

Labelings for Decreasing Diagrams

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Overview

- Preliminaries
- Decreasing Diagrams
- Labelings for Decreasing Diagrams
- Implementation
- Experiments
- Conclusion

Preliminaries

Definition (ARS)

$$\mathcal{A} = (A, \rightarrow)$$

Definition (Confluence)

$$*\leftarrow \cdot \rightarrow^* \subseteq \rightarrow^* \cdot *\leftarrow$$

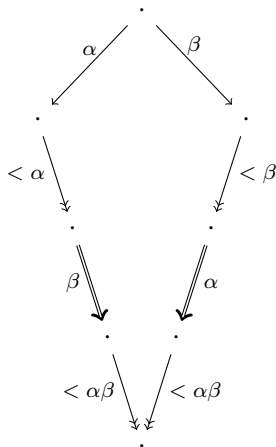
Definition (Local Confluence)

$$\leftarrow \cdot \rightarrow \subseteq \rightarrow^* \cdot *\leftarrow$$

Theorem (van Oostrom, 94)

every locally decreasing ARS is confluent

Decreasing Diagrams



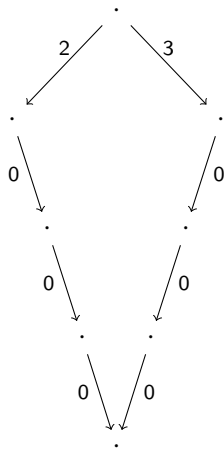
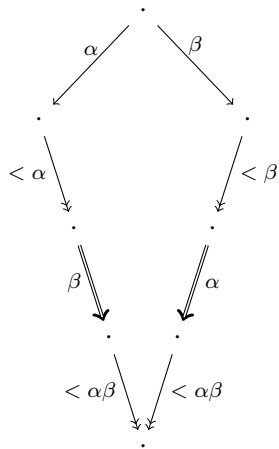
Definition (local diagram)

$$\leftarrow \cdot \rightarrow \subseteq \rightarrow^* \cdot \cdot^* \leftarrow$$

Definition (local decreasing)

$$\alpha \leftarrow \cdot \rightarrow \beta \subseteq \overset{\vee}{\rightarrow}^*_{\alpha} \cdot \rightarrow \overline{\beta} \cdot \overset{\vee}{\rightarrow}^*_{\alpha\beta} \cdot \alpha\beta \overset{\vee}{\leftarrow}^* \cdot \overline{\alpha} \leftarrow \cdot \overset{\vee}{\leftarrow}^*_{\beta}$$

Examples



Term Rewriting

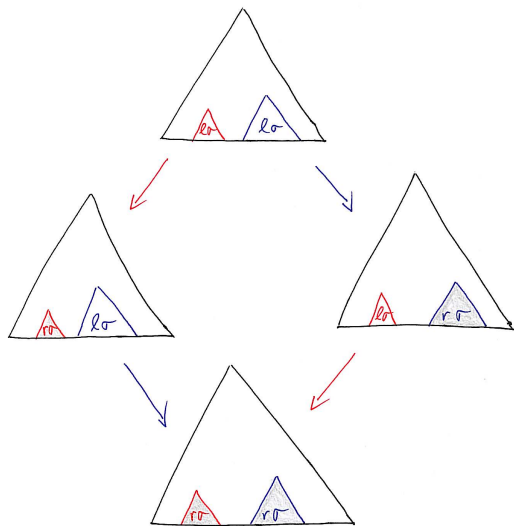
$$t \leftarrow s \rightarrow u$$

Three possibilities (modulo symmetry)

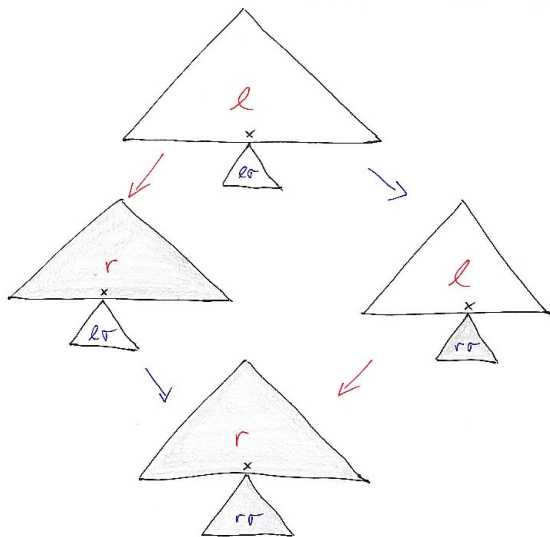
$$t \xrightarrow{p, l \rightarrow r} s[l\sigma]_p = s = s[l\sigma]_q = l \xrightarrow{q, l \rightarrow r} u$$

- $p \parallel q$ (parallel overlap)
- $p < q$ and $q \leq \text{Pos}_V(s[l]_p)$ (variable overlap)
- $p \leq q$ and $q \in \text{Pos}_{\mathcal{F}}(s[l]_p)$ (critical overlap)

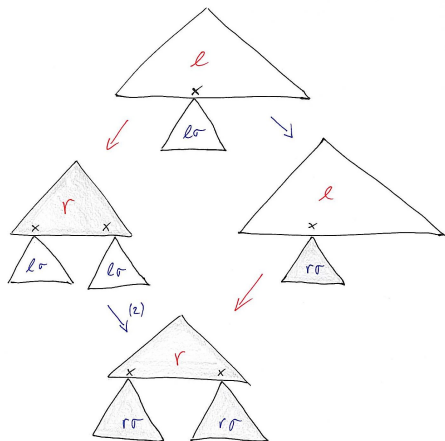
parallel



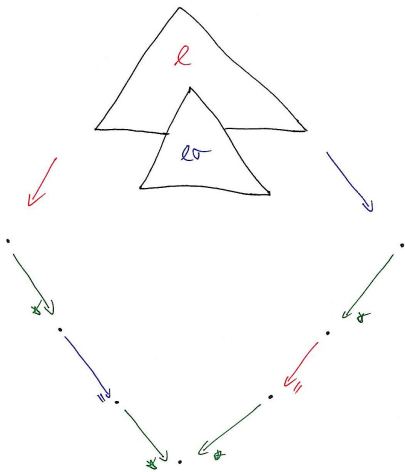
variable (linear)



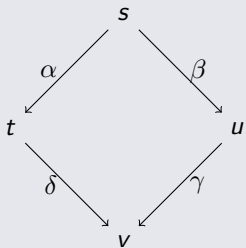
variable (left-linear)



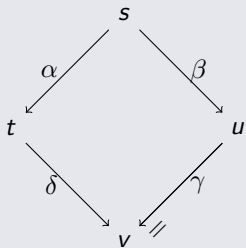
critical



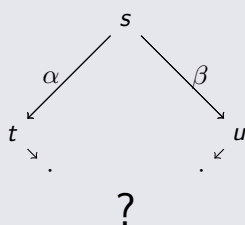
linear TRSs (3 kinds of peaks)



(a) parallel



(b) variable linear



(c) critical

Definition (labeling)

$$\Gamma = s \rightarrow_{p,l} t \quad \Delta = u \rightarrow_{q,l'} v$$

$(l, \geq, >)$ labeling if $\geq \cdot > \cdot \geq \subseteq >$

- $l(\Gamma) \geq l(\Delta) \longrightarrow l(C[\Gamma\sigma]) \geq l(C[\Delta\sigma])$
- $l(\Gamma) > l(\Delta) \longrightarrow l(C[\Gamma\sigma]) > l(C[\Delta\sigma])$

L-labeling

Definition (L-labeling)

labeling ℓ is L-labeling if $\alpha \geq \gamma, \beta \geq \delta$ (parallel, variable linear)

Example

$(\ell_{rl}, \geq_{\mathbb{N}}, >_{\mathbb{N}})$ is L-labeling ($\ell_{rl}: \mathcal{R} \rightarrow \mathbb{N}$)

$(\ell_{sn}, \rightarrow_{\mathcal{R}}^*, \rightarrow_{\mathcal{S}/\mathcal{R}}^+)$ is L-labeling ($\ell_{sn}(s \rightarrow t) = s$)

if $\rightarrow_{\mathcal{S}} \subseteq \rightarrow_{\mathcal{R}}$ and \mathcal{S}/\mathcal{R} is terminating

Lemma

ℓ_1, ℓ_2 L-labelings $\longrightarrow \ell_1 \times \ell_2$ L-labeling

Theorem

\mathcal{R} linear, critical peaks decreasing for L-labeling $\longrightarrow \mathcal{R}$ confluent

Example (van Oostrom, 2008)

1 : $\text{nats} \rightarrow 0 : \text{inc}(\text{nats})$

2 : $\text{hd}(x : y) \rightarrow x$

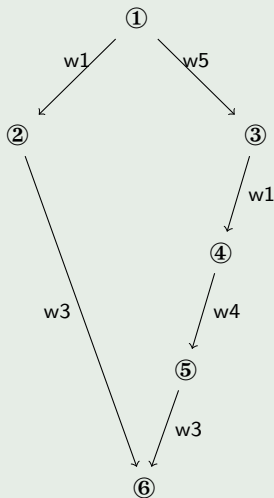
3 : $\text{tl}(x : y) \rightarrow y$

4 : $\text{inc}(x : y) \rightarrow \text{s}(x) : \text{inc}(y)$

5 : $\text{inc}(\text{tl}(\text{nats})) \rightarrow \text{tl}(\text{inc}(\text{nats}))$

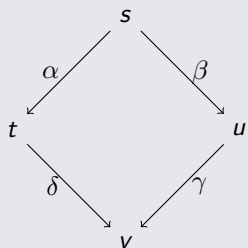
- ① $\text{inc}(\text{tl}(\text{nats}))$
- ② $\text{inc}(\text{tl}(0 : \text{inc}(\text{nats})))$
- ③ $\text{tl}(\text{inc}(\text{nats}))$
- ④ $\text{tl}(\text{inc}(0 : \text{inc}(\text{nats})))$
- ⑤ $\text{tl}(\text{s}(0) : \text{inc}(\text{inc}(\text{nats})))$
- ⑥ $\text{inc}(\text{inc}(\text{nats}))$

$w1 >_{\mathbb{N}} w3, w4$ shows confluence

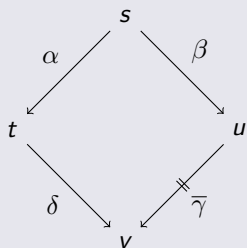


LL-labeling

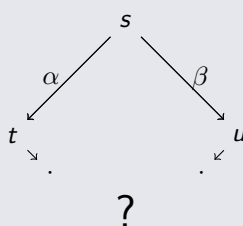
left-linear TRSs (3 kinds of peaks)



(a) parallel



(b) variable left-linear



(c) critical

Definition (LL-labeling)

L-labeling ℓ is LL-labeling if $\alpha > \gamma_i$, $\beta \geq \delta$ (variable left-linear)

Example

ℓ_{rl} is not *LL*-labeling

ℓ_{sn} is *LL*-labeling if $\mathcal{R}_d/\mathcal{R}_{nd}$ terminating

Definition (*LL*-labeling)

L-labeling ℓ is **weak** *LL*-labeling if $\alpha \geq \gamma_i, \beta \geq \delta$ (variable left-linear)

Example

ℓ_{rl} weak *LL*-labeling

every *LL*-labeling is weak *LL*-labeling

Lemma

LL-labeling ℓ_1 , weak *LL*-labeling $\ell_2 \longrightarrow \ell_1 \times \ell_2, \ell_2 \times \ell_1$ *LL*-labelings

Theorem

\mathcal{R} left-linear, critical peaks decreasing for *LL*-labeling $\longrightarrow \mathcal{R}$ confluent

First result

Corollary

\mathcal{R} left-linear, critical peaks decreasing for weak LL-labeling,
 $\mathcal{R}_d/\mathcal{R}_{nd}$ terminating \longrightarrow \mathcal{R} confluent

Example (OO03 cont'd)

1: $x + (y + z) \rightarrow (x + y) + z$

2: $(x + y) + z \rightarrow x + (y + z)$

3: $s(x) + y \rightarrow x + s(y)$

4: $x + s(y) \rightarrow s(x) + y$

5: $x \times s(y) \rightarrow x + (x \times y)$

6: $s(x) \times y \rightarrow (x \times y) + y$

7: $x + y \rightarrow y + x$

8: $x \times y \rightarrow y \times x$

9: $sq(x) \rightarrow x \times x$

10: $sq(s(x)) \rightarrow (x \times x) + s(x + x)$

$\mathcal{R}_d/\mathcal{R}_{nd}$ terminating

$$+_{\mathbb{N}}(x, y) = x + y \quad s_{\mathbb{N}}(x) = x + 1 \quad \times_{\mathbb{N}}(x, y) = x^2 + xy + y^2 \quad sq_{\mathbb{N}}(x) = 3x^2 + 1$$

weak LL-labeling $l_{r1}(8) = l_{r1}(9) = 2$, $l_{r1}(6) = l_{r1}(10) = 1$, $l_{r1}(\cdot) = 0$

Towards a second result

Definition

$$\star(f(t_1, \dots, t_n), p) = \begin{cases} f_i(\star(t_i, q)) & \text{if } p = iq \\ x & \text{if } p = \epsilon \end{cases}$$

$$\mathcal{R}_\circ^\star = \{\star(l, p) \rightarrow \star(r, q) \mid l \rightarrow r \in \mathcal{R}, l|_p = r|_q \in \mathcal{V}, \text{ and } |r|_y \circ 1\}$$

$$\circ \in \{>, =\}$$

$$\star(\mathcal{R}) := \mathcal{R}_>^\star / \mathcal{R}_=^\star$$

Example

$$1 : \mathbf{f}(g(x, a)) \rightarrow \mathbf{g}(f(x), f(x)) \quad 2 : a \rightarrow b \quad 3 : b \rightarrow a \quad 4 : \mathbf{h}(x) \rightarrow \mathbf{h}(x)$$

$$\mathcal{R}_>^\star = \{\mathbf{f}_1(g_1(x)) \rightarrow \mathbf{g}_1(f_1(x)), \mathbf{f}_1(g_1(x)) \rightarrow \mathbf{g}_2(f_1(x))\}$$

$$\mathcal{R}_=^\star = \{\mathbf{h}_1(x) \rightarrow \mathbf{h}_1(x)\}$$

Definition

$$\ell_\star(s \rightarrow_{p,l \rightarrow r} t) = \star(s, p)$$

Lemma

$(\geq, >)$ monotone reduction pair, $\mathcal{R}_>^\star \subseteq >$, $\mathcal{R}_>^\star \cup \mathcal{R}_=^\star \subseteq \geq \longrightarrow (\ell_\star, \geq, >)$
LL-labeling

Corollary

\mathcal{R} left-linear, $\mathcal{R}_>^\star / \mathcal{R}_=^\star$ terminating, ℓ weak LL-labeling, critical peaks decreasing for $\ell_\star \times \ell \longrightarrow \mathcal{R}$ confluent

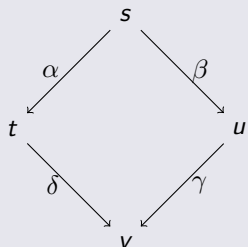
Example (OO03)

$$\begin{array}{ll} 1: x + (y + z) \rightarrow (x + y) + z & 2: (x + y) + z \rightarrow x + (y + z) \\ 3: s(x) + y \rightarrow x + s(y) & 4: x + s(y) \rightarrow s(x) + y \\ 5: x \times s(y) \rightarrow x + (x \times y) & \dots \end{array}$$

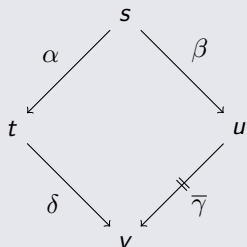
$\mathcal{R}_>^\star$ contains nonterminating $\times_1(x) \rightarrow +_2(\times_1(x))$

LL-labeling

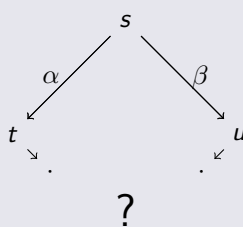
left-linear TRSs (3 kinds of peaks)



(a) parallel



(b) variable left-linear



(c) critical

Definition (LL-labeling)

L-labeling ℓ is LL-labeling if $\alpha > \gamma_i$, $\beta \geq \delta$ (variable left-linear)

$$\alpha \geq \gamma_1, \alpha > \gamma_i \ (i > 1)$$

LL-labeling improved

Definition

$$\star(\mathcal{R}) := \mathcal{R}_{>}^{\star\star} / \mathcal{R}_{=}^{\star\star}$$

Lemma

$(\geq, >)$ monotone reduction pair, $\mathcal{R}_{>}^{\star\star} \subseteq >$, $\mathcal{R}_{>}^{\star\star} \cup \mathcal{R}_{=}^{\star\star} \subseteq \geq \longrightarrow$
 $(\ell_\star, \geq, >)$ LL-labeling

Corollary

ℓ weak LL-labeling, $\star(\mathcal{R})$ terminating, critical peaks decreasing for $\ell_\star \times \ell$
 $\longrightarrow \mathcal{R}$ confluent

Example (OO03)

 $\mathcal{R}_>^*$

$$\text{sq}_1(s_1(x)) \rightarrow +_1(\times_1(x))$$

$$\text{sq}_1(s_1(x)) \rightarrow +_1(\times_2(x))$$

$$\text{sq}_1(s_1(x)) \rightarrow +_2(s_1(+_1(x)))$$

$$\text{sq}_1(s_1(x)) \rightarrow +_2(s_1(+_2(x)))$$

$$\text{sq}_1(x) \rightarrow \times_1(x)$$

$$\text{sq}_1(x) \rightarrow \times_2(x)$$

$$\times_2(y) \rightarrow +_1(\times_2(y))$$

$$\times_2(y) \rightarrow +_2(y)$$

$$\times_1(x) \rightarrow +_1(x)$$

$$\times_1(x) \rightarrow +_2(\times_1(x))$$

$$\times_1(x) \rightarrow \times_2(x)$$

$$\times_2(y) \rightarrow \times_1(y)$$

$$+_1(x) \rightarrow +_2(x)$$

$$+_2(y) \rightarrow +_1(y)$$

$$+_1(x) \rightarrow +_1(s_1(x))$$

$$+_1(x) \rightarrow +_1(+_1(x))$$

$$+_2(z) \rightarrow +_2(+_2(z))$$

$$+_2(y) \rightarrow +_2(s_1(y))$$

$$\times_1(x) \rightarrow +_2(\times_1(x))$$

 $\mathcal{R}_=^*$

$$\times_1(s_1(x)) \rightarrow +_1(\times_1(x))$$

$$\times_2(s_1(y)) \rightarrow +_2(\times_2(y))$$

$$+_1(s_1(x)) \rightarrow +_1(x)$$

$$+_1(+_1(x)) \rightarrow +_1(x)$$

$$+_1(+_2(y)) \rightarrow +_2(+_1(y))$$

$$+_2(+_2(z)) \rightarrow +_2(z)$$

$$+_2(s_1(y)) \rightarrow +_2(y)$$

$$+_2(+_1(y)) \rightarrow +_1(+_2(y))$$

$$\times_2(y) \rightarrow +_1(\times_2(y))$$

$$\text{sq}_{1\mathbb{N}}(x) = x + 2 \quad \times_{1\mathbb{N}}(x) = \times_{2\mathbb{N}}(x) = x + 1 \quad +_{1\mathbb{N}}(x) = +_{2\mathbb{N}}(x) = s_1(x) = x$$

Implementation

Label with \mathbb{N} (for lexicographic combination)

l_{rl} ✓ l_{sn} ? ✓ l_{\star} ? ? ✓

l_{rl}

(Aoto, 2010), (Hirokawa & Middeldorp, 2010)

l_{sn}

peak $t \leftarrow s \rightarrow u$

partial termination proof of $\text{CDS}(\mathcal{R})/\mathcal{R} \longrightarrow \mathcal{S}'/\mathcal{R}'$

$l_{sn}(t_i \rightarrow t_{i+1}) = 1$ if $s \rightarrow_{\mathcal{S}'}^* t_i$

$l_{sn}(t_i \rightarrow t_{i+1}) = 0$ otherwise

l_{\star}

prove termination of $\star_{\star}(\mathcal{R})$ with matrix interpretations

Experiments

53 left-linear TRSs (Aoto, 2010)

method	pre	$CR(\ell_{rl})$	$CR(\ell_{sn})$	CR
rule labeling	40	35	–	35
$SN(\mathcal{R}_d/\mathcal{R}_{nd})$	45	40	37	42
$SN(\star(\mathcal{R}))$	45	42	34	42
$SN(\star_{\neq}(\mathcal{R}))$	48	45	36	45
ACP	–	42	–	48
CSI	–	–	–	49

106 TRSs (Aoto, 2010)

tool	CR	not CR
ACP	64	18
CSI	61	20

Conclusion

Summary

- decreasing diagrams
- (linear) $l_{rl}, l_{sn}, l_{\star}$
- (left-linear) $SN(\mathcal{R}_d/\mathcal{R}_{nd}), SN(\star(\mathcal{R})), SN(\star^{\dagger}(\mathcal{R}))$

Future Work

- study parallel reduction