Fixed Base Elementary Interpretations

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September 18, 2013

Overview

- Motivation
- Encoding
- Implementation
- Conclusion

Why elementary interpretations?

Some history

method	theory	implementation
polynomials	Lankford'79	

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Which functions

- ✓ $+$, $\times$, 2^x
- ? x^2, y^x
- × $\sin(x)$, $\log(x)$, $\sqrt[n]{m}$

Which shape?

Linear elementary interpretation (LEI)

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Example (Factorial)

$0 + x \to x$	$0 \cdot x \to 0$	$\mathsf{fact}(0) \to \mathsf{s}(0)$
$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$	$\mathsf{s}(x) \cdot y \to x \cdot y + y$	$\mathsf{fact}(\mathsf{s}(x)) \to \mathsf{s}(x) \cdot \mathsf{fact}(x)$

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$$x + 5 > x \qquad 4x > 2 \qquad 2^{2^2} > 4$$

$$2x + y + 5 > 2x + y + 3 \qquad 2^{x+2} y > 2^{x+1} y + y + 1 \qquad 2^{2^{x+2}} > 2^{x+2} 2^{2^x}$$

FBIs - General idea

$$f(\vec{x}) = \sum_{1 \leqslant i \leqslant n} x_i f_i + f_0 + b^{f'(\vec{x})}(\sum_{1 \leqslant i \leqslant n} x_i \dot{f}_i + \dot{f}_0)$$

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Problem (FBIs not closed under typical operations)

- addition: $2^x + 2^y$ ✗
- multiplication: $2^x(x + 2^y) = 2^x x + 2^{x+y}$ ✗
- composition: $(x + 2^x x) \circ 2^x = 2^x + 2^{2^x} 2^x = 2^x + 2^{2^x + x}$ ✗
- scalar multiplication: $(x + 2^x y)4 = 4x + 2^x 4y$ ✓

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Solution

- use approximations
- if $[\alpha]^{\nu}_{\mathcal{A}}(\ell) > [\alpha]^{\mu}_{\mathcal{A}}(r)$ for all $\ell \to r \in \mathcal{R}$ then $\mathcal{R}$ is terminating

Desired properties

$$f(\vec{x}) = \sum_{1 \leqslant i \leqslant n} x_i f_i + f_0 + b^{f'(\vec{x})}(\sum_{1 \leqslant i \leqslant n} x_i \dot{f}_i + \dot{f}_0) = \overline{f}(\vec{x}) + b^{f'(\vec{x})} \dot{f}(\vec{x})$$

Well-definedness ($\mathbb{N}_{\geqslant 1}$)

$wd(f) := \bigvee_{0 \leqslant i \leqslant n} f_i > 0 \vee \bigvee_{0 \leqslant i \leqslant n} \dot{f}_i > 0$

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Monotonicity

$mon(f) := \bigwedge_{1 \leqslant i \leqslant n} (f_i > 0 \vee \dot{f}_i > 0 \vee (\dot{f} > 0 \wedge mon(f')))$

Addition

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Addition

$$f(\vec{x}) = \sum_{1\leqslant i\leqslant n} x_i f_i + f_0 + b^{f'(\vec{x})}\Big(\sum_{1\leqslant i\leqslant n} x_i \dot{f}_i + \dot{f}_0\Big) \qquad g(\vec{x}) = \sum_{1\leqslant i\leqslant n} x_i g_i + g_0 + b^{g'(\vec{x})}\Big(\sum_{1\leqslant i\leqslant n} x_i \dot{g}_i + \dot{g}_0\Big)$$

$$\begin{aligned} f(\vec{x}) +_\nu g(\vec{x}) = & \sum_{1\leqslant i\leqslant n} x_i(f_i + g_i) + (f_0 + g_0) \\ & + b^{\dot{f}(\vec{x})=0?g'(\vec{x}):\dot{g}(\vec{x})=0?f'(\vec{x}):\max(f'(\vec{x}),g'(\vec{x}))}\left(\sum_{1\leqslant i\leqslant n} x_i(\dot{f}_i + \dot{g}_i) + (\dot{f}_0 + \dot{g}_0)\right) \end{aligned}$$

Example

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Multiplication

$$f(\vec{x}) = \sum_{1 \leqslant i \leqslant n} x_i f_i + f_0 + b^{f'(\vec{x})}(\sum_{1 \leqslant i \leqslant n} x_i \dot{f}_i + \dot{f}_0)$$

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$$:$$

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Example

- $2^x \cdot_\mu y =$
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Multiplication

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Example

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Composition

Let $f(\vec{g})(\vec{x}) := f(g_1(\vec{x}), \ldots, g_n(\vec{x}))$ and $\sum_{1 \leqslant i \leqslant n}^{\mu} h_i := h_1 +_\mu \cdots +_\mu h_n$.

$$f(\vec{g})_\mu(\vec{x}) = \sum_{1 \leqslant i \leqslant n}^{\mu} g_i(\vec{x}) f_i +_\mu f_0 +_\mu b^{f'(\vec{g})_\mu(\vec{x})} \cdot_\mu \Big(\sum_{1 \leqslant i \leqslant n}^{\mu} g_i(\vec{x}) \dot{f}_i +_\mu \dot{f}_0 \Big)$$

$$f(\vec{g})_\nu(\vec{x}) = \sum_{1 \leqslant i \leqslant n}^{\nu} g_i(\vec{x}) f_i +_\nu f_0 +_\nu b^{f'(\vec{g})_\nu(\vec{x})} \cdot_\nu \Big(\sum_{1 \leqslant i \leqslant n}^{\nu} g_i(\vec{x}) \dot{f}_i +_\nu \dot{f}_0 \Big)$$

Comparison

$$[f(\vec{x}) \geqslant g(\vec{x})]$$

Comparison

$$[f(\vec{x}) \geqslant g(\vec{x})] := (\dot{g}(\vec{x}) > 0 \rightarrow [f'(\vec{x}) \geqslant g'(\vec{x})]) \wedge (① \vee ② \vee ③)$$

Comparison

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① $\dot{f}(\vec{x}) > 0 \wedge [f'(\vec{x})b \geqslant g(\vec{x})]$

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③ $\lfloor b^{\overline{f'}(\vec{x})-\overline{g'}(\vec{x})} \rfloor \dot{f}(\vec{x}) \geqslant \dot{g}(\vec{x}) + \alpha(\vec{x}) \wedge \overline{f}(\vec{x}) + \lfloor b^{\overline{g'}(\vec{x})} \rfloor \alpha(\vec{x}) \geqslant \overline{b}(\vec{x})$

Comparison

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$$p(\vec{x}) + b^{p'(\vec{x})} \dot{p}(\vec{x}) \geqslant q(\vec{x}) + b^{q'(\vec{x})} \dot{q}(\vec{x})$$

Comparison

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$$\frac{p(\vec{x})}{b^{q'(\vec{x})}} + b^{p'(\vec{x}) - q'(\vec{x})} \dot{p}(\vec{x}) \geqslant \frac{q(\vec{x})}{b^{q'(\vec{x})}} + \dot{q}(\vec{x})$$

Comparison

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$$b^{p'(\vec{x})-q'(\vec{x})}\dot{p}(\vec{x}) \geqslant \dot{q}(\vec{x}) + \alpha(\vec{x}) \land p(\vec{x}) + b^{q'(\vec{x})}\alpha(\vec{x}) \geqslant q(\vec{x})$$

Comparison

$$[f(\vec{x}) \geqslant g(\vec{x})] := (\dot{g}(\vec{x}) > 0 \to [f'(\vec{x}) \geqslant g'(\vec{x})]) \wedge (① \vee ② \vee ③)$$

① $\dot{f}(\vec{x}) > 0 \wedge [f'(\vec{x})b \geqslant g(\vec{x})]$ ② $\overline{f}(\vec{x}) \geqslant \overline{g}(\vec{x}) \wedge \dot{f}(\vec{x}) \geqslant \dot{g}(\vec{x})$

③ $\lfloor b^{\overline{f'}(\vec{x})-\overline{g'}(\vec{x})} \rfloor \dot{f}(\vec{x}) \geqslant \dot{g}(\vec{x}) + \alpha(\vec{x}) \wedge \overline{f}(\vec{x}) + \lfloor b^{\overline{g'}(\vec{x})} \rfloor \alpha(\vec{x}) \geqslant \overline{b}(\vec{x})$

$$p(\vec{x}) + b^{p'(\vec{x})}\dot{p}(\vec{x}) \geqslant q(\vec{x}) + b^{q'(\vec{x})}\dot{q}(\vec{x})$$

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$\lfloor b^{p'(\vec{x})} \rfloor := p'(\vec{x}) = 0 \ ?\ 1 : p'(\vec{x})b$

Heuristics

1-5 limit coefficients

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6 avoid non-linear polynomials

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d limit degree of FBIs (based on call-graph)

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Example (Factorial)

$$0 + x \to x \qquad 0 \cdot x \to 0 \qquad \mathsf{fact}(0) \to \mathsf{s}(0)$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \cdot y \to x \cdot y + y \qquad \mathsf{fact}(\mathsf{s}(x)) \to \mathsf{s}(x) \cdot \mathsf{fact}(x)$$

function symbol	0	s	$+$	$\cdot$	fact
degree					

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degree	0	0	0		

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function symbol	0	s	$+$	$\cdot$	fact
degree	0	0	0	1	

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Example (Factorial)

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function symbol	0	s	$+$	$\cdot$	fact
degree	0	0	0	1	2

Experimental results

method	TPDB 8.0.6[1] YES	avg. time	Example (Factorial) avg. time
poly	125	0.4	-

[1]TRS Standard

Experimental results

method	TPDB 8.0.6[1] YES	TPDB 8.0.6[1] avg. time	Example (Factorial) avg. time
poly	125	0.4	-
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[1] TRS Standard

Experimental results

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method	YES	avg. time	avg. time
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[1] TRS Standard

Experimental results

method	TPDB 8.0.6[1] YES	avg. time	Example (Factorial) avg. time
poly	125	0.4	-
fbi	44	30.7	*2857.1*
fbi(d)	161	5.2	25.3
fbi(d1)	163	5.6	24.9
fbi(d2)	159	5.1	13.8
fbi(d3)	161	5.1	24.5
fbi(d4)	161	5.0	24.3
fbi(d5)	164	5.4	20.8
fbi(d6)	158	4.3	25.5
fbi(d123456)	154	3.3	10.1

[1] TRS Standard

Conclusion

Summary

- implementation of elementary functions
- harder than ordinal arithmetic

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Future work

- suitable for AC
- suitable for ordered completion