# Lazy Termination Analysis

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23 November 2009

# Term Rewrite Systems

## Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

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$$
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$$

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## Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

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$\exists$ *reduction order* $>$ *s.t.* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}$ *terminating*

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# SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

## Example

```
(Real a)
(Real b)
(formula (and (or (b + b > 3 * b) (a * a = 2)) (b > -5)))
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| undecidable over $\mathbb{N}$, $\mathbb{Z}$ | (Hilbert's 10th problem, Matiyasevic 1970) |
| decidable over $\mathbb{R}$ | (Tarski 1951, Collins 1973) |
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## Idea

reduce (bounded) arithmetic to SAT (fast solvers exist)

# The Global Picture

## Overal Idea

$$\text{TRS } \mathcal{R} \quad \rightsquigarrow \quad \varphi$$

$$\alpha \models \varphi \quad \rightsquigarrow \quad \mathcal{R} \text{ terminating}$$

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# Main Contributions

Tyrolean Termination Tool 2
Knuth-Bendix Order
Finding Loops
SAT via Termination
Matrix Interpretations
argument filterings
Constraints
Solving Arithmetic
usable rules
Certification
Real Arithmetic
Increasing Interpretations

## Relevant Literature


- H. Zankl, N. Hirokawa and A. Middeldorp.
  KBO orientability, *J Automated Reasoning* 43(2):173–201, Springer, 2009.
- H. Zankl, N. Hirokawa and A. Middeldorp.
  Constraints for Argument Filterings, *33rd SOFSEM*, LNCS 4362, 579–590, 2007.
- H. Zankl and A. Middeldorp.
  Satisfying KBO constraints, *17th RTA*, LNCS 4533, 389–403, 2007.

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Increasing Interpretations, *Annals of Mathematics and Artificial Intelligence* 56(1):87-108, Springer, 2009.

H. Zankl and A. Middeldorp.
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## Relevant Literature

H. Zankl and A. Middeldorp.
Solving Non-Linear Arithmetic for Matrix Interpretations over the Reals, 2009.
Submitted.

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## Relevant Literature

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## Relevant Literature

H. Zankl, C. Sternagel and A. Middeldorp.
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## Relevant Literature

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# Overview

# Monotone Algebras

## Monotone Algebra

$(A, [\cdot], >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i > b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) > f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded

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## Example $(\mathbb{N},[\cdot],>)$

$$\mathsf{add}(0, y) \to y \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

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## Example $(\mathbb{N}, [\cdot], >)$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f$

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$$\mathsf{add}(\mathsf{0}, y) \to y \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

$$\mathsf{add}_{\mathbb{N}}(x, y) = 2x + y \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{0}_{\mathbb{N}} = 1$$

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$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f$ $\quad$ ($f_i \geqslant 1 \longrightarrow$ monotone)

$$\mathsf{add}(0, y) \to y \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

$$\mathsf{add}_{\mathbb{N}}(x, y) = 2x + 1y \qquad \mathsf{s}_{\mathbb{N}}(x) = 1x + 1 \qquad 0_{\mathbb{N}} = 1$$

# Monotone Algebras

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## Example $(\mathbb{N},[\cdot],>)$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f \quad (f_i \geqslant 1 \longrightarrow \text{monotone})$

$2 + y \quad > y \qquad 2 + 2x + y \quad > 1 + 2x + y$

$\mathsf{add}_{\mathbb{N}}(x, y) = 2x + \ y \qquad \mathsf{s}_{\mathbb{N}}(x) = \ x + 1 \qquad \mathsf{0}_{\mathbb{N}} = 1$

# Matrix Interpretations

## Example $(\mathbb{R}_+^d, [\cdot], >_{\mathbb{R}^d}^{\delta})$

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$$\mathsf{add}_{\mathbb{R}}(\vec{x}, \vec{y}) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} \quad \mathsf{s}_{\mathbb{R}}(\vec{x}) = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathsf{0}_{\mathbb{R}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

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## Example $(\mathbb{R}_+^d, [\cdot], >_{\mathbb{R}^d}^{\delta})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f$ $\qquad (f_{i(1,1)} \geqslant 1 \longrightarrow$ monotone)

$$\mathsf{add}(0, y) \to 0 \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

$$\mathsf{add}_{\mathbb{R}}(\vec{x}, \vec{y}) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} \quad \mathsf{s}_{\mathbb{R}}(\vec{x}) = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad 0_{\mathbb{R}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} >_{\mathbb{R}^2}^{\delta} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

# Matrix Interpretations

## Example $(\mathbb{R}^d_+, [\cdot], >^\delta_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f \qquad (f_{i(1,1)} \geqslant 1 \longrightarrow \text{monotone})$

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$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} >^\delta_{\mathbb{R}^2} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{y}$$

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$$? > ? \;\wedge\; ? \geqslant ? \;\wedge\; ? \geqslant ? \;\wedge\; ? \geqslant ? \;\wedge\; ? \geqslant ? \;\wedge\; ? \geqslant ?$$

# Exemplary Constraints

## As SMT problem

```
(benchmark ttt2
:logic QF_NIA
:status unknown
:extrafuns ((x17 Real) ... (x0 Real))
:formula (and (and (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3
x7))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11
1))) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0
(+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11)))))) (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3 x7
))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11 1)
)) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0 (
+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
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x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
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```

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1))) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0
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(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
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```

## As SAT problem (5 bits for variables)

65,420 clauses 149,755 variables $<$ 1 second solving time

# Overview

- Introduction
- Monotone Algebras/Matrix Interpretations
- SMT Solving
- Experimental Results
- Conclusion

# Arithmetic over $\mathbb{N}$

$\vec{a}_k = \langle a_k, \ldots, a_1 \rangle$

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$\vec{a}_k = \langle a_k, \ldots, a_1 \rangle$ $\quad\quad$ $\langle \top, \bot, \top, \top \rangle \equiv 2^3 + 2^1 + 2^0 = 11$

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## Definition (Addition)

$$\vec{a}_k >_{\mathbf{N}} \vec{b}_k = \begin{cases} \bot & \text{if } k = 0 \\ (a_k \wedge \neg b_k) \vee \big((a_k \leftrightarrow b_k) \wedge \vec{a}_{k-1} >_{\mathbf{N}} \vec{b}_{k-1}\big) & \text{if } k > 1 \end{cases}$$

$$\vec{a}_k =_{\mathbf{N}} \vec{b}_k = \bigwedge_{i=1}^{k} (a_i \leftrightarrow b_i)$$

$$\vec{a}_k +_{\mathbf{N}} \vec{b}_k = \langle c_k, s_k, \ldots, s_1 \rangle$$

$$\text{for } 1 \leqslant i \leqslant k \text{ with } \begin{aligned} c_0 &= \bot \\ s_i &= a_i \oplus b_i \oplus c_{i-1} \\ c_i &= (a_i \wedge b_i) \vee (a_i \wedge c_{i-1}) \vee (b_i \wedge c_{i-1}) \end{aligned}$$

$\oplus$ is exclusive or

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# Arithmetic over $\mathbb{N}$ (cont'd)

## Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1 \rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1 \rangle$$

$$\langle a_3, a_2, a_1 \rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot \rangle$$

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## Definition (Multiplication)

$$\vec{a}_m \times_{\mathbf{N}} \vec{b}_n \quad = \quad \left((\vec{a}_m \cdot b_1 \ll 0) +_{\mathbf{N}} \cdots +_{\mathbf{N}} (\vec{a}_m \cdot b_n \ll (n-1))\right)_{m+n}$$

# Arithmetic over $\mathbb{N}$ (cont'd)

## Scalar Multiplication and Shifting

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## Example

$$\langle \top, \top \rangle \;+_{\mathbf{N}}\; \langle \top, \bot, \top \rangle = \langle \top, \bot, \bot, \bot \rangle \qquad 3 + 5 = 8$$
$$\langle \top, \top \rangle \;\times_{\mathbf{N}}\; \langle \top, \bot, \top \rangle = \langle \bot, \top, \top, \top, \top \rangle \qquad 3 \times 5 = 15$$
$$\langle \bot, \top, \top, \top, \top \rangle \;\times_{\mathbf{N}}\; \langle \bot, \bot, \top \rangle = \langle \bot, \bot, \bot, \bot, \top, \top, \top, \top \rangle \qquad 15 \times 1 = 15$$

# Arithmetic over $\mathbb{N}$ (cont'd)

## Scalar Multiplication and Shifting

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## Example

| | | | | | |
|---|---|---|---|---|---|
| $\langle \top, \top \rangle$ | $+_{\mathbf{N}}$ | $\langle \top, \bot, \top \rangle$ | $=$ | $\langle \top, \bot, \bot, \bot \rangle$ | $3 + 5 = 8$ |
| $\langle \top, \top \rangle$ | $\times_{\mathbf{N}}$ | $\langle \top, \bot, \top \rangle$ | $=$ | $\langle \bot, \top, \top, \top, \top \rangle$ | $3 \times 5 = 15$ |
| $\langle \bot, \top, \top, \top, \top \rangle$ | $\times_{\mathbf{N}}$ | $\langle \bot, \bot, \top \rangle$ | $=$ | $\langle \bot, \bot, \bot, \bot, \top, \top, \top, \top \rangle$ | $15 \times 1 = 15$ |

# Arithmetic over $\mathbb{N}$ (cont'd)

## Scalar Multiplication and Shifting

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$$\vec{a}_m \times_{\mathbf{N}} \vec{b}_n \quad = \quad ((\vec{a}_m \cdot b_1 \ll 0) +_{\mathbf{N}} \cdots +_{\mathbf{N}} (\vec{a}_m \cdot b_n \ll (n-1)))_{m+n}$$

## Example

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# Arithmetic over $\mathbb{N}$ (cont'd)

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## Remark

restrict bit-width using side-constraints

# Arithmetic over $\mathbb{Q}$

$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

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\begin{aligned}
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(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}
$$

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$$
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## Definition

$$\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &= (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}$$

# Arithmetic over $\mathbb{Q}$

$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

## Definition

$$\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &= (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}$$

## Problem

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} =$$

# Arithmetic over $\mathbb{Q}$

$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

## Definition

$$
\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &= (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}
$$

## Problem

$$
\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} =
$$

# Arithmetic over $\mathbb{Q}$

$(\langle\top,\bot,\top,\top\rangle, 2) \equiv \frac{11}{2}$

## Definition

$$
\begin{array}{rcl}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) & = & \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) & = & \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) & = & (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') & = & (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{array}
$$

## Problem

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} = \frac{12}{8} + \frac{1}{2} =$$

# Arithmetic over $\mathbb{Q}$

$$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$$

## Definition

$$\begin{array}{rcl}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) & = & \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) & = & \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) & = & (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') & = & (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{array}$$

## Problem

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} = \frac{12}{8} + \frac{1}{2} = \frac{16}{8}$$

# Fast Arithmetic for $\mathbb{Q}$

## Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

## Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

## Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{6}{4} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = {\color{red}\frac{3}{2}} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

**Problem: if denominator chosen too small (here: 1)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

**Problem: if denominator chosen too small (here: 1)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

**Problem: if denominator chosen too small (here: 1)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

**Problem: if denominator chosen too small (here: 1)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

**Problem: if denominator chosen too small (here: 1)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{?}{1} + \frac{1}{2} =$$

# Fast Arithmetic for $\mathbb{Q}$

**Solution: cancel if denominator exceeds limit (here: 2)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

**Problem: if denominator chosen too small (here: 1)**

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{?}{1} + \frac{1}{2} = \frac{?}{1}$$

# Fast Arithmetic for $\mathbb{Q}$

## Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

## Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{?}{1} + \frac{1}{2} = \frac{?}{1}$$

## Remarks

cancel fractions by side-constraints

# Arithmetic over $\mathbb{R}$

$(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3},\mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$? $\qquad 6 + 3\sqrt{2} >_{\mathbb{R}} 2 + 5\sqrt{2}$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$? $\quad 6 + 3\sqrt{2} >_{\mathbb{R}} 2 + 5\sqrt{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3},\mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

## Definition

$$
\begin{aligned}
(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5},4) >_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3},2) \\
(\mathbf{c},\mathbf{d}) =_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \wedge \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c},\mathbf{d}) +_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c},\mathbf{d}) \times_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}
$$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3},\mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

## Definition

$$
\begin{aligned}
(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5},4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3},2) \\
(\mathbf{c},\mathbf{d}) =_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \;\wedge\; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
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(\mathbf{c},\mathbf{d}) \times_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}
$$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

## Definition

$$\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \;\wedge\; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
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\end{aligned}$$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3},\mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

## Definition

$$(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) = \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5},4) >_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3},2)$$

$$(\mathbf{c},\mathbf{d}) =_{\mathbf{R}} (\mathbf{e},\mathbf{f}) = \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \wedge \mathbf{d} =_{\mathbf{Q}} \mathbf{f}$$

$$(\mathbf{c},\mathbf{d}) +_{\mathbf{R}} (\mathbf{e},\mathbf{f}) = (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f})$$

$$(\mathbf{c},\mathbf{d}) \times_{\mathbf{R}} (\mathbf{e},\mathbf{f}) = (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})$$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

## Definition

$$\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
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(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}$$

## Example

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2} \equiv (\mathbf{5}, \mathbf{11})$

$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = 66 + 28\sqrt{2} \equiv (\mathbf{66}, \mathbf{28})$

# Arithmetic over $\mathbb{R}$

$(\mathbf{3},\mathbf{5}) \equiv 3 + 5\sqrt{2}$

## Problem

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

## Definition

$$
\begin{aligned}
(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
(\mathbf{c},\mathbf{d}) =_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \;\wedge\; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c},\mathbf{d}) +_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c},\mathbf{d}) \times_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}
$$

## Example

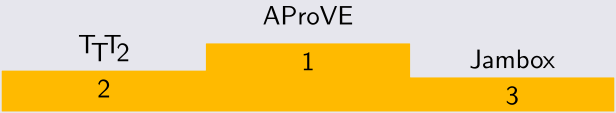
$(\mathbf{3},\mathbf{5}) +_{\mathbf{R}} (\mathbf{2},\mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2} \equiv (\mathbf{5},\mathbf{11})$

$(\mathbf{3},\mathbf{5}) \times_{\mathbf{R}} (\mathbf{2},\mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = 66 + 28\sqrt{2} \equiv (\mathbf{66},\mathbf{28})$

# Overview

# TermComp 2008

## TRS category (1391 problems)



# TermComp 2008

## TRS category (1391 problems)

| | TTT2 | AProVE | Jambox |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 792 | 995 | 750 |
| no | 178 | 231 | 60 |
| time | 2,797 | 20,533 | 27,925 |

# TermComp 2008

## TRS category (1391 problems)

| | TTT2 | AProVE | Jambox |
|---|---|---|---|
| rank | 2 | 1 | 3 |
| yes | 792 | 995 | 750 |
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| time | 2,797 | 20,533 | 27,925 |

## SRS category (732 problems)

| | AProVE | TTT2 | Jambox |
|---|---|---|---|
| rank | 2 | 1 | 3 |

# TermComp 2008

## TRS category (1391 problems)

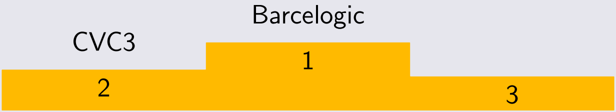
| | $T_TT_2$ | AProVE | Jambox |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 792 | 995 | 750 |
| no | 178 | 231 | 60 |
| time | 2,797 | 20,533 | 27,925 |

## SRS category (732 problems)

| | AProVE | $T_TT_2$ | Jambox |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 501 | 512 | 252 |
| no | 22 | 40 | 0 |
| time | 18,101 | 14,747 | 29,160 |

# SMT Comp 2009

## QF_NIA category (200 problems)



# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 47 | 106 | |
| no | 89 | 87 | |
| time | 3,890 | 554 | |

# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | MiniSmt |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 47 | 106 | 106 |
| no | 89 | 87 | - |
| time | 3,890 | 554 | 3,535 |

# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | MiniSmt |
|---|---|---|---|
| rank | 2 | 1 | 3 |
| yes | 47 | 106 | 106 |
| no | 89 | 87 | - |
| time | 3,890 | 554 | 3,535 |

## 1391 matrix constraints ($\mathbb{N}$)

| | MiniSmt | Barcelogic | CVC3 |
|---|---|---|---|
| rank | 2 | 1 | 3 |

# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | MiniSmt |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 47 | 106 | 106 |
| no | 89 | 87 | - |
| time | 3,890 | 554 | 3,535 |

## 1391 matrix constraints ($\mathbb{N}$)

| | MiniSmt | Barcelogic | CVC3 |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 403 | 407 | 117 |
| no | - | 3 | 130 |
| time | 7,103 | 40,514 | 17,787 |

# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | MiniSmt |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 47 | 106 | 106 |
| no | 89 | 87 | - |
| time | 3,890 | 554 | 3,535 |

## 1391 matrix constraints ($\mathbb{Q}$)

| | CVC3 | MiniSmt | |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | | | |
| no | | | |
| time | | | |

# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | MiniSmt |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 47 | 106 | 106 |
| no | 89 | 87 | - |
| time | 3,890 | 554 | 3,535 |

## 1391 matrix constraints ($\mathbb{Q}$)

| | CVC3 | MiniSmt | |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 58 | 408 | |
| no | 125 | - | |
| time | 69,485 | 6,190 | |

# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | MiniSmt |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 47 | 106 | 106 |
| no | 89 | 87 | - |
| time | 3,890 | 554 | 3,535 |

## 1391 matrix constraints ($\mathbb{R}$)

| | | MiniSmt | |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | | | |
| no | | | |
| time | | | |

# SMT Comp 2009

## QF_NIA category (200 problems)

| | CVC3 | Barcelogic | MiniSmt |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | 47 | 106 | 106 |
| no | 89 | 87 | - |
| time | 3,890 | 554 | 3,535 |

## 1391 matrix constraints ($\mathbb{R}$)

| | | MiniSmt | |
|---|---|---|---|
| | 2 | 1 | 3 |
| yes | | 354 | |
| no | | - | |
| time | | 19,892 | |

# Conclusion

## Termination

SMT solving is suitable

# Conclusion

## Termination

SMT solving is suitable

arctic interpretations, argument filterings, below zero, delta-restricted interpretations, increasing interpretations, Knuth-Bendix order, lexicographic path order, loops, matrix interpretations, multiset path order, polynomial interpretations, polynomial path order, predictive labeling, recursive path order, semantic labeling, usable rules wrt argument filtering, triangular matrices.

# Conclusion

## Termination

SMT solving is suitable

arctic interpretations, argument filterings, below zero, delta-restricted interpretations, increasing interpretations, Knuth-Bendix order, lexicographic path order, loops, matrix interpretations, multiset path order, polynomial interpretations, polynomial path order, predictive labeling, recursive path order, semantic labeling, usable rules wrt argument filtering, triangular matrices.

## SMT Solving

SAT solving is suitable

# Conclusion

## Termination

SMT solving is suitable
arctic interpretations, argument filterings, below zero, delta-restricted interpretations, increasing interpretations, Knuth-Bendix order, lexicographic path order, loops, matrix interpretations, multiset path order, polynomial interpretations, polynomial path order, predictive labeling, recursive path order, semantic labeling, usable rules wrt argument filtering, triangular matrices.

## SMT Solving ($T$ = non-linear arithmetic)

SAT solving is suitable

# Appendix

TRS from [Lucas 06]

$$k(x,x,b_1) \to k(g(x),b_2,b_2) \qquad g(c(x)) \to f(c(f(x)))$$
$$k(x,a_2,b_1) \to k(a_1,x,b_1) \qquad f(f(x)) \to g(x)$$
$$k(a_4,x,b_1) \to k(x,a_3,b_1) \qquad f(f(f(f(x)))) \to k(x,x,x)$$
$$k(g(x),b_3,b_3) \to k(x,x,b_4)$$

$\mathsf{T_TT_2}$ finds the interpretation that orients all rules strictly

$$a_{1\mathbb{R}} = 0 \qquad b_{1\mathbb{R}} = 2+\sqrt{2} \qquad f_{\mathbb{R}}(x) = \sqrt{2}x+\sqrt{2}$$
$$a_{2\mathbb{R}} = 1+2\sqrt{2} \qquad b_{2\mathbb{R}} = 0 \qquad g_{\mathbb{R}}(x) = 2x+1+\sqrt{2}$$
$$a_{3\mathbb{R}} = 0 \qquad b_{3\mathbb{R}} = 1+\sqrt{2} \qquad c_{\mathbb{R}}(x) = x+1+2\sqrt{2}$$
$$a_{4\mathbb{R}} = 1+\sqrt{2} \qquad b_{4\mathbb{R}} = \sqrt{2} \qquad k_{\mathbb{R}}(x,y,z) = x+y+\sqrt{2}z+3\sqrt{2}$$

within a fraction of a second