Increasing Interpretations

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Example (Term Rewrite System)

$$\begin{aligned} 0 + y &\to y & \qquad 0 \times y &\to 0 \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) & \qquad \mathsf{s}(x) \times y &\to (x \times y) + y \end{aligned}$$

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Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

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Theorem

$\exists$ *reduction order* $>$ *s.t.* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}$ *terminating*

Example (Term Rewrite System)

$$0 + y > y \qquad 0 \times y > 0$$
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Definition (Dependency Pairs [Arts, Giesl 2000])

$\mathrm{DP}(\mathcal{R}) = \{l^\sharp \to u^\sharp \mid l \to r \in \mathcal{R}, r \trianglerighteq u, \mathit{root}(u) \text{ defined}\}$

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Theorem (Termination)

$\mathcal{R}$ terminating iff no sequence $s_0 \to_{\mathrm{DP}(\mathcal{R})} t_0 \to^*_{\mathcal{R}} s_1 \to_{\mathrm{DP}(\mathcal{R})} t_1 \to^*_{\mathcal{R}} \cdots$

Definition

$(\geq, >)$ is reduction pair if

- $>$ is rewrite-order, stable, well-founded
- $\geq$ is rewrite-preorder, monotone, stable
- $\geq \cdot > \cdot \geq \subseteq >$

Theorem

$\exists$ *reduction pair* $(\geq,\ >)$, $\mathcal{R} \subseteq \geq$, $\mathsf{DP}(\mathcal{R}) \subseteq >$ *then* $\mathcal{R}$ *terminating*

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$$\mathsf{s}(x) + y \geq \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \geq (x \times y) + y$$

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$$\mathsf{s}(x) +^\sharp y > x +^\sharp y \quad \mathsf{s}(x) \times^\sharp y > (x \times y) +^\sharp y \quad \mathsf{s}(x) \times^\sharp y > x \times^\sharp y$$

Definition (Dependency Graph)

graph DG $:= (N, E)$ with

- $N = \mathrm{DP}(\mathcal{R})$
- $E = \{(s \to t,\ u \to v) \in N \times N \mid \exists \sigma, \tau : t\sigma \to_{\mathcal{R}}^{*} u\tau\}$

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$$\circlearrowright 1 \longleftarrow 2 \longleftarrow 3 \circlearrowleft$$

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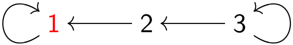

Theorem

$\forall$ *cycle* $\mathcal{C}$ $\exists$ $(\geq, >)$ *s.t.* $\mathcal{R} \cup \mathcal{C} \subseteq \geq$ *and* $\mathcal{C} \cap > \neq \varnothing$ *then* $\mathcal{R}$ *terminating*

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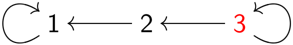

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$\forall$ cycle $\mathcal{C}$ $\exists$ $(\geq, >)$ s.t. $\mathcal{R} \cup \mathcal{C} \subseteq \geq$ and $\mathcal{C} \cap > \neq \varnothing$ then $\mathcal{R}$ terminating

Problems

- DG not computable $\longrightarrow$ sound approximations ($E' \supseteq E$)
- $2^n - 1$ possible cycles for n DPs (but only n SCCs)
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Theorem (Recursive SCC [HM05])

$\forall$ *SCC* $\mathcal{S}$ $\exists$ $(\geq, >)$ *s.t.* $\mathcal{R} \cup \mathcal{S} \subseteq \geq$ *and theorem holds for SCCs of* $\mathcal{S} \setminus >$ *then* $\mathcal{R}$ *terminating*

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Theorem ([HM05])

recursive SCC "equivalent to" cycle treatment

Weakly-Monotone Algebras

$(\mathcal{A}, >)$ with non-empty $\mathcal{F}$-algebra $\mathcal{A}$ and

- well-founded order $>$ on carrier of $\mathcal{A}$
- if $a_i \geq b$ then $f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \geq f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$
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Definition

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Linear Polynomials

- carrier $\mathbb{N}$, $> := >_\mathbb{N}$
- $f_\mathbb{N}(x_0, \ldots, x_n) = f_0 x_0 + \cdots + f_n x_n + f_{n+1}$ linear polynomial over $\mathbb{N}$

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A Simple Example

Example

$$\begin{aligned} \mathsf{f}(0, x) &\to \mathsf{f}(1, \mathsf{g}(x)) & \mathsf{g}(\mathsf{g}(0)) &\to 1 \\ \mathsf{f}(1, \mathsf{g}(\mathsf{g}(x))) &\to \mathsf{f}(0, x) & \mathsf{g}(1) &\to \mathsf{g}(\mathsf{g}(0)) \end{aligned}$$

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$3 : \mathsf{F}(0, x) \to \mathsf{G}(x)$ $\quad 4 : \mathsf{G}(1) \to \mathsf{G}(\mathsf{g}(0))$ $\quad 5 : \mathsf{G}(1) \to \mathsf{G}(0)$

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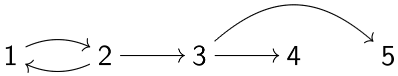

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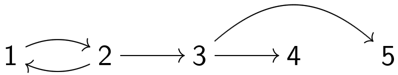

$$F_0 0_0 + F_1 x + F_2 \geq F_0 1_0 + F_1(g_0 x + g_1) + F_2$$
$$F_0 1_0 + F_1(g_0(g_0 x + g_1) + g_1) + F_2 \geq F_0 0_0 + F_1 x + F_2$$
$$g_0(g_0 0_0 + g_1) + g_1 \geq 1_0$$
$$g_0 1_0 + g_1 \geq g_0(g_0 0_0 + g_1) + g_1$$

A Simple Example (cont'd)

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F_0 0_0 + F_1 x &\geq F_0 1_0 + F_1 g_0 x + F_1 g_1 \\
F_0 1_0 + F_1 g_0 g_0 x + F_1 (g_0 g_1 + g_1) &\geq F_0 0_0 + F_1 x \\
g_0 g_0 0_0 + g_0 g_1 + g_1 &\geq 1_0 \\
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\end{aligned}$$

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$$

learn: $F_1 > 0$

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F_0 0_0 + F_1 x &\geq F_0 1_0 + F_1 g_0 x + F_1 g_1 \\
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learn: $F_1 > 0$, $g_0 \leq 1$

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\end{aligned}
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learn: $F_1 > 0$, $g_0 = 1$

$$
\begin{aligned}
F_0 0_0 &\geq F_0 1_0 + F_1 g_1 \\
F_0 1_0 + F_1 (g_1 + g_1) &\geq F_0 0_0 \\
0_0 + g_1 + g_1 &\geq 1_0 \\
1_0 &\geq 0_0 + g_1
\end{aligned}
$$

A Simple Example (cont'd)

$$
\begin{aligned}
F_0 0_0 + F_1 x &\geq F_0 1_0 + F_1 g_0 x + F_1 g_1 \\
F_0 1_0 + F_1 g_0 g_0 x + F_1 (g_0 g_1 + g_1) &\geq F_0 0_0 + F_1 x \\
g_0 g_0 0_0 + g_0 g_1 + g_1 &\geq 1_0 \\
g_0 1_0 &\geq g_0 g_0 0_0 + g_0 g_1
\end{aligned}
$$

learn: $F_1 > 0$, $g_0 = 1$

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A Simple Example (cont'd)

$$\begin{aligned} F_0 0_0 + F_1 x &\geq F_0 1_0 + F_1 g_0 x + F_1 g_1 \\ F_0 1_0 + F_1 g_0 g_0 x + F_1 (g_0 g_1 + g_1) &\geq F_0 0_0 + F_1 x \\ g_0 g_0 0_0 + g_0 g_1 + g_1 &\geq 1_0 \\ g_0 1_0 &\geq g_0 g_0 0_0 + g_0 g_1 \end{aligned}$$

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learn: $g_1 > 0,\ F_0 > 0$

$$F_0 0_0 \geq F_0 1_0 + F_1 g_1 \geq F_0(0_0 + g_1) + F_1 g_1 = F_0 0_0 + F_0 g_1 + F_1 g_1$$

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Running Example

Problematic SCC

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Problematic SCC

$$1 \rightleftarrows 2$$

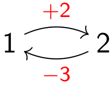
(Increasing) Interpretation

$$\mathsf{F}_{\mathbb{N}}(x,y) = x + y \quad \mathsf{g}_{\mathbb{N}}(x) = x + 1 \quad \mathsf{0}_{\mathbb{N}} = 0 \quad \mathsf{1}_{\mathbb{N}} = 1$$

$$\begin{aligned}
1 : \mathsf{F}(\mathsf{0}, x) &\to \mathsf{F}(\mathsf{1}, \mathsf{g}(x)) & x &\geq x + 2 \\
2 : \mathsf{F}(\mathsf{1}, \mathsf{g}(\mathsf{g}(x))) &\to \mathsf{F}(\mathsf{0}, x) & x + 3 &\geq x \\
\mathsf{g}(\mathsf{g}(\mathsf{0})) &\to \mathsf{1} & 2 &\geq 1 \\
\mathsf{g}(\mathsf{1}) &\to \mathsf{g}(\mathsf{g}(\mathsf{0})) & 2 &\geq 2
\end{aligned}$$

Running Example

Problematic SCC



(Increasing) Interpretation

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$$\begin{aligned} 1 : \mathsf{F}(0, x) &\to \mathsf{F}(1, \mathsf{g}(x)) & 2 + x &\geq x + 2 \\ 2 : \mathsf{F}(1, \mathsf{g}(\mathsf{g}(x))) &\to \mathsf{F}(0, x) & -3 + x + 3 &\geq x \\ \mathsf{g}(\mathsf{g}(0)) &\to 1 & 2 &\geq 1 \\ \mathsf{g}(1) &\to \mathsf{g}(\mathsf{g}(0)) & 2 &\geq 2 \end{aligned}$$

Theorem

if $\forall$ *cycle* $\mathcal{C}$ $\exists$ *reduction pair* $(\geq_{\mathcal{A}}, >_{\mathcal{A}})$ *based on interpretation* $\mathcal{A}$ *with*

$$l \geq_{\mathcal{A}} r \qquad l \to r \in \mathcal{R}$$

$$l \geq_{\mathcal{A}} r \qquad l \to r \in \mathcal{C}, \text{ and}$$

$$\mathcal{C} \cap >_{\mathcal{A}} \neq \varnothing$$

then $\mathcal{R}$ *is terminating*

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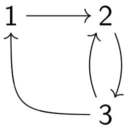
?

Slowly ...



cycles $\{2, 3\}$ and $\{1, 2, 3\}$

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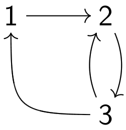

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assume $\{2, 3\} = [2, 3, 2, 3, 2, \ldots]$ already handled

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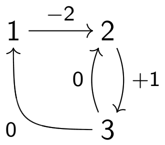

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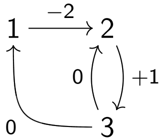

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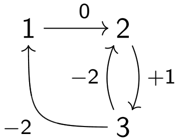

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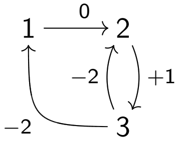

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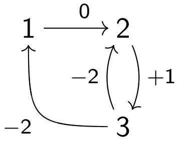

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Automation

SAT encoding produces abstract labels

Labeled Graphs

Definition

labeled graph $\mathcal{G}_\ell = (\mathcal{G}, \ell)$ is graph $\mathcal{G} = (N, E)$ with labeling $\ell \colon N \to \mathbb{Z}$

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Maximal Distance

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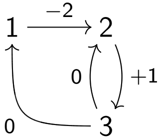
Example



$R_{220} = -\infty,\ R_{211} = 1,\ R_{222} = 2,\ R_{110} = R_{111} = -\infty,\ R_{112} = -1$

Observations

- computing R_{abk} for $k = \lceil \log_2(|N|) \rceil$ suffices
- $\mathsf{DG}_\ell = (\mathsf{DG}, \ell)$ with $\ell(l \to r) = \mathsf{cp}([r]_\mathcal{A} - [l]_\mathcal{A})$

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Theorem

$\mathcal{R}$ is terminating if $\forall$ SCC $\mathcal{S}$ in DG *$\exists$ reduction pair $(\geq_\mathcal{A}, >_\mathcal{A})$ based on interpretation $\mathcal{A}$ with*

$$
\begin{aligned}
l \geq_\mathcal{A} r & \qquad l \to r \in \mathcal{R} \\
\ell(l \to r) + l \geq_\mathcal{A} r & \qquad l \to r \in \mathcal{S} \\
\forall a \in \mathcal{S} : R_{aak} \leq 0 &
\end{aligned}
$$

and theorem holds for all SCCs in $\mathcal{S} \setminus \{a \in \mathcal{S} \mid R_{aak} < 0\}$

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Computing R_{abk} is of complexity $\mathcal{O}(n^2 log(n))$

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Simple SCCs

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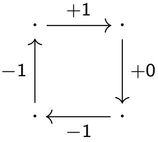

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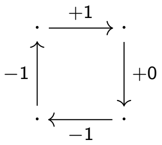

Almost Simple SCCs

$\exists$ node a s.t. $(N, E \setminus \{(a,b) \mid b \in N\})$
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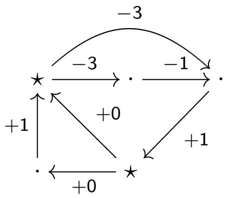

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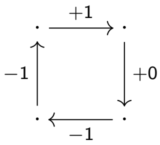

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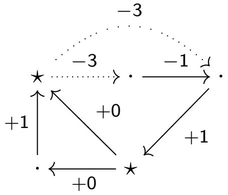

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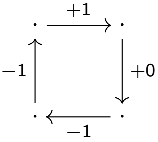

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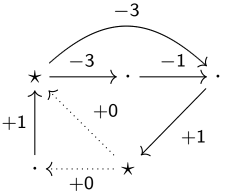

Example

$$
\begin{aligned}
\mathsf{f}(0,0,0,x) &\to \mathsf{f}(0,0,1,\mathsf{g}(x)) & \mathsf{f}(0,0,1,x) &\to \mathsf{f}(0,1,0,\mathsf{g}(x)) \\
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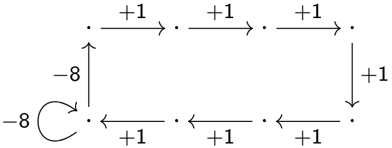

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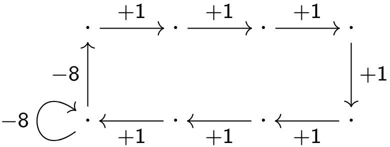

AProVE, CiME, Jambox, μTerm, Matchbox, TPA, TTT, TTT2, ...

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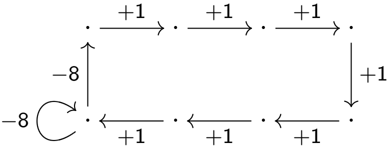

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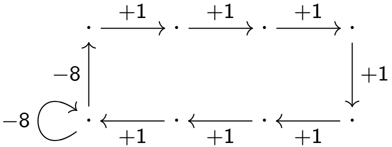


AProVE, CiME, Jambox, μTerm, Matchbox, TPA, TTT, TTT2, ... fail

TTT2$^{\text{II}}$: 23sec, TTT2$^{\text{II(b)}}$: 0.27sec

Future Work

Generalizing to Variables

$$\mathsf{F}(\mathsf{s}(x)) \to \mathsf{G}(\mathsf{s}(x))$$
$$\mathsf{G}(x) \to \mathsf{F}(x)$$

with increasing interpretation

$$\mathsf{F}_{\mathbb{N}}(x) = 2x \qquad \mathsf{G}_{\mathbb{N}}(x) = x \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1$$

Future Work

Generalizing to Variables

$$\mathsf{F}(\mathsf{s}(x)) \to \mathsf{G}(\mathsf{s}(x)) \qquad 2x + 2 \geq x + 1$$
$$\mathsf{G}(x) \to \mathsf{F}(x) \qquad x \geq 2x$$

with increasing interpretation

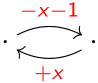
$$\mathsf{F}_{\mathbb{N}}(x) = 2x \qquad \mathsf{G}_{\mathbb{N}}(x) = x \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1$$

Future Work

Generalizing to Variables

$$\mathsf{F}(\mathsf{s}(x)) \to \mathsf{G}(\mathsf{s}(x)) \qquad -x - 1 + 2x + 2 \geq x + 1$$

$$\mathsf{G}(x) \to \mathsf{F}(x) \qquad x + x \geq 2x$$



with increasing interpretation

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Future Work

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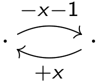


with increasing interpretation

$$\mathsf{F}_{\mathbb{N}}(x) = 2x \qquad \mathsf{G}_{\mathbb{N}}(x) = x \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1$$

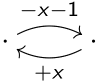
Generalizing to Other Methods

- Matrix Interpretations
- KBO

Future Work

Generalizing to Variables

$$\mathsf{F}(\mathsf{s}(x)) \to \mathsf{G}(\mathsf{s}(x)) \qquad\qquad -x - 1 + 2x + 2 \geq x + 1$$
$$\mathsf{G}(x) \to \mathsf{F}(x) \qquad\qquad x + x \geq 2x$$



with increasing interpretation

$$\mathsf{F}_{\mathbb{N}}(x) = 2x \qquad \mathsf{G}_{\mathbb{N}}(x) = x \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1$$

Generalizing to Other Methods

- Matrix Interpretations
- KBO

Is this Artificial?

$$\begin{aligned}
\mathsf{ack}(0, n) &\rightarrow \mathsf{s}(n) \\
\mathsf{ack}(\mathsf{s}(m), 0) &\rightarrow \mathsf{ack}(m, \mathsf{s}(0)) \\
\mathsf{ack}(\mathsf{s}(m), \mathsf{s}(n)) &\rightarrow \mathsf{ack}(m, \mathsf{ack}(\mathsf{s}(m), n))
\end{aligned}$$