Epsilon Calculus I

"In the ε-calculus it is hard to understand anything"[1]

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[1] © Michel Parigot

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator
- if $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term
- $\varepsilon_x A(x)$ is an indefinite description:
  $\varepsilon_x A(x)$ is some x for which $A(x)$ is true
- ε can replace $\exists$: $\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x))$
- axioms of ε-calculus:

  1. propositional tautologies
  2. equality axioms
  3. $A(t) \rightarrow A(\varepsilon_x A(x))$

predicate logic can be embedded in the ε-calculus

Why Should You Care?

1. basis of proof theory
2. interesting logical formalism
   - trade logical structure for term structure
   - formalisation of choice; recognised in its use in proof assistants, like Coq, Isabelle, etc.
   - full potential for linguistics and computer science is unexplored
3. foundation of noteworthy proof-theoretic results
   - ε-theorems and Herbrand's theorem (this lecture)
   - ε-substitution method and its connection to learning
   - Kreisel's no-counterexample interpretation
   - foundation of "unsound, but short proofs" (Matthias' lecture)
4. applications and great interest in linguistics
   - choice functions
   - *anaphora*, that is, an expression whose interpretation depends upon another expression in context

Outline

- Historical Remarks
- Axiomatisation
- The Embedding Lemma
- The First Epsilon Theorem

Rough Timeline

1922 introduced by Hilbert in 1921, as the basis for his formulation of mathematics for which Hilbert's Program was supposed to be carried out

1930s original work in proof theory (pre-Gentzen) concentrated on ε-calculus and ε-substitution method (Ackermann [Ack25, Ack40], von Neumann [vN27], Bernays [HB39], see also [Zac03, MZ06, Mos06, Zac17])

1950s ε-substitution method used by Kreisel for no-counterexample interpretation [Kre52, Kre58, Koh99] leading to work on proof analysis by Kreisel, Luckhardt, Kohlenbach

1990s use of the ε-substitution method for ordinal analysis by Arai, Avigad, Mints, Tait

recent renewed interest in connection to update procedures and learning [Avi02, Asc11, Pow16] and computational interpretations

Axioms of the Epsilon Calculus

Definitions

- AxEC: all substitution instances of propositional tautologies
- $\mathrm{AxEC}_\varepsilon$: AxEC + all substitution instances of

$$\underbrace{A(t) \to A(\varepsilon_x A(x))}_{\text{critical formula}}$$

- AxPC: AxEC + all substitution instances of

$$A(a) \to \exists x\, A(x) \qquad \forall x\, A(x) \to A(a)$$

- $\mathrm{AxPC}_\varepsilon$: AxPC + all substitution instances of critical formulas

Definitions

- a proof in EC (EC_ε) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxEC ($AxEC_\varepsilon$) or it follows from formulas preceding it by modus ponens
- a proof in PC (PC_ε) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxPC ($AxPC_\varepsilon$) or follows from formulas preceding it by modus ponens or generalisation
- if A is provable in say EC_ε we write $EC_\varepsilon \vdash_\pi A$
- the size $\mathrm{sz}(\pi)$ of a proof π is the number of steps in π
- the critical count $\mathrm{cc}(\pi)$ of π is the number of distinct critical formulas and quantifier axioms in π (plus 1)

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition

define a mapping $^\varepsilon$:

$$f(t_1,\dots,t_n)^\varepsilon = f(t_1^\varepsilon,\dots,t_n^\varepsilon) \qquad P(t_1,\dots,t_n)^\varepsilon = P(t_1^\varepsilon,\dots,t_n^\varepsilon)$$

$$x^\varepsilon = x \qquad (A \to B)^\varepsilon = A^\varepsilon \to B^\varepsilon \qquad [\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A^\varepsilon(x)$$

$$a^\varepsilon = a \qquad (A \vee B)^\varepsilon = A^\varepsilon \vee B^\varepsilon \qquad (\exists x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x A^\varepsilon(x))$$

$$(\neg A)^\varepsilon = \neg A^\varepsilon \qquad (A \wedge B)^\varepsilon = A^\varepsilon \wedge B^\varepsilon \qquad (\forall x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x \neg A^\varepsilon(x))$$

Lemma (Embedding Lemma)

if π is a regular PC_ε-proof of A then there is an EC_ε-proof π^ε of A^ε with $\mathrm{sz}(\pi^\varepsilon) \leqslant 3 \cdot \mathrm{sz}(\pi)$ and $\mathrm{cc}(\pi^\varepsilon) \leqslant \mathrm{cc}(\pi)$

Example: Epsilon Mapping

Example

$$[\exists x(P(x) \vee \forall y Q(y))]^{\varepsilon} =$$

$$= [P(x) \vee \forall y Q(y)]^{\varepsilon} \quad \{x \leftarrow \varepsilon_x [P(x) \vee \forall y Q(y)]^{\varepsilon}\}$$

$$[P(x) \vee \forall y Q(y)]^{\varepsilon} = P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})$$

$$= P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1}) \quad \{x \leftarrow \underbrace{\varepsilon_x [P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})]}_{e_2}\}$$

$$= P(\underbrace{\varepsilon_x [P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})]}_{e_2}) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})$$

Drinker's Paradox

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a), \forall y P(y)}}{\Rightarrow P(a) \rightarrow \forall y P(y), P(a)}}{\Rightarrow \exists x(P(x) \rightarrow \forall y P(y)), P(a)}}{\Rightarrow \exists x(P(x) \rightarrow \forall y P(y)), \forall y P(y)}}{P(b) \Rightarrow \exists x(P(x) \rightarrow \forall y P(y)), \forall y P(y)}}{\Rightarrow \exists x(P(x) \rightarrow \forall y P(y)), P(b) \rightarrow \forall y P(y)}}{\Rightarrow \exists x(P(x) \rightarrow \forall y P(y)), \exists x(P(x) \rightarrow \forall y P(y))}}{\Rightarrow \exists x(P(x) \rightarrow \forall y P(y))}$$

where we employ

$$[\forall y P(y)]^\varepsilon = P(\varepsilon_y \neg P(y))$$

$$[\exists x(P(x) \rightarrow \forall y P(y)]^\varepsilon = P(\underbrace{\varepsilon_x(P(x) \rightarrow P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \rightarrow P(\varepsilon_y \neg P(y))$$

Drinker's Paradox (II)

Example (cont'd)

1	$P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))$	TAUT
2	$(P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))) \to$	
	$\to (P(\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y)))$	critical axiom
3	$P(\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y))$	$1, 2$, *MP*

Remarks

- ε-calculus allows proof compression
- propositional inferences and structural rules become irrelevant
- focus on quantifier inferences

Lemma

for any semi-formula A, a fresh, any term $t \in \mathrm{L}(\mathrm{PC}_\varepsilon)$ and any substitution for free variables, we have

1. $(A\{a \mapsto t\})^\varepsilon = A^\varepsilon\{a \mapsto t^\varepsilon\}$
2. $(A\{a \mapsto t\})\sigma = A\sigma\{a \mapsto t\sigma\}$

Proof.

by induction on A ■

Corollary

for any formula A and any substitution from free variables into $\mathrm{L}(\mathrm{EC}_\varepsilon)$, we have $(A\sigma)^\varepsilon = A^\varepsilon\sigma$

Lemma

for any semi-formula A: $\mathsf{Var}(A) = \mathsf{Var}(A^\varepsilon)$

(Corrected) Proof of the Embedding Lemma[1]

Definition

- let $\pi \colon A_1, \dots, A_n$ be a regular proof in PC_ε
- let $E_1, \dots, E_p$ be a subsequence of π consisting of conclusions of quantifier rules
- for each $j \leqslant p$, let a_j and $C_j(a_j)$ denote the eigenvariable and formula associated to E_j

then we define a substitution σ from free variables to $\mathrm{L}(\mathrm{EC}_\varepsilon)$:

$$\sigma = \{a_1 \mapsto t_1\} \circ \cdots \circ \{a_p \mapsto t_p\}$$

$$\text{where } t_j = \begin{cases} \varepsilon_x \neg C_j^\varepsilon(x) & C_j(a) \text{ associated to } \forall i \\ \varepsilon_x C_j^\varepsilon(x) & C_j(a) \text{ associated to } \exists i \end{cases}$$

[1] Existing proofs in the literature [HB39, MZ06, Zac17] are false, correction due to M. Parigot.

Definition

for $q \leqslant p$, define

$$\sigma_{<q} = \{a_1 \mapsto t_1\} \circ \cdots \circ \{a_{q-1} \mapsto t_{q-1}\}$$

define $\sigma_{>q}$ similarly

Observations

for $q \leqslant p$ and $j \leqslant q$

- for any formula A: $(A\sigma)^\varepsilon = A^\varepsilon \sigma$
- C_q doesn't contain a_j
- $a_q\sigma = t_q\sigma_{>q}$

Proof.

using the regularity assumption and the previous corollary ∎

Proof of Embedding Lemma.

- we show $\forall$ regular proofs $\pi \colon A_1, \dots, A_n$
  $\exists$ proof π^ε containing $A_1^\varepsilon\sigma, \dots, A_n^\varepsilon\sigma$ (+ extra formulas)
- we use by induction on n
- base case is trivial and if $A_n =: A$ is a propositional tautology, $A^\varepsilon\sigma$ is also a tautology
- Case A an instance of a quantifier axiom; suppose
  $A = A(t) \to \exists x A(x)$; hence

  $$[A(t) \to \exists x\, A(x)]^\varepsilon\sigma = A^\varepsilon\sigma(t^\varepsilon\sigma) \to A^\varepsilon\sigma(\varepsilon_x A^\varepsilon\sigma(x))$$

  the latter is an instance of a critical axiom
- Case A follows by modus ponens from A_i and $A_j \equiv A_i \to A$
  applying IH there exists a proof π^* containing $A_i^\varepsilon\sigma$ and $A_i^\varepsilon\sigma \to A_j^\varepsilon\sigma$;
  we add $A^\varepsilon\sigma$ to π^*

Proof (cont'd).

- Case A follows by quantifier rule; e.g. $A = B \to \forall x\, C_q(x)$ and there exists $A_i = B \to C_q(a)$; a eigenvariable

  by IH there exists a proof π^* containing $A_i^\varepsilon \sigma \equiv B^\varepsilon \sigma \to C_q^\varepsilon(a)\sigma$; by definition σ replacing the eigenvariable a by $\varepsilon_x \neg A^\varepsilon(x)\sigma$. Hence

$$
\begin{aligned}
A_i^\varepsilon \sigma &= B^\varepsilon \sigma \to C_q^\varepsilon \sigma(\varepsilon_x \neg A^\varepsilon(x)\sigma) \\
&= B^\varepsilon \sigma_{>q} \to C_q^\varepsilon \sigma_{>q}(\varepsilon_x \neg A^\varepsilon(x)\sigma_{>q}) \\
&= B^\varepsilon \sigma_{>q} \to [\forall x\ C_q(x)]^\varepsilon \sigma_{>q} \\
&= B^\varepsilon \sigma \to [\forall x\ C_q(x)]^\varepsilon \sigma \\
&= A^\varepsilon \sigma
\end{aligned}
$$

  and thus we can set $\pi^\varepsilon = \pi^*$

The First Epsilon Theorem

Theorem

suppose $E(e_1, \dots, e_m)$ *is a quantifier-free formula containing only the* ε*-terms* $s_1, \dots, s_m$*, and*

$$\mathrm{EC}_\varepsilon \vdash_\pi E(s_1, \dots, s_m)$$

then there are ε*-free terms* t^i_j *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$$

where $n \leqslant 2^{3 \cdot \mathrm{cc}(\pi)}_{2 \cdot \mathrm{cc}(\pi)}$

number of instances independent off # of propositional inferences

Herbrand's Theorem

Theorem

if $\exists x_1 \dots \exists x_m E(x_1, \dots, x_m)$ *is a purely existential formula containing only the bound variables* $x_1, \dots, x_m$*, and*

$$\mathrm{PC} \vdash_\pi \exists x_1 \dots \exists x_m E(x_1, \dots, x_m) \ ,$$

then there are ε*-free terms* t^i_j *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$$

where $n \leqslant 2^{3\cdot \mathrm{cc}(\pi)}_{2\cdot \mathrm{cc}(\pi)}$

length of Herbrand disjunction independent off # of propositional inferences

Degree and Rank

Definition (degree)

- $\deg(\varepsilon_x A(x)) = 1$ if $A(x)$ contains no ε-subterms
- If $e_1, \ldots, e_n$ are all immediate ε-subterms of $A(x)$, then $\deg(\varepsilon_x A(x)) = \max\{\deg(e_1), \ldots, \deg(e_n)\} + 1$

Definition (rank)

- An ε-expression e is subordinate to $\varepsilon_x A$ if e is a proper sub-expression of A and contains x
- $\mathrm{rk}(e) = 1$ if no sub-ε-expression of e is subordinate to e
- If $e_1, \ldots, e_n$ are all the ϵ-expressions subordinate to e, then $\mathrm{rk}(e) = \max\{\mathrm{rk}(e_1), \ldots, \mathrm{rk}(e_n)\} + 1$

Example

consider

$$P(\underbrace{\varepsilon_x[P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})]}_{e_2}) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})$$

then

$$\deg(e_1) = 1 \qquad \deg(e_2) = 2 \qquad \mathrm{rk}(e_1) = \mathrm{rk}(e_2) = 1$$

Example

$$A(\underbrace{\varepsilon_x A(x, \varepsilon_z A(x,z))}_{e_2}, \underbrace{\varepsilon_y A(\underbrace{\varepsilon_x A(x, \varepsilon_z A(x,z))}_{e_2}, y)}_{e_1(e_2)})$$

then $\deg(e_2) = 1$, $\deg(e_1(e_2)) = 2$, $\mathrm{rk}(e_2) = 2$, $\mathrm{rk}(e_1(e_2)) = 1$

Rank of Critical Formulas and Derivations

Definitions

- rank of a critical formula $A(t) \to A(\varepsilon_x A(x))$ is $\mathrm{rk}(\varepsilon_x A(x))$
- rank of a derivation $\mathrm{rk}(\pi)$: maximum rank of its critical formulas
- critical ϵ-term of a derivation: ε-term e so that $A(t) \to A(e)$ is a critical formula
- degree of a derivation $\deg(\pi)$: maximum degree of its critical ε-terms of maximal rank
- order of a derivation $o(\pi, r)$ wrt. rank r: number of different critical ε-terms of rank r

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