The Epsilon Calculus

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August 26 & 27, 2003
CSL’03, Vienna
Brief Reminder

We defined the language $L_\varepsilon$ of the Epsilon Calculus, whose key-feature is the term-forming operator $\varepsilon$.

Governed by instances of so-called *critical axioms*.

$$A(t) \rightarrow A(\varepsilon x A(x)) .$$

Definition of an intensional semantics.

Axiomatization of the $\varepsilon$-calculus.

Informally $EC^\varepsilon$ accounts for the elementary calculus ($EC$) plus critical axioms.

$PC^\varepsilon$ accounts for the predicate calculus ($PC$) plus critical axioms.

Embedding Lemma:

$$PC \vdash A \text{ implies } EC^\varepsilon \vdash A^\varepsilon .$$

This was shown by showing, if $PC^\varepsilon \vdash A$ then $EC^\varepsilon \vdash A^\varepsilon$. 
Overview

1. The First Epsilon Theorem
2. The Second Epsilon Theorem and Herbrand’s Theorem
3. Generalizations of the Epsilon Theorems
4. (Intermediate) Conclusion
5. Hilbert’s “Ansatz”
The Epsilon Theorems

**First Epsilon Theorem.** If \( A \) is a formula without bound variables (no quantifiers, no epsilons) and \( PC^\epsilon \vdash A \) then \( EC \vdash A \).

**Extended First Epsilon Theorem.** If \( \exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n) \) is a purely existential formula containing only the bound variables \( x_1, \ldots, x_n \), and

\[
PC^\epsilon \vdash \exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n),
\]

then there are terms \( t_{ij} \) such that

\[
EC \vdash \bigvee_i A(t_{i1}, \ldots, t_{in}).
\]

**Second Epsilon Theorem.** If \( A \) is an \( \epsilon \)-free formula and \( PC^\epsilon \vdash A \) then \( PC \vdash A \).
Degree and Rank

Degree of an $\varepsilon$-Term

- $\deg(\varepsilon x A(x)) = 1$ if $A(x)$ contains no $\varepsilon$-subterms.
- If $e_1, \ldots, e_n$ are all immediate $\varepsilon$-subterms of $A(x)$, then $\deg(\varepsilon x A(x)) = \max\{\deg(e_1), \ldots, \deg(e_n)\} + 1$.

Rank of an $\varepsilon$-Expression

An $\varepsilon$-expression $e$ is subordinate to $\varepsilon x A$ if $e$ is a proper sub-expression of $A$ and contains $x$.

- $\rk(e) = 1$ if no sub-$\varepsilon$-expression of $e$ is subordinate to $e$.
- If $e_1, \ldots, e_n$ are all the $\varepsilon$-expressions subordinate to $e$, then $\rk(e) = \max\{\rk(e_1), \ldots, \rk(e_n)\} + 1$.
Examples

\[ P(\varepsilon x [P(x) \lor Q(\varepsilon y \neg Q(y))] \lor Q(\varepsilon y \neg Q(y)) ) \]
\[ e_2 \]
\[ e_1 \]

\[ \deg(e_1) = 1, \deg(e_2) = 2 \]

\[ \text{rk}(e_1) = \text{rk}(e_2) = 1 \]

\[ A(\varepsilon x A(x, \varepsilon z A(x, z)), \varepsilon y A(\varepsilon x A(x, \varepsilon z A(x, z)), y)) \]
\[ e_2 \]
\[ e_1(e_2) \]
\[ e_2 \]

\[ \deg(e_2) = 1, \deg(e_1(e_2)) = 2 \]

\[ \text{rk}(e_2) = 2, \text{rk}(e_1(e_2)) = 1 \]
Rank of Critical Formulas and Derivations

Rank of a critical formula \( A(t) \rightarrow A(\varepsilon x A(x)) \) is \( \text{rk}(\varepsilon x A(x)) \).

Rank of a derivation \( \text{rk}(\pi) \): maximum rank of its critical formulas.

Critical \( \varepsilon \)-term of a derivation: \( \varepsilon \)-term \( e \) so that \( A(t) \rightarrow A(e) \) is a critical formula.

Degree of a derivation \( \text{deg}(\pi) \): maximum degree of its critical \( \varepsilon \)-terms of maximal rank.

Order of a derivation \( o(\pi, r) \) wrt. rank \( r \): number of different critical formulas of rank \( r \).
The First Epsilon Theorem

(Proof for case without =)

Suppose \( \mathcal{P} \mathcal{C}^\varepsilon \vdash \pi \varepsilon \) and \( \varepsilon \) contains no bound variables. We show that \( \mathcal{E} \mathcal{C} \vdash \varepsilon \) by induction on the rank and degree of \( \pi \).

First, w.l.o.g. we assume \( \pi \) is actually a derivation in \( \mathcal{E} \mathcal{C}^\varepsilon \). Since \( \varepsilon \) contains no bound variables, \( \varepsilon^\varepsilon = \varepsilon \).

Second, w.l.o.g. we assume \( \pi \) doesn’t contain any free variables (replace free variables by new constants—may be resubstituted later).

**Lemma.** Let \( e \) be a critical \( \varepsilon \)-term of \( \pi \) of maximal degree among the critical \( \varepsilon \)-terms of maximal rank. Then there is \( \pi_e \) with end formula \( A \) so that \( \text{rk}(\pi_e) \leq \text{rk}(\pi) \), \( \text{deg}(\pi_e) \leq \text{deg}(\pi) \) and \( o(\pi_e, \text{rk}(e)) = o(\pi, \text{rk}(e)) - 1. \)
The First Epsilon Theorem: Main Lemma

Proof. Construct $\pi_e$ as follows:

1. Suppose $A(t_1) \rightarrow A(e), \ldots, A(t_n) \rightarrow A(e)$ are all the critical formulas belonging to $e$. For each critical formula

   $$A(t_i) \rightarrow A(e),$$

   we obtain a derivation

   $$\pi_i \vdash A(t_i) \rightarrow E :$$

   - Replace $e$ everywhere it occurs by $t_i$. Every critical formula $A(t) \rightarrow A(e)$ belonging to $e$ turns into a formula of the form $B \rightarrow A(t_i)$.
   - Add $A(t_i)$ to the axioms. Now every such formula is derivable using the propositional tautology

     $$A(t_i) \rightarrow (B \rightarrow A(t_i)),$$

     and modus ponens.
   - Apply the deduction theorem for the propositional calculus to obtain $\pi_i$.  


The First Epsilon Theorem: Main Lemma

2. Obtain a derivation $\pi'$ of $\bigwedge \neg A(t_i) \rightarrow E$ by:
   - Add $\bigwedge \neg A(t_i)$ to the axioms. Now every critical formula $A(t_i) \rightarrow A(e)$ belonging to $e$ is derivable using the propositional tautology $\neg A(t_i) \rightarrow (A(t_i) \rightarrow A(e))$.
   - Apply the deduction theorem.

3. Combine the proofs

   $$\pi_i \vdash A(t_i) \rightarrow E,$$

   and

   $$\pi' \vdash \bigwedge \neg A(t_i) \rightarrow E,$$

   to get $\pi_e \vdash E$ (case distinction)
Why is this correct?

Verify that the resulting derivation is indeed a derivation in $EC^e$ with the required properties.

We started with critical formulas of the form

$$A(t_i) \rightarrow A(e).$$

Facts:

- The proof $\pi'$ contains no critical formulas belonging to $e$, and all other critical formulas remain unchanged.
- In the construction of $\pi_i$, we substituted $e$ by $t$ throughout the proof. Such uniform substitution of a term by another are proof-preserving.
- Replacing $e$ by $t_i$ in $A(e)$ indeed results in $A(t_i)$, since $e$ cannot occur in $A(x)$—else $e = \varepsilon x A(x)$ would be a proper subterm of itself, which is impossible.
- If $e$ appears in another critical formula $B(s) \rightarrow B(\varepsilon y B(y))$, we have three cases.
Case I

*Case*: $e$ occurs only in $s$.

Replacing $e$ by $t_i$ results in a critical formula $B(s') \rightarrow B(\varepsilon y B(y))$.

The new critical critical formula belongs to the same $\varepsilon$-term as the original formula.

Hence maximal rank and order of $\pi_i$ wrt. $rk(e)$ remain unaffected.
Case II

**Case:** $e$ may occur in $B(y)$ and perhaps also in $s$, but contains neither $s$ nor $\varepsilon y B(y)$.

In other words, the critical formula has the form

$$B'(s'(e), e) \rightarrow B'(\varepsilon y B'(y, e), e).$$

But then the $\varepsilon$-term belonging to this critical formula

$$e' = \varepsilon y B'(y, e),$$

is of higher degree than $e$.

By our assumptions, this implies that $\text{rk}(\varepsilon y B'(y, e)) < \text{rk}(e)$.

Replacing $e$ by $t_i$ results in a different critical formula

$$B'(s'(t_i), t_i) \rightarrow B'(\varepsilon y B'(y, t_i), t_i),$$

belonging to the $\varepsilon$-term $\varepsilon y B'(y, t_i)$ which has the same rank as $e'$ and hence a lower rank than $e$ itself.
Case III

Case: \( e \) does contain \( s \) or \( \varepsilon y B(y) \).

Then \( e \) is of the form \( e'(s) \) or \( e'((\varepsilon y B(y)) \), and

\( B(a) \) is really of the form \( B'(e'(a)) \) where \( e'(a) \) is an \( \varepsilon \)-term of the same rank as \( e \).

Then \( \varepsilon y B(y) \) has the form \( \varepsilon y B'(e'(y)) \), to which the \( \varepsilon \)-expression \( e'(y) \) is subordinate.

But then \( \varepsilon y B'(e'(y)) \) has higher rank than \( e'(y) \), which has the same rank as \( e \). This cannot happen.
The First Epsilon Theorem: Proof

By induction on $\text{rk}(\pi)$.

If $\text{rk}(\pi) = 0$, there is nothing to prove (no critical formulas).

If $\text{rk}(\pi) > 0$ and the order of $\pi$ wrt. $\text{rk}(\pi)$ is $m$, then $m$-fold application of the lemma results in a derivation $\pi'$ of rank $< \text{rk}(\pi)$.

Note that by the 2nd case above there may be $n \cdot c$ critical $\varepsilon$-terms in $\pi_e$ (where $c$ is number of critical formulas in $\pi$); these are all of rank $< \text{rk}(e)$, however.

To remove all $\varepsilon$-terms of rank $\text{rk}(e)$ increases the number of critical $\varepsilon$-terms by $2^{O(c)}$. Bound for the necessary steps in the $\varepsilon$-elimination: $2^{O(c)}_{\text{rk}(\pi)}$. 
The Extended First Epsilon Theorem

**Theorem.** If $\exists x_1 \ldots \exists x_k A(x_1, \ldots, x_k)$ is a purely existential formula containing only the bound variables $x_1, \ldots, x_k$, and

$$\text{PC}^\varepsilon \vdash \exists x_1 \ldots \exists x_k A(x_1, \ldots, x_k),$$

then there are terms $t_{ij}$ such that

$$\text{EC} \vdash \bigvee_i A(t_{i1}, \ldots, t_{ik}).$$

Consider proofs in $\text{PC}^\varepsilon$ of $\exists x_1 \ldots \exists x_k A(x_1, \ldots, x_k)$, where $A(a_1, \ldots, a_k)$ contains no bound variables.

Employing embedding we obtain a derivation $\pi$ of $A(s_1, \ldots, s_k)$, where $s_1, \ldots, s_k$ are terms (containing $\varepsilon$’s).

**Proof Sketch.** We employ the same sequence of elimination steps as in the proof of the First Epsilon Theorem. The difference being that now the end-formula $A(s_1, \ldots, s_k)$ may contain $\varepsilon$-terms.

Hence the first elimination step transform the end-formula into a disjunction.

$$A(s_{01}, \ldots, s_{0k}) \lor \ldots \lor A(s_{n1}, \ldots, s_{nk}).$$
The Second Epsilon Theorem

**Theorem.** If $A$ is an $\varepsilon$-free formula and $PC^\varepsilon \vdash A$ then $PC \vdash A$.

Assume $A$ has the form

$$\exists x \forall y \exists z B(x, y, z),$$

with $B(x, y, z)$ quantifier-free and no other than the indicated variables occur in $A$.

**Herbrand Normal Form.** Suppose $A = \exists x \forall y \exists z B(x, y, z)$. If $f$ is a new function symbol, then the Herbrand normal form $A^H$ of $A$ is $\exists x \exists z B(x, f(x), z)$.

**Lemma.** Suppose $PC^\varepsilon \vdash A$. Then $PC^\varepsilon \vdash A^H$. 
Second Epsilon Theorem: Proof

The Strong First Epsilon Theorem yields:

There are \( \varepsilon \)-free terms \( r_i, s_i \) so that

\[
\text{EC} \vdash \bigvee_i B(r_i, f(r_i), s_i) \quad (1)
\]

We now can replace the \( t_i \) by new free variables \( a_i \) and obtain from (1), that

\[
\bigvee_i B(r_i, a_i, s_i) \quad (2)
\]

is deducible in \( \text{EC} \).

Then the original prenex formula \( A \) can be obtained from (2) if we employ the following rules (deducible in \( \text{PC} \))

\[
(\mu) : F \lor G(t) \vdash F \lor \exists y G(y)
\]

\[
(\nu) : F \lor G(a) \vdash F \lor \forall z G(z), \text{ provided } a \text{ appears only in } G(a) \text{ at the displayed occurrences.}
\]
Corollaries

Conservative Extension. Due to the Second Epsilon Theorem the Epsilon Calculus (with equality) is a conservative extension of pure predicate logic.

Equivalence. Due to the Embedding Lemma we have $PC^\varepsilon \vdash A$ implies $EC^\varepsilon \vdash A^\varepsilon$. Due to the Second Epsilon Theorem we obtain $EC^\varepsilon \vdash A^\varepsilon$ implies $PC^\varepsilon \vdash A$.

Herbrand’s Theorem. Assume $A = \exists x \forall y \exists z B(x, y, z)$. Iff $PC^\varepsilon \vdash A$, then there are terms $r_i, s_i$ such that $EC \vdash \lor_i B(r_i, f(r_i), s_i)$. 
Generalizations

**First Epsilon Theorem.** Let $A$ be a formula without bound variables (no quantifiers, no epsilons) but possible including $=$. Then

$$PC^e \cup Ax \vdash A \implies EC \cup Ax \vdash A,$$

where $Ax$ includes instances of quantifier-free (and $e$-free) axioms.

**Extended First Epsilon Theorem.** Let $\exists \bar{x}A(\bar{x})$ be a purely existential formula (possibly containing $=$). Then

$$PC^e \cup Ax \vdash \exists \bar{x}A(\bar{x}) \implies EC \cup Ax \vdash \bigvee_{i} A(t_{i1}, \ldots, t_{in}),$$

where $Ax$ is defined as above.
Generalizations (cont’d)

Second Epsilon Theorem. If \( A \) is an \( \varepsilon \)-free formula (possibly containing \( = \)) and

\[
\text{PC}^\varepsilon \cup \text{Ax} \vdash A \text{ implies } \text{PC} \cup \text{Ax} \vdash A,
\]

where \( \text{Ax} \) includes instances of \( \varepsilon \)-free axioms.
Some facts in favour of the Epsilon Calculus:

- The input parameter for the proof of Herbrand’s Theorem is the collection of critical formulas \( C \) used in the derivation. E.g. this gives a bound depending only on \( C \).
- The Epsilon Calculus allows a condensed representation of proofs.
  
  \( \text{Why: Assume } \text{EC}^\varepsilon \vdash A^\varepsilon. \text{ Then there exists a tautology of the form} \)
  
  \[
  \bigwedge_{i,j} (B_i(t_j) \rightarrow B_i(\varepsilon x B_i(x))) \rightarrow A^\varepsilon. \tag{3}
  \]
  
  Thus as soon as the critical formulas \( B_i(t_j) \rightarrow B_i(\varepsilon x B_i(x)) \) are known, we only need to verify that (3) is a tautology to infer that \( A^\varepsilon \) is provable in \( \text{EC}^\varepsilon \).
- Formalization of proofs should be simpler in the Epsilon Calculus.
A bluffer’s guide to Hilbert’s “Ansatz”

Assume we work within number theory and let \( \mathcal{N} \) denote the standard model of number theory.

(For conciseness we ignore induction.)

- The initial substitution \( S_0 \): Assign the constant function 0 to all \( \varepsilon \)-terms (well, to precise \( \varepsilon \)-matrices).
- Assume the substitution \( S_n \) has already been defined. Define \( S_{n+1} \): Pick a false critical axiom, e.g.

\[
P(t) \rightarrow P(\varepsilon x P(x)).
\]

(False means wrt. to \( \mathcal{N} \) and the current substitution \( S_n \)).
- Let \( z \in \mathbb{N} \) denote the value of \( t \) under \( S_n \). Then the next substitution \( S_{n+1} \) is obtained by assigning the value \( z \) to \( \varepsilon x P(x) \).

Note that the critical axiom \( P(t) \rightarrow P(\varepsilon x P(x)) \) is true wrt. to \( \mathcal{N} \) and the current substitution \( S_{n+1} \).
Peano Arithmetic: Results

1-consistency. Every purely existential formula derivable in $PA^\varepsilon$ is true.

Provable Recursive Functions. The numerical content of proofs of purely existential formulas in $PA^\varepsilon$ is extractable.

Put differently: The provable recursive functions of $PA^\varepsilon$ are exactly the $<\varepsilon_0$-recursive functions.

Assume $PA^\varepsilon \vdash \forall x \exists y A(x, y)$ with $A(a, b)$ quantifier-free and without free variables other than the shown. Then we can find a $<\varepsilon_0$-recursive function $f$ such that $\forall x A(x, f(x))$ holds.
Non-counter example interpretation. Let
\[ \exists x \forall y \exists z A(a,x,y,z) \]
be deducible in PA^ε such that only the indicated free variable \(a\) occurs. Let \( \exists x \exists z A(a,x,f(x),z) \) denote the Herbrand normal form of \(A\).

Then there exists \(< \varepsilon_0\)-recursive functionals \(G\) and \(H\) such that for all functions \(f\),
\[ A(n, G(f,n), f(G), H(f,n)) , \]
holds.

The transformation \(\varepsilon^\varepsilon: L^\varepsilon \rightarrow L_{PC}\) defined yesterday, can be employed to show that Peano Arithmetic embeds into PA^ε:

Then if \(PA \vdash A\), then \(PA^\varepsilon \vdash A^\varepsilon\).