Uncurrying for Innermost Termination and Derivational Complexity*

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In this paper we investigate the uncurrying transformation from (Hirokawa et al., 2008) for innermost termination and derivational complexity.

1 Introduction

Proving termination of first-order applicative term rewrite systems is challenging since the rules lack sufficient structure. But these systems are important since they provide a natural framework for modeling higher-order aspects found in functional programming languages some of which (e.g., OCaml) have an eager evaluation strategy. Since proving termination is easier for innermost than for full rewriting we lift some of the recent results from [2] from full to innermost termination. For the properties that do not transfer to the innermost setting we provide counterexamples. Furthermore we show that the uncurrying transformation is suitable for (innermost) derivational complexity analysis.

The remainder of this paper is organised as follows. After recalling the uncurrying transformation from [2] in Section 2 we show that it preserves innermost nontermination (but not innermost termination) in Section 3. In Section 4 we show that it preserves polynomial complexities of programs.

2 Uncurrying

We assume familiarity with term rewriting [1, 7]. This section recalls definitions and results from [2]. An *applicative* term rewrite system (ATRS for short) is a TRS over a signature that consists of constants and a single binary function symbol called application denoted by the infix and left-associative symbol $\circ$. In examples we often use juxtaposition instead of $\circ$. Every ordinary TRS can be transformed into an ATRS by currying. Let $\mathcal{F}$ be a signature. The currying system $\mathcal{C}(\mathcal{F})$ consists of the rewrite rules $f_{i+1}(x_1,\ldots,x_i,y) \to f_i(x_1,\ldots,x_i) \circ y$ for every n-ary function symbol $f \in \mathcal{F}$ and every $0 \leqslant i < n$. Here $f_n = f$ and, for every $0 \leqslant i < n$, f_i is a fresh function symbol of arity i. The currying system $\mathcal{C}(\mathcal{F})$ is confluent and terminating. Hence every term t has a unique normal form $t{\downarrow_{\mathcal{C}(\mathcal{F})}}$. For instance, $\mathsf{f}(\mathsf{a},\mathsf{b})$ is transformed into $\mathsf{f}\ \mathsf{a}\ \mathsf{b}$. Let $\mathcal{R}$ be a TRS over the signature $\mathcal{F}$. The *curried* system $\mathcal{R}{\downarrow_{\mathcal{C}(\mathcal{F})}}$ is the ATRS consisting of the rules $l{\downarrow_{\mathcal{C}(\mathcal{F})}} \to r{\downarrow_{\mathcal{C}(\mathcal{F})}}$ for every $l \to r \in \mathcal{R}$. The signature of $\mathcal{R}{\downarrow_{\mathcal{C}(\mathcal{F})}}$ contains the application symbol $\circ$ and a constant f_0 for every function symbol $f \in \mathcal{F}$. In the following we write $\mathcal{R}{\downarrow_{\mathcal{C}}}$ for $\mathcal{R}{\downarrow_{\mathcal{C}(\mathcal{F})}}$ whenever $\mathcal{F}$ can be inferred from the context or is irrelevant. Moreover, we write f for f_0.

Next we recall the uncurrying transformation from [2]. Let $\mathcal{R}$ be an ATRS over a signature $\mathcal{F}$. The *applicative arity* $\mathsf{aa}(f)$ of a constant $f \in \mathcal{F}$ is defined as the maximum n such that $f \circ t_1 \circ \cdots \circ t_n$ is

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a subterm in the left- or right-hand side of a rule in $\mathcal{R}$. This notion is extended to terms as follows: $\mathrm{aa}(t) = \mathrm{aa}(f)$ if t is a constant f and $\mathrm{aa}(t_1) - 1$ if $t = t_1 \circ t_2$. Note that $\mathrm{aa}(t)$ is undefined if the head symbol of t is a variable. The uncurrying system $\mathcal{U}(\mathcal{F})$ consists of the rewrite rules $f_i(x_1,\ldots,x_i) \circ y \to f_{i+1}(x_1,\ldots,x_i,y)$ for every constant $f \in \mathcal{F}$ and every $0 \leqslant i < \mathrm{aa}(f)$. Here $f_0 = f$ and, for every $i > 0$, f_i is a fresh function symbol of arity i. We say that $\mathcal{R}$ is *left head variable free* if $\mathrm{aa}(t)$ is defined for every non-variable subterm t of a left-hand side of a rule in $\mathcal{R}$. This means that no subterm of a left-hand side in $\mathcal{R}$ is of the form $t_1 \circ t_2$ where t_1 is a variable. The uncurrying system $\mathcal{U}(\mathcal{F})$, or simply $\mathcal{U}$, is confluent and terminating. Hence every term t has a unique normal form $t{\downarrow_\mathcal{U}}$. The *uncurried* system $\mathcal{R}{\downarrow_\mathcal{U}}$ is the TRS consisting of the rules $l{\downarrow_\mathcal{U}} \to r{\downarrow_\mathcal{U}}$ for every $l \to r \in \mathcal{R}$. However the rules of $\mathcal{R}{\downarrow_\mathcal{U}}$ are not enough to simulate an arbitrary rewrite sequence in $\mathcal{R}$. The natural idea is now to add $\mathcal{U}(\mathcal{F})$. In the following we write $\mathcal{U}^+(\mathcal{R},\mathcal{F})$ for $\mathcal{R}{\downarrow_{\mathcal{U}(\mathcal{F})}} \cup \mathcal{U}(\mathcal{F})$. If $\mathcal{F}$ can be inferred from the context or is irrelevant, $\mathcal{U}^+(\mathcal{R},\mathcal{F})$ is abbreviated to $\mathcal{U}^+(\mathcal{R})$.

Let $\mathcal{R}$ be a left head variable free ATRS. The *η-saturated* ATRS $\mathcal{R}_\eta$ is the smallest extension of $\mathcal{R}$ such that $l \circ x \to r \circ x \in \mathcal{R}_\eta$ whenever $l \to r \in \mathcal{R}_\eta$ and $\mathrm{aa}(l) > 0$. Here x is a variable that does not appear in $l \to r$. For a term t over the signature of the TRS $\mathcal{U}^+(\mathcal{R})$, we denote by $t{\downarrow_{\mathcal{C}'}}$ the result of identifying different function symbols in $t{\downarrow_\mathcal{C}}$ that originate from the same function symbol in $\mathcal{F}$. The notation ${\downarrow_{\mathcal{C}'}}$ is extended to TRSs and substitutions in the obvious way. For a substitution σ, we write $\sigma{\downarrow_\mathcal{U}}$ for the substitution $\{x \mapsto \sigma(x){\downarrow_\mathcal{U}} \mid x \in \mathcal{V}\}$. Next we recall some results from [2].

Lemma 1. *Let σ be a substitution. If t is head variable free then $t{\downarrow_\mathcal{U}}\sigma{\downarrow_\mathcal{U}} = (t\sigma){\downarrow_\mathcal{U}}$.* □

Lemma 2. *Let $\mathcal{R}$ be a left head variable free ATRS. If s and t are terms over the signature of $\mathcal{U}^+(\mathcal{R})$ then (1) $s \to_{\mathcal{R}\downarrow_\mathcal{U}} t$ if and only if $s{\downarrow_{\mathcal{C}'}} \to_\mathcal{R} t{\downarrow_{\mathcal{C}'}}$ and (2) $s \to_\mathcal{U} t$ implies $s{\downarrow_{\mathcal{C}'}} = t{\downarrow_{\mathcal{C}'}}$.* □

Theorem 3. *A left head variable free ATRS $\mathcal{R}$ is terminating if and only if $\mathcal{U}^+(\mathcal{R}_\eta)$ is terminating.* □

3 Innermost Uncurrying

Before showing that our transformation reflects innermost termination we show that it does not preserve innermost termination.

Example 4. Consider the ATRS $\mathcal{R} = \{\mathsf{f}\ x \to \mathsf{f}\ x, \mathsf{f} \to \mathsf{g}\}$. In an innermost sequence the first rule is never applied and hence $\mathcal{R}$ is innermost terminating. We have $\mathcal{U}^+(\mathcal{R}_\eta) = \{\mathsf{f}_1(x) \to \mathsf{f}_1(x), \mathsf{f} \to \mathsf{g}, \mathsf{f}_1(x) \to \mathsf{g} \circ x, \mathsf{f} \circ x \to \mathsf{f}_1(x)\}$ which is not innermost terminating due to the rule $\mathsf{f}_1(x) \to \mathsf{f}_1(x)$.

The next example shows that $s \xrightarrow{\mathsf{i}}_\mathcal{R} t$ does not imply $s{\downarrow_\mathcal{U}} \xrightarrow{\mathsf{i}}{}^+_{\mathcal{U}^+(\mathcal{R}_\eta)} t{\downarrow_\mathcal{U}}$. This is not a counterexample to soundness of uncurrying for innermost termination, but it shows that the proof for the "if-direction" of Theorem 3 (see [2] for details) cannot be adopted for the innermost case without further ado.

Example 5. Consider the ATRS $\mathcal{R} = \{\mathsf{f} \to \mathsf{g}, \mathsf{a} \to \mathsf{b}, \mathsf{g}\ x \to \mathsf{h}\}$ and the innermost step $s = \mathsf{f}\ \mathsf{a} \xrightarrow{\mathsf{i}}_\mathcal{R} \mathsf{g}\ \mathsf{a} = t$. We have $s{\downarrow_\mathcal{U}} = \mathsf{f} \circ \mathsf{a}$ and $t{\downarrow_\mathcal{U}} = \mathsf{g}_1(\mathsf{a})$. In the TRS $\mathcal{U}^+(\mathcal{R}_\eta) = \{\mathsf{f} \to \mathsf{g}, \mathsf{a} \to \mathsf{b}, \mathsf{g}_1(x) \to \mathsf{h}, \mathsf{g} \circ x \to \mathsf{g}_1(x)\}$ we have $s{\downarrow_\mathcal{U}} \xrightarrow{\mathsf{i}}_{\mathcal{U}^+(\mathcal{R}_\eta)} \mathsf{g} \circ \mathsf{a}$ but the step from $\mathsf{g} \circ \mathsf{a}$ to $t{\downarrow_\mathcal{U}}$ is not innermost.

The above problems can be solved if we consider terms that are not completely uncurried. The next two lemmata prepare for our proof. Below we write $s \rhd t$ if t is a proper subterm of s.

Lemma 6. *Let $\mathcal{R}$ be a left head variable free ATRS. If s is a term over the signature of $\mathcal{R}$, $s \in NF(\mathcal{R})$, and $s \to_\mathcal{U}^* t$ then $t \in NF(\mathcal{R}_\eta{\downarrow_\mathcal{U}})$.* □

Lemma 7. $\to_\mathcal{U}^* \cdot \rhd \subseteq \rhd \cdot \to_\mathcal{U}^*$ □

Lemma 8. *For every left head variable free ATRS $\mathcal{R}$ the inclusion ${}_\mathcal{U}^*\!\leftarrow \cdot \xrightarrow{\mathsf{i}}{}^\epsilon_\mathcal{R} \subseteq \xrightarrow{\mathsf{i}}{}^+_{\mathcal{U}^+(\mathcal{R}_\eta)} \cdot {}_\mathcal{U}^*\!\leftarrow$ holds.*

Proof. We prove that $s \xrightarrow{\mathsf{i}}{}^{+}_{\mathcal{U}^+(\mathcal{R}_\eta)} r{\downarrow_\mathcal{U}}\sigma{\downarrow_\mathcal{U}} \;{}^{*}_{\mathcal{U}}{\leftarrow}\; r\sigma$ whenever $s \;{}^{*}_{\mathcal{U}}{\leftarrow}\; l\sigma \xrightarrow{\mathsf{i}}{}^{\epsilon}_{\mathcal{R}} r\sigma$ for some rewrite rule $l \to r$ in $\mathcal{R}$. By Lemma 1 and the confluence of $\mathcal{U}$, $s \xrightarrow{\mathsf{i}}{}^{*}_{\mathcal{U}} (l\sigma){\downarrow_\mathcal{U}} = l{\downarrow_\mathcal{U}}\sigma{\downarrow_\mathcal{U}} \to_{\mathcal{U}^+(\mathcal{R}_\eta)} r{\downarrow_\mathcal{U}}\sigma{\downarrow_\mathcal{U}} \;{}^{*}_{\mathcal{U}}{\leftarrow}\; r\sigma$. It remains to show that the sequence $s \xrightarrow{\mathsf{i}}{}^{*}_{\mathcal{U}} (l\sigma){\downarrow_\mathcal{U}}$ and the step $l{\downarrow_\mathcal{U}}\sigma{\downarrow_\mathcal{U}} \to_{\mathcal{U}^+(\mathcal{R}_\eta)} r{\downarrow_\mathcal{U}}\sigma{\downarrow_\mathcal{U}}$ are innermost with respect to $\mathcal{U}^+(\mathcal{R}_\eta)$. For the former, let $s \xrightarrow{\mathsf{i}}{}^{*}_{\mathcal{U}} C[u] \xrightarrow{\mathsf{i}}_{\mathcal{U}} C[u'] \xrightarrow{\mathsf{i}}{}^{*}_{\mathcal{U}} (l\sigma){\downarrow_\mathcal{U}}$ with $u \xrightarrow{\mathsf{i}}{}^{\epsilon}_{\mathcal{U}} u'$ and let t be a proper subterm of u. Obviously $l\sigma \to^*_{\mathcal{U}} C[u] \rhd t$. According to Lemma 7, $l\sigma \rhd v \to^*_{\mathcal{U}} t$ for some term v. Since $l\sigma \xrightarrow{\mathsf{i}}_{\mathcal{R}} r\sigma$, the term v is a normal form of $\mathcal{R}$. Hence $t \in \mathit{NF}(\mathcal{R}_\eta{\downarrow_\mathcal{U}})$ by Lemma 6. Since $u \xrightarrow{\mathsf{i}}{}^{\epsilon}_{\mathcal{U}} u'$, t is also a normal form of $\mathcal{U}$. Hence $t \in \mathit{NF}(\mathcal{U}^+(\mathcal{R}_\eta))$ as desired. For the latter, let t be a proper subterm of $(l\sigma){\downarrow_\mathcal{U}}$. According to Lemma 7, $l\sigma \rhd u \to^*_{\mathcal{U}} t$. The term u is a normal form of $\mathcal{R}$. Hence $t \in \mathit{NF}(\mathcal{R}_\eta{\downarrow_\mathcal{U}})$ by Lemma 6. Obviously, $t \in \mathit{NF}(\mathcal{U})$ and thus also $t \in \mathit{NF}(\mathcal{U}^+(\mathcal{R}_\eta))$. □

The next example shows that for Lemma 8 the $\mathcal{R}$-step must take place at the root position.

Example 9. If $\mathcal{R} = \{\mathsf{f} \to \mathsf{g}, \mathsf{f}\ x \to \mathsf{g}\ x, \mathsf{a} \to \mathsf{b}\}$ then $\mathsf{f}_1(\mathsf{a}) \;{}^{*}_{\mathcal{U}}{\leftarrow}\; \mathsf{f} \circ \mathsf{a} \xrightarrow{\mathsf{i}}_{\mathcal{R}} \mathsf{g} \circ \mathsf{a}$ but $\mathsf{f}_1(\mathsf{a}) \not\xrightarrow{\mathsf{i}}{}^{+}_{\mathcal{U}^+(\mathcal{R}_\eta)} \cdot \;{}^{*}_{\mathcal{U}}{\leftarrow}\; \mathsf{g} \circ \mathsf{a}$.

In order to extend Lemma 8 to non-root positions, we have to use *rightmost innermost step* $\xrightarrow{\mathsf{ri}}$. This avoids the situation in the above example where parallel redexes become nested by uncurrying.

Lemma 10. *For every left head variable free ATRS $\mathcal{R}$ the inclusion ${}^{*}_{\mathcal{U}}{\leftarrow} \cdot \xrightarrow{\mathsf{ri}}_{\mathcal{R}} \subseteq \xrightarrow{\mathsf{i}}{}^{+}_{\mathcal{U}^+(\mathcal{R}_\eta)} \cdot {}^{*}_{\mathcal{U}}{\leftarrow}$ holds.*

Proof. Let $s \;{}^{*}_{\mathcal{U}}{\leftarrow}\; t = C[l\sigma] \xrightarrow{\mathsf{ri}}_{\mathcal{R}} C[r\sigma] = u$ with $l\sigma \xrightarrow{\mathsf{i}}{}^{\epsilon}_{\mathcal{R}} r\sigma$. We use induction on C. If $C = \Box$ then $s \;{}^{*}_{\mathcal{U}}{\leftarrow}\; t \xrightarrow{\mathsf{i}}{}^{\epsilon}_{\mathcal{R}} u$. Lemma 8 yields $s \xrightarrow{\mathsf{i}}{}^{+}_{\mathcal{U}^+(\mathcal{R}_\eta)} \cdot \;{}^{*}_{\mathcal{U}}{\leftarrow}\; u$. For the induction step we consider two cases.

- Suppose $C = \Box \circ s_1 \circ \cdots \circ s_n$ and $n > 0$. Since $\mathcal{R}$ is left head variable free, $\mathrm{aa}(l)$ is defined. If $\mathrm{aa}(l) = 0$ then $s = t' \circ s'_1 \circ \cdots \circ s'_n \;{}^{*}_{\mathcal{U}}{\leftarrow}\; l\sigma \circ s_1 \circ \cdots \circ s_n \xrightarrow{\mathsf{i}}_{\mathcal{R}} r\sigma \circ s_1 \circ \cdots \circ s_n$ with $t' \;{}^{*}_{\mathcal{U}}{\leftarrow}\; l\sigma$ and $s'_j \;{}^{*}_{\mathcal{U}}{\leftarrow}\; s_j$ for $1 \leqslant j \leqslant n$. The claim follows using Lemma 8 and the fact that innermost rewriting is closed under contexts. If $\mathrm{aa}(l) > 0$ then the head symbol of l cannot be a variable. We have to consider two cases. In the case where the leftmost $\circ$ symbol in C has not been uncurried we proceed as when $\mathrm{aa}(l) = 0$. If the leftmost $\circ$ symbol of C has been uncurried, we reason as follows. We may write $l\sigma = f \circ u_1 \circ \cdots \circ u_k$ where $k < \mathrm{aa}(f)$. We have $t = f \circ u_1 \circ \cdots \circ u_k \circ s_1 \circ \cdots \circ s_n$ and $u = r\sigma \circ s_1 \circ \cdots \circ s_n$. There exists an i with $1 \leqslant i \leqslant \min\{\mathrm{aa}(f), k+n\}$ such that $s = f_i(u'_1, \ldots, u'_k, s'_1, \ldots, s'_{i-k}) \circ s'_{i-k+1} \circ \cdots \circ s'_n$ with $u'_j \;{}^{*}_{\mathcal{U}}{\leftarrow}\; u_j$ for $1 \leqslant j \leqslant k$ and $s'_j \;{}^{*}_{\mathcal{U}}{\leftarrow}\; s_j$ for $1 \leqslant j \leqslant n$. Because of rightmost innermost rewriting, the terms $u_1, \ldots, u_k, s_1, \ldots, s_n$ are normal forms of $\mathcal{R}$. According to Lemma 6 the terms $u'_1, \ldots, u'_k, s'_1, \ldots, s'_n$ are normal forms of $\mathcal{R}_\eta{\downarrow_\mathcal{U}}$. Since $i - k \leqslant \mathrm{aa}(l)$, $\mathcal{R}_\eta$ contains the rule $l \circ x_1 \circ \cdots \circ x_{i-k} \to r \circ x_1 \circ \cdots \circ x_{i-k}$ where $x_1, \ldots, x_{i-k}$ are pairwise distinct variables not occurring in l. Hence the substitution $\tau = \sigma \cup \{x_1 \mapsto s_1, \ldots, x_{i-k} \mapsto s_{i-k}\}$ is well-defined. We obtain

$$\begin{array}{lll} s & \xrightarrow{\mathsf{i}}{}^{*}_{\mathcal{U}^+(\mathcal{R}_\eta)} & f_i(u_1{\downarrow_\mathcal{U}}, \ldots, u_k{\downarrow_\mathcal{U}}, s_1{\downarrow_\mathcal{U}}, \ldots, s_{i-k}{\downarrow_\mathcal{U}}) \circ s'_{i-k+1} \circ \cdots \circ s'_n \\ & \xrightarrow{\mathsf{i}}_{\mathcal{U}^+(\mathcal{R}_\eta)} & (r \circ x_1 \circ \cdots \circ x_{i-k}){\downarrow_\mathcal{U}}\tau{\downarrow_\mathcal{U}} \circ s'_{i-k+1} \circ \cdots \circ s'_n \\ & {}^{*}_{\mathcal{U}}{\leftarrow} & (r \circ x_1 \circ \cdots \circ x_{i-k})\tau \circ s_{i-k+1} \circ \cdots \circ s_n \;=\; r\sigma \circ s_1 \circ \cdots \circ s_n \;=\; t \end{array}$$

  where we use the confluence of $\mathcal{U}$ in the first sequence.

- In the second case we have $C = s_1 \circ C'$. Clearly $C'[l\sigma] \xrightarrow{\mathsf{ri}}_{\mathcal{R}} C'[r\sigma]$. If $\mathrm{aa}(s_1) \leqslant 0$ or if $\mathrm{aa}(s_1)$ is undefined or if $\mathrm{aa}(s_1) > 0$ and the outermost $\circ$ has not been uncurried in the sequence from t to s then $s = s'_1 \circ s' \;{}^{*}_{\mathcal{U}}{\leftarrow}\; s_1 \circ C'[l\sigma] \xrightarrow{\mathsf{ri}}_{\mathcal{R}} s_1 \circ C'[r\sigma] = u$ with $s'_1 \;{}^{*}_{\mathcal{U}}{\leftarrow}\; s_1$ and $s' \;{}^{*}_{\mathcal{U}}{\leftarrow}\; C'[l\sigma]$. If $\mathrm{aa}(s_1) > 0$ and the outermost $\circ$ has been uncurried in the sequence from t to s then we may write $s_1 = f \circ u_1 \circ \cdots \circ u_k$ where $k < \mathrm{aa}(f)$. We have $s = f_{k+1}(u'_1, \ldots, u'_k, s')$ for some term s' with $s' \;{}^{*}_{\mathcal{U}}{\leftarrow}\; C'[l\sigma]$ and $u'_i \;{}^{*}_{\mathcal{U}}{\leftarrow}\; u_i$ for $1 \leqslant i \leqslant k$. In both cases the induction hypothesis yields $s' \xrightarrow{\mathsf{i}}{}^{+}_{\mathcal{U}^+(\mathcal{R}_\eta)} \cdot \;{}^{*}_{\mathcal{U}}{\leftarrow}\; C'[r\sigma]$ and, since innermost rewriting is closed under contexts, we obtain $s \xrightarrow{\mathsf{i}}{}^{+}_{\mathcal{U}^+(\mathcal{R}_\eta)} \cdot \;{}^{*}_{\mathcal{U}}{\leftarrow}\; u$ as desired. □

By Lemma 10 and the equivalence of rightmost innermost and innermost termination [6] we obtain the main result of this section.

Theorem 11. *A left head variable free ATRS $\mathcal{R}$ is innermost terminating if $\mathcal{U}^{+}(\mathcal{R}_\eta)$ is.* □

4 Derivational Complexity

Hofbauer and Lautemann [4] introduced the concept of derivational complexity for terminating TRSs. The idea is to measure the maximal length of rewrite sequences (derivations) depending on the size of the starting term. Formally, the *derivation length* of t (with respect to $\to$) is defined as $\mathrm{dl}(t,\to) = \max\{m \in \mathbb{N} \mid t \to^m u \text{ for some } u\}$. The *derivational complexity* $\mathrm{dc}_\mathcal{R}(n)$ of a TRS $\mathcal{R}$ is then defined as $\mathrm{dc}_\mathcal{R}(n) = \max\{\mathrm{dl}(t,\to_\mathcal{R}) \mid |t| \leqslant n\}$ where $|t|$ denotes the *size* of t. Similarly we define $\mathrm{idc}_\mathcal{R}(n) = \max\{\mathrm{dl}(t,\xrightarrow{\mathrm{i}}_\mathcal{R}) \mid |t| \leqslant n\}$. Since we regard only finite TRSs, these functions are well-defined only when $\mathcal{R}$ is (innermost) terminating. If $\mathrm{dc}_\mathcal{R}(n)$ is bounded by a linear, quadratic, cubic, ... polynomial, $\mathcal{R}$ is said to have linear, quadratic, cubic ... (or polynomial) derivational complexity. A similar definition applies for $\mathrm{idc}_\mathcal{R}(n)$.

4.1 Full Rewriting

It is sound to use uncurrying as a preprocessor for proofs of derivational complexity:

Theorem 12. *Let $\mathcal{R}$ be a left head variable free and terminating ATRS. Then $\mathrm{dc}_\mathcal{R}(n) \leqslant \mathrm{dc}_{\mathcal{U}^{+}(\mathcal{R}_\eta)}(n)$ for all $n \in \mathbb{N}$.*

Proof. Consider an arbitrary maximal rewrite sequence $t_0 \to_\mathcal{R} t_1 \to_\mathcal{R} t_2 \to_\mathcal{R} \cdots \to_\mathcal{R} t_m$ which we can transform into the sequence $t_0{\downarrow_\mathcal{U}} \to^{+}_{\mathcal{U}^{+}(\mathcal{R}_\eta)} t_1{\downarrow_\mathcal{U}} \to^{+}_{\mathcal{U}^{+}(\mathcal{R}_\eta)} t_2{\downarrow_\mathcal{U}} \to^{+}_{\mathcal{U}^{+}(\mathcal{R}_\eta)} \cdots \to^{+}_{\mathcal{U}^{+}(\mathcal{R}_\eta)} t_m{\downarrow_\mathcal{U}}$ just as the "if-direction" of the proof of Theorem 3 (see [2]). Moreover, $t_0 \to^{*}_{\mathcal{U}^{+}(\mathcal{R}_\eta)} t_0{\downarrow_\mathcal{U}}$ holds. Therefore, $\mathrm{dl}(t_0,\to_\mathcal{R}) \leqslant \mathrm{dl}(t_0,\to_{\mathcal{U}^{+}(\mathcal{R}_\eta)})$. Hence $\mathrm{dc}_\mathcal{R}(n) \leqslant \mathrm{dc}_{\mathcal{U}^{+}(\mathcal{R}_\eta)}(n)$ holds for all $n \in \mathbb{N}$. □

Next we show that uncurrying preserves polynomial complexities. Hence we disregard duplicating (exponential complexity, cf. [3]) and empty (constant complexity) ATRSs. A TRS $\mathcal{R}$ is called *length-reducing* if $\mathcal{R}$ is non-duplicating and $|l| > |r|$ for all rules $l \to r \in \mathcal{R}$. The following lemma is an easy consequence of [3, Theorem 23]. Here $\to_{\mathcal{R}/\mathcal{S}}$ denotes $\to^{*}_\mathcal{S} \cdot \to_\mathcal{R} \cdot \to^{*}_\mathcal{S}$.

Lemma 13. *Let $\mathcal{R}$ be a non-empty non-duplicating TRS over a signature containing at least one symbol of arity at least two and let $\mathcal{S}$ be a length-reducing TRS. If $\mathcal{R} \cup \mathcal{S}$ is terminating then $\mathrm{dc}_{\mathcal{R}\cup\mathcal{S}}(n) \in \mathcal{O}(\mathrm{dc}_{\mathcal{R}/\mathcal{S}}(n))$.* □

Note that the above lemma does not hold if the TRS $\mathcal{R}$ is empty.

Theorem 14. *Let $\mathcal{R}$ be a non-empty, non-duplicating, left head variable free, and terminating ATRS. If $\mathrm{dc}_\mathcal{R}(n)$ is in $\mathcal{O}(n^k)$ then $\mathrm{dc}_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}}(n)$ and $\mathrm{dc}_{\mathcal{U}^{+}(\mathcal{R}_\eta)}(n)$ are in $\mathcal{O}(n^k)$.*

Proof. Let $\mathrm{dc}_\mathcal{R}(n)$ be in $\mathcal{O}(n^k)$ and consider a maximal rewrite sequence of $\to_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}}$ from a term t_0: $t_0 \to_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}} t_1 \to_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}} \cdots \to_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}} t_m$. By Lemma 2 (2) we obtain the sequence $t_0{\downarrow_{\mathcal{C}'}} \to_\mathcal{R} t_1{\downarrow_{\mathcal{C}'}} \to_\mathcal{R} \cdots \to_\mathcal{R} t_m{\downarrow_{\mathcal{C}'}}$. Thus, $\mathrm{dl}(t_0,\to_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}}) \leqslant \mathrm{dl}(t_0{\downarrow_{\mathcal{C}'}},\to_\mathcal{R})$. Because $|t_0{\downarrow_{\mathcal{C}'}}| \leqslant 2|t_0|$ holds, we obtain $\mathrm{dc}_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}}(n) \leqslant \mathrm{dc}_\mathcal{R}(2n)$. From the assumption the right-hand side is in $\mathcal{O}(n^k)$, hence $\mathrm{dc}_{\mathcal{R}_\eta{\downarrow_\mathcal{U}}/\mathcal{U}}(n)$ is in $\mathcal{O}(n^k)$. Because $\mathcal{U}$ is length-reducing, $\mathrm{dc}_{\mathcal{U}^{+}(\mathcal{R}_\eta)}(n)$ is also in $\mathcal{O}(n^k)$, by Lemma 13. □

In practice it is recommendable to investigate $\mathrm{dc}_{\mathcal{R}_\eta\downarrow_{\mathcal{U}}/\mathcal{U}}(n)$ instead of $\mathrm{dc}_{\mathcal{U}^+(\mathcal{R}_\eta)}(n)$, see [8]. The next example shows that uncurrying might be useful to enable criteria for polynomial complexity.

Example 15. Consider the ATRS $\mathcal{R} = \{\mathsf{add}\ x\ 0 \to x, \mathsf{add}\ x\ (\mathsf{s}\ y) \to \mathsf{s}\ (\mathsf{add}\ x\ y)\}$. It is easy to see that there exists a triangular matrix interpretation of dimension 2 that orients all rules in $\mathcal{U}^+(\mathcal{R}_\eta)$ strictly, inducing quadratic derivational complexity of $\mathcal{U}^+(\mathcal{R}_\eta)$ (see [5]) and by Theorem 12 also of $\mathcal{R}$. In contrast, the rule $\mathsf{add}\ x\ (\mathsf{s}\ y) \to \mathsf{s}\ (\mathsf{add}\ x\ y)$ does not admit such an interpretation of dimension 2. To see this we encoded the required condition as a satisfaction problem in non-linear arithmetic constraint over the integers. MiniSmt[1] can prove this problem unsatisfiable.

4.2 Innermost Rewriting

Next we consider innermost derivational complexity. Let $\mathcal{R}$ be an innermost terminating TRS. From a result by Krishna Rao [6, Section 5.1] we infer that $\mathrm{dl}(t, \xrightarrow{\mathsf{i}}_{\mathcal{R}}) = \mathrm{dl}(t, \xrightarrow{\mathsf{ri}}_{\mathcal{R}})$ holds for all terms t.

Theorem 16. *Let $\mathcal{R}$ be a left head variable free and innermost terminating ATRS. We have* $\mathrm{idc}_{\mathcal{R}}(n) \leqslant \mathrm{idc}_{\mathcal{U}^+(\mathcal{R}_\eta)}(n)$ *for all* $n \in \mathbb{N}$.

Proof. Consider a maximal rightmost innermost rewrite sequence $t_0 \xrightarrow{\mathsf{ri}}_{\mathcal{R}} t_1 \xrightarrow{\mathsf{ri}}_{\mathcal{R}} t_2 \xrightarrow{\mathsf{ri}}_{\mathcal{R}} \cdots \xrightarrow{\mathsf{ri}}_{\mathcal{R}} t_m$. Using Lemma 10 we obtain a sequence $t_0 \xrightarrow{\mathsf{i}}{}^+_{\mathcal{U}^+(\mathcal{R}_\eta)} t'_1 \xrightarrow{\mathsf{i}}{}^+_{\mathcal{U}^+(\mathcal{R}_\eta)} t'_2 \xrightarrow{\mathsf{i}}{}^+_{\mathcal{U}^+(\mathcal{R}_\eta)} \cdots \xrightarrow{\mathsf{i}}{}^+_{\mathcal{U}^+(\mathcal{R}_\eta)} t'_m$ for terms $t'_1, t'_2, \ldots, t'_m$ such that $t_i \to^*_{\mathcal{U}} t'_i$ for all $1 \leqslant i \leqslant m$. Thus, $\mathrm{dl}(t_0, \xrightarrow{\mathsf{i}}_{\mathcal{R}}) = \mathrm{dl}(t_0, \xrightarrow{\mathsf{ri}}_{\mathcal{R}}) \leqslant \mathrm{dl}(t_0, \xrightarrow{\mathsf{i}}_{\mathcal{U}^+(\mathcal{R}_\eta)})$. Hence, we conclude $\mathrm{idc}_{\mathcal{R}}(n) \leqslant \mathrm{idc}_{\mathcal{U}^+(\mathcal{R}_\eta)}(n)$. ☐

As Example 4 shows, uncurrying does not preserve innermost termination. Similarly, it does not preserve polynomial complexities even if the original ATRS has linear innermost derivational complexity.

Example 17. Consider the non-duplicating ATRS $\mathcal{R} = \{\mathsf{f} \to \mathsf{s}, \mathsf{f}\ (\mathsf{s}\ x) \to \mathsf{s}\ (\mathsf{s}\ (\mathsf{f}\ x))\}$. Since the second rule is never used in innermost rewriting, $\mathrm{idc}_{\mathcal{R}}(n) \leqslant n$ is easily shown by induction on n. We show that the innermost derivational complexity of $\mathcal{U}^+(\mathcal{R}_\eta)$ is at least exponential. We have $\mathcal{U}^+(\mathcal{R}_\eta) = \{\mathsf{f} \to \mathsf{s}, \mathsf{f}_1(\mathsf{s}_1(x)) \to \mathsf{s}_1(\mathsf{s}_1(\mathsf{f}_1(x))), \mathsf{f} \circ x \to \mathsf{f}_1(x), \mathsf{f}_1(x) \to \mathsf{s}_1(x), \mathsf{s} \circ x \to \mathsf{s}_1(x)\}$ and one can verify that $\mathrm{dl}(\mathsf{f}_1^n(\mathsf{s}_1(x)), \xrightarrow{\mathsf{i}}_{\mathcal{U}^+(\mathcal{R}_\eta)}) \geqslant 2^n$ for all $n \geqslant 1$. Hence, $\mathrm{idc}_{\mathcal{U}^+(\mathcal{R}_\eta)}(n+3) \geqslant 2^n$ for all $n \geqslant 0$.

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