x))) \to p^2(f(g(x))) \)  \quad 20: \( q^2(g(g(x))) \to p^2(g(f(x))) \)

Hence for WDG(\( \mathcal{R} \))/\( \equiv \) the set \( \text{(9.5)} \) consists of 4 trivial paths: \( \{17\} \), \( \{18\} \), \( \{19\} \), and \( \{20\} \). Exemplarily we treat one of the trivial paths.

- Consider 17: \( p^2(f(f(x))) \to q^2(f(g(x))) \). Observe that \( U(\{17\}) = \emptyset \). In order to orient this weak dependency pair it suffices to employ the following polynomial interpretation \( p^2_B(x) = 1 \) and \( q^2_B(x) = f_B(x) = g_B(x) = 0 \).

In a similar fashion, we can treat the remaining three paths. It is not difficult to argue that this implies that the runtime complexity function of \( \mathcal{R} \) is constant. Note that none of the other techniques in the analysis of runtime complexities can obtain this (optimal) bound.

Moreover, to deal efficiently with polynomial interpretations, the issuing constraints are encoded in \textit{propositional logic} in a similar spirit as in \[^51\]. Assignments are found by employing a state-of-the-art SAT solver, in our case MiniSat\[^7\]. Furthermore, SLIs

\[^7\text{http://minisat.se/}\]
are handled by linear programming. Based on these ideas we implemented a complexity analyser. These techniques have also been incorporated into the Tyrolean Complexity Tool (TCT for short) that incorporates the most powerful techniques to analyse the complexity of rewrite systems that are currently at hand. For compilation of the here presented experimental data we used the latter implementation.

As suitable test bed we used the rewrite systems in the Termination Problem Data Base version 4.0. This test bed comprises 1739 TRSs. The presented tests were performed on a server with 8 Dual-Core 2.6 GHz AMD® Opteron® Processor 8220 CPUs, for a total of 16 cores. 64 GB of RAM are available. For each system we used a timeout of 60 seconds, the times in the tables are given in seconds. In interpreting defined and dependency pair symbols, we restrict to polynomials whose coefficients are in the range \( \{0, 1, \ldots, 5\} \). Table 9.1 (9.2) shows the experimental results for linear (quadratic) runtime complexities based on linear (quadratic) restricted interpretations. Text written in *italics* below the number of successes or failures indicates the total time (in seconds) of success cases or failure cases, respectively. The columns marked “Prop. 9.3.3” and “Cor. 9.4.1” refer to the applicability of the respective results. Moreover “S”, “F”, “T” denotes success, failure, or timeout respectively. For the sake of comparison, in the parentheses we indicate the number of successes by the method of the column or by Proposition 9.3.1.

In concluding, we observe that the experimental data shows that the here introduced dependency graph refinement for complexity analysis extends the analytic power of the methods introduced in [74] (see also Chapter 8). Note the significant difference between those TRSs that can be handled by Propositions 9.3.1 and 9.3.3 in contrast to those that can be handled either by Proposition 9.3.1 or by Corollary 9.4.1. Moreover observe the gain in power in relation to direct methods, compare also [26, 11].

---

**Table 9.1: Results for Linear Runtime Complexities**

<table>
<thead>
<tr>
<th></th>
<th>full rewriting</th>
<th>innermost rewriting</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>direct Prop. 9.3.1 Prop. 9.3.3 Cor. 9.4.1 Prop. 9.3.1 Prop. 9.3.3 Cor. 9.4.1</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>139</td>
<td>139 93 108 144 102 117</td>
</tr>
<tr>
<td></td>
<td>(147) (162) (166) (181)</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>1591</td>
<td>1582 1646 1630 1577 1637 1619</td>
</tr>
<tr>
<td></td>
<td>2474 4789 456 607 4699 462 536</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>8</td>
<td>17 0 1 17 0 3</td>
</tr>
</tbody>
</table>

---

8 [http://cl-informatik.uibk.ac.at/software/tct/](http://cl-informatik.uibk.ac.at/software/tct/)
9 See [http://termcomp.uibk.ac.at](http://termcomp.uibk.ac.at).
10 For full experimental evidence see [http://www.jaist.ac.jp/~hirokawa/08b/](http://www.jaist.ac.jp/~hirokawa/08b/) or [http://cl-informatik.uibk.ac.at/software/tct/](http://cl-informatik.uibk.ac.at/software/tct/).
Table 9.2: Results for Quadratic Runtime Complexities

<table>
<thead>
<tr>
<th></th>
<th>direct</th>
<th>Prop 9.3.1</th>
<th>Prop 9.3.3</th>
<th>Cor 9.4.1</th>
<th>Prop 9.3.1</th>
<th>Prop 9.3.3</th>
<th>Cor 9.4.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>182</td>
<td>182</td>
<td>93</td>
<td>108</td>
<td>183</td>
<td>102</td>
<td>117</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(186)</td>
<td>(202)</td>
<td></td>
<td>(193)</td>
<td>(208)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>152</td>
<td>329</td>
<td>97</td>
<td>74</td>
<td>324</td>
<td>98</td>
<td>54</td>
</tr>
<tr>
<td>F</td>
<td>524</td>
<td>473</td>
<td>1636</td>
<td>1624</td>
<td>492</td>
<td>1627</td>
<td>1613</td>
</tr>
<tr>
<td></td>
<td>5469</td>
<td>5436</td>
<td>793</td>
<td>850</td>
<td>5500</td>
<td>825</td>
<td>781</td>
</tr>
<tr>
<td>T</td>
<td>864</td>
<td>951</td>
<td>10</td>
<td>7</td>
<td>924</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>


Bibliography


certify termination of left-linear term rewriting systems. In Proceedings of the 16th
International Conference on Rewriting Techniques and Applications, volume 3467

[56] A. Geser, D. Hofbauer, J. Waldmann, and H. Zantema. On tree automata that
certify termination of left-linear term rewriting systems. Inf. and Comput., 205

[57] J. Giesl, T. Arts, and E. Ohlebusch. Modular termination proofs for rewriting

proofs with Aprove. In Proceedings of the 15th International Conference on Rewrite
Techniques and Applications, volume 3091 of LNCS, pages 210–220. Springer Ver-

[59] J. Giesl, R. Thiemann, and P. Schneider-Kamp. Proving and disproving termina-
tion of higher-order functions. In Proceedings of the 5th International Workshop on
Frontiers of Combining Systems, volume 3717 of LNAI, pages 216–231. Springer Ver-
lag, 2005.

[60] J. Giesl, P. Schneider-Kamp, and R. Thiemann. AProVE 1.2: Automatic ter-
mination proofs in the dependency pair framework. In Proceedings of the 3rd
International Joint Conference on Automated Reasoning, volume 4130 of LNCS,

[61] J. Giesl, R. Thiemann, and P. Schneider-Kamp. Proving and disproving termina-
tion in the dependency pair framework. In F. Baader, P. Baumgartner, R. Nieuwen-
huis, and A. Voronkov, editors, Deduction and Applications, volume 05431 of
Dagstuhl Seminar Proceedings, Germany, 2006. Internationales Begegnungs- und
Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl.


[65] G. Gómez and Y. A. Liu. Automatic time-bound analysis for a higher-order lan-
and Semantics-Based Program Manipulation, pages 75–86. ACM, 2002.

International EACSL Conference on Computer Science Logic, volume 2142 of


Bibliography


