Challenges in Automation of Rewriting

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21 April 2015

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} & \mathsf{bits}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0)))$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\underbrace{\mathsf{s}(0)}_{x}))$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\underbrace{\mathsf{s}(0)}_{x})) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\underbrace{\mathsf{s}(0)}_{x}))))$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0)))))$$

Example (Term Rewrite System)

$$\mathsf{half}(\mathsf{0}) \to \mathsf{0} \qquad \mathsf{bits}(\mathsf{0}) \to \mathsf{0}$$
$$\mathsf{half}(\mathsf{s}(\mathsf{0})) \to \mathsf{0} \qquad \mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x))$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(\mathsf{0}))))$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(0)))$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0)))))$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$
$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0)))$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Example (Term Rewrite System)

$$\mathsf{half}(0) \to 0 \qquad \mathsf{bits}(0) \to 0$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0 \qquad \mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x))$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) > \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) > \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) >$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) > \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) > \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) > \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?

Example (Term Rewrite System)

$$\mathsf{half}(0) > 0 \qquad \mathsf{bits}(0) > 0$$
$$\mathsf{half}(\mathsf{s}(0)) > 0 \qquad \mathsf{bits}(\mathsf{s}(x)) > \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) > \mathsf{s}(\mathsf{half}(x))$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?
- Complexity: What is the cost of producing a result?

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?
- Complexity: What is the cost of producing a result?
- Confluence: Do we always get the same result?

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

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Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?
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Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?
- Complexity: What is the cost of producing a result?
- Confluence: Do we always get the same result?
- Completion: Make equational logic deterministic?

Example (Equational System)

$$\begin{aligned} \mathsf{half}(0) &\approx 0 & \mathsf{bits}(0) &\approx 0 \\ \mathsf{half}(\mathsf{s}(0)) &\approx 0 & \mathsf{bits}(\mathsf{s}(x)) &\approx \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\approx \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \approx \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \approx \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \approx$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \approx \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \approx \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \approx \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?
- Complexity: What is the cost of producing a result?
- Confluence: Do we always get the same result?
- Completion: Make equational logic deterministic?

Example (Term Rewrite System)

$$\mathsf{half}(0) \to 0 \qquad \mathsf{bits}(0) \to 0$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0 \qquad \mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x))$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result?
- Complexity: What is the cost of producing a result?
- Confluence: Do we always get the same result?
- Completion: Make equational logic deterministic?

- Automation: Automatic and reliable analysis

Example (Term Rewrite System)

$$
\begin{aligned}
\mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0\\
\mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))\\
\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x))
\end{aligned}
$$

$$
\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))
$$

static program analysis

Properties

- Termination: Do all computations produce a result? TTT2
- Complexity: What is the cost of producing a result? CaT
- Confluence: Do we always get the same result? CSI
- Completion: Make equational logic deterministic? KBCV

- Automation: Automatic and reliable analysis

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to \mathsf{s}(\mathsf{s}(0))$$

static program analysis

Properties

- Termination: Do all computations produce a result? $\mathsf{T_TT_2}$
- Complexity: What is the cost of producing a result? CaT
- Confluence: Do we always get the same result? CSI
- Completion: Make equational logic deterministic? KBCV

- Automation: Automatic and reliable analysis CeTA

Termination

Example (Addition)

$$x + 0 \to x$$
$$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$

Termination

Example (Addition)

$$x + 0 \to x$$
$$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$

$$0_{\mathbb{N}} = 1 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Addition)

$$x + 0 \to x \qquad\qquad x + 2 > x$$

$$x + s(y) \to s(x + y) \qquad\qquad x + 2y + 2 > x + 2y + 1$$

$$0_{\mathbb{N}} = 1 \quad s_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Multiplication)

$$x + 0 \to x \qquad\qquad x + 2 > x$$

$$x + s(y) \to s(x + y) \qquad\qquad x + 2y + 2 > x + 2y + 1$$

$$0 \cdot x \to 0$$

$$s(x) \cdot y \to (x \cdot y) + y$$

$$0_{\mathbb{N}} = 1 \quad s_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Multiplication)

$$x + 0 \to x \qquad\qquad x + 2 > x$$

$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad\qquad x + 2y + 2 > x + 2y + 1$$

$$0 \cdot x \to 0$$

$$\mathsf{s}(x) \cdot y \to (x \cdot y) + y$$

$$0_{\mathbb{N}} = 1 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y \quad x \cdot_{\mathbb{N}} y = 2xy + 2x + y$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Multiplication)

$$
\begin{aligned}
x + 0 &\to x & x + 2 &> x \\
x + s(y) &\to s(x + y) & x + 2y + 2 &> x + 2y + 1 \\
0 \cdot x &\to 0 & 3x + 2 &> 1 \\
s(x) \cdot y &\to (x \cdot y) + y & 2xy + 2x + 3y + 2 &> 2xy + 2x + 3y
\end{aligned}
$$

$$0_{\mathbb{N}} = 1 \quad s_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y \quad x \cdot_{\mathbb{N}} y = 2xy + 2x + y$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Factorial)

$$x + 0 \to x \qquad x + 2 > x$$
$$x + s(y) \to s(x + y) \qquad x + 2y + 2 > x + 2y + 1$$
$$0 \cdot x \to 0 \qquad 3x + 2 > 1$$
$$s(x) \cdot y \to (x \cdot y) + y \qquad 2xy + 2x + 3y + 2 > 2xy + 2x + 3y$$
$$!(0) \to s(0)$$
$$!(s(x)) \to s(x) \cdot !(x)$$

$$0_{\mathbb{N}} = 1 \quad s_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y \quad x \cdot_{\mathbb{N}} y = 2xy + 2x + y$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Factorial)

$$\begin{aligned} x + 0 &\to x & x + 2 &> x \\ x + s(y) &\to s(x + y) & x + 2y + 2 &> x + 2y + 1 \\ 0 \cdot x &\to 0 & 3x + 2 &> 1 \\ s(x) \cdot y &\to (x \cdot y) + y & 2xy + 2x + 3y + 2 &> 2xy + 2x + 3y \\ !(0) &\to s(0) \\ !(s(x)) &\to s(x) \cdot !(x) \end{aligned}$$

$$0_{\mathbb{N}} = 1 \quad s_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y \quad x \cdot_{\mathbb{N}} y = 2xy + 2x + y$$

$$!_{\mathbb{N}}(x) = (2x + 2)^{2x+1}$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Factorial)

$$
\begin{array}{rcl}
x + 0 \to x & & x + 2 > x \\
x + \mathsf{s}(y) \to \mathsf{s}(x + y) & & x + 2y + 2 > x + 2y + 1 \\
0 \cdot x \to 0 & & 3x + 2 > 1 \\
\mathsf{s}(x) \cdot y \to (x \cdot y) + y & & 2xy + 2x + 3y + 2 > 2xy + 2x + 3y \\
!(0) \to \mathsf{s}(0) & & 4^3 > 2 \\
!(\mathsf{s}(x)) \to \mathsf{s}(x) \cdot !(x) & & (2x+4)^{2x+3} > (2x+3)(2x+2)^{2x+1} + 2x + 2
\end{array}
$$

$$
0_{\mathbb{N}} = 1 \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad x +_{\mathbb{N}} y = x + 2y \quad x \cdot_{\mathbb{N}} y = 2xy + 2x + y
$$

$$
!_{\mathbb{N}}(x) = (2x+2)^{2x+1}
$$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination

Example (Factorial)

$$
\begin{aligned}
x + 0 &\to x & ? > ?\\
x + s(y) &\to s(x + y) & ? > ?\\
0 \cdot x &\to 0 & ? > ?\\
s(x) \cdot y &\to (x \cdot y) + y & ? > ?\\
!(0) &\to s(0) & ? > ?\\
!(s(x)) &\to s(x) \cdot !(x) & ? > ?
\end{aligned}
$$

$0_{\mathbb{N}} = ?$ $\quad s_{\mathbb{N}}(x) = ?$ $\quad x +_{\mathbb{N}} y = ?$ $\quad x \cdot_{\mathbb{N}} y = ?$

$!_{\mathbb{N}}(x) = ?$

Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)

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- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form

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Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form

RTA List of open problems #28 (Lescanne, 1991)

Develop effective methods to decide whether a system decreases with respect to some exponential interpretation

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Very long computations

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Very looooooooooooong computations

Ackermann Function

$$\mathsf{ack}(0, y) \to \mathsf{s}(y)$$
$$\mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$
$$\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Very loooooooooooooong computations

Sudan Function

$$\begin{aligned} \mathsf{f}_0(x, y) &\to x + y \\ \mathsf{f}_{n+1}(x, \mathsf{0}) &\to x \\ \mathsf{f}_{n+1}(x, \mathsf{s}(y)) &\to \mathsf{f}_n(\mathsf{f}_{n+1}(x, y), \mathsf{f}_{n+1}(x, y) + \mathsf{s}(y)) \end{aligned}$$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2		

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$1 \cdot 3^3 + 0 \cdot 3^1 + 0 \cdot 3^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$1 \cdot 3^3 + 0 \cdot 3^1 + 0 \cdot 3^0 - 1$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		$2 \cdot 4^2 + 2 \cdot 4^1 + 1 \cdot 4^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		$2 \cdot 4^2 + 2 \cdot 4^1 + 1 \cdot 4^0 - 1$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		$2 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
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- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		$2 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$
5		

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		$2 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$
5		$2 \cdot 5^2 + 2 \cdot 5^1 + 0 \cdot 5^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		$2 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$
5		$2 \cdot 5^2 + 2 \cdot 5^1 + 0 \cdot 5^0 - 1$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3		$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4		$2 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$
5		$2 \cdot 5^2 + 1 \cdot 5^1 + 4 \cdot 5^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

Goodstein Sequence

base	number	hereditary notation
2	4	$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$
3	26	$2 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$
4	41	$2 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$
5	60	$2 \cdot 5^2 + 1 \cdot 5^1 + 4 \cdot 5^0$

Termination (2)

Challenges

- Hilbert's 10th problem (Diophantine constraints)
- Hilbert's 17th problem (SOS – Positiveness of polynomials)
- Exponentiation does not admit canonical form
- Veeeeeeeeeeeeery looooooooooooong computations

T_TT_2

- Elementary interpretations
- Ordinal interpretations: Goodstein's theorem
  Hydra game
  Worm principle

Complexity

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

Complexity

Example

$$\mathsf{half}(\mathsf{0}) \to \mathsf{0} \qquad 2 > 1$$
$$\mathsf{half}(\mathsf{s}(\mathsf{0})) \to \mathsf{0} \qquad 4 > 1$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad 2x + 4 > 2x + 1$$

$\mathsf{0}_{\mathbb{N}} = 1$ $\qquad$ $\mathsf{s}_{\mathbb{N}}(x) = x + 1$ $\qquad$ $\mathsf{half}_{\mathbb{N}}(x) = 2x$

Complexity

Example

$$\mathsf{half}(0) \to 0 \qquad 2 > 1$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0 \qquad 4 > 1$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad 2x + 4 > 2x + 1$$

$0_{\mathbb{N}} = 1$ $\qquad$ $\mathsf{s}_{\mathbb{N}}(x) = x + 1$ $\qquad$ $\mathsf{half}_{\mathbb{N}}(x) = 2x$

$$\mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) \quad \to \quad \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) \quad \to \quad \mathsf{s}(0)$$

Complexity

Example

$$\mathsf{half}(0) \to 0 \qquad 2 > 1$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0 \qquad 4 > 1$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad 2x + 4 > 2x + 1$$

$0_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$

$$\underset{8}{\mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))} \quad \to \quad \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) \quad \to \quad \mathsf{s}(0)$$

Complexity

Example

$$\mathsf{half}(0) \to 0 \qquad 2 > 1$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0 \qquad 4 > 1$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad 2x + 4 > 2x + 1$$

$$0_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$$

$$\begin{array}{ccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) \\ 8 & > & 5 & & \end{array}$$

Complexity

Example

$$\mathsf{half}(\mathsf{0}) \to \mathsf{0} \qquad 2 > 1$$

$$\mathsf{half}(\mathsf{s}(\mathsf{0})) \to \mathsf{0} \qquad 4 > 1$$

$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad 2x + 4 > 2x + 1$$

$$\mathsf{0}_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$$

$$\begin{array}{ccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(\mathsf{0}))) & \to & \mathsf{s}(\mathsf{0}) \\ 8 & > & 5 & > & 2 \end{array}$$

Complexity

Example

$$\mathsf{half}(0) \to 0 \qquad 2 > 1$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0 \qquad 4 > 1$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x)) \qquad 2x + 4 > 2x + 1$$

$$0_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$$

$$\begin{array}{ccccccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) & & \\ 8 & > & 5 & > & 2 & & \\ \mathsf{half}^n(0) & \to & \mathsf{half}^{n-1}(0) & \to & \cdots & \to & 0 \end{array}$$

Complexity

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & 2 &> 1 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & 4 &> 1 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & 2x+4 &> 2x+1 \end{aligned}$$

$0_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x+1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$

$$\begin{array}{ccccccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) & & \\ 8 & > & 5 & > & 2 & & \\ \mathsf{half}^n(0) & \to & \mathsf{half}^{n-1}(0) & \to & \cdots & \to & 0 \\ 2^n & & & & & & \end{array}$$

Complexity

Example

$$\mathsf{half}(0) \to 0 \qquad 2 > 1$$
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$0_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$

$$\begin{array}{ccccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) & & \\ 8 & > & 5 & > & 2 & & \\ \mathsf{half}^n(0) & \to & \mathsf{half}^{n-1}(0) & \to & \cdots & \to & 0 \\ 2^n & > & 2^{n-1} & & & & \end{array}$$

Complexity

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & 2 &> 1 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & 4 &> 1 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & 2x+4 &> 2x+1 \end{aligned}$$

$0_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x+1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$

$$\begin{array}{ccccccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) & & \\ 8 & > & 5 & > & 2 & & \\ \mathsf{half}^n(0) & \to & \mathsf{half}^{n-1}(0) & \to & \cdots & \to & 0 \\ 2^n & > & 2^{n-1} & > & \cdots & & \end{array}$$

Complexity

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & 2 &> 1 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & 4 &> 1 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) & 2x+4 &> 2x+1 \end{aligned}$$

$0_{\mathbb{N}} = 1 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{half}_{\mathbb{N}}(x) = 2x$

$$\begin{array}{ccccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) & & \\ 8 & > & 5 & > & 2 & & \\ \mathsf{half}^n(0) & \to & \mathsf{half}^{n-1}(0) & \to & \cdots & \to & 0 \\ 2^n & > & 2^{n-1} & > & \cdots & > & 1 \end{array}$$

Complexity

Example

$$\mathsf{half}(0) \to 0$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x))$$

$0_{\mathbb{N}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \mathsf{s}_{\mathbb{N}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $\quad \mathsf{half}_{\mathbb{N}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{array}{ccccccc} \mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) & & \\ 8 & > & 5 & > & 2 & & \\ \mathsf{half}^n(0) & \to & \mathsf{half}^{n-1}(0) & \to & \cdots & \to & 0 \\ 2^n & > & 2^{n-1} & > & \cdots & > & 1 \end{array}$$

Complexity

Example

$$\mathsf{half}(0) \to 0$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x))$$

$$0_{\mathbb{N}} = \begin{pmatrix}0\\1\end{pmatrix} \qquad \mathsf{s}_{\mathbb{N}}(x) = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}x + \begin{pmatrix}0\\2\end{pmatrix} \qquad \mathsf{half}_{\mathbb{N}}(x) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}x + \begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{array}{ccccccccc}
\mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) & \to & \mathsf{s}(\mathsf{half}(\mathsf{s}(0))) & \to & \mathsf{s}(0) & & \\
8 & > & 5 & > & 2 & & \\
\mathsf{half}^n(0) & \to & \mathsf{half}^{n-1}(0) & \to & \cdots & \to & 0 \\
2^n & > & 2^{n-1} & > & \cdots & > & 1 \\
\begin{pmatrix}n\\1\end{pmatrix} & > & \begin{pmatrix}n-1\\1\end{pmatrix} & > & \cdots & > & \begin{pmatrix}0\\1\end{pmatrix}
\end{array}$$

Complexity (2)

Challenges

- polynomial growth of entries in matrix products

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\longrightarrow$ spectral radius

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\longrightarrow$ spectral radius
- avoid monolithic proofs:

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\longrightarrow$ spectral radius
- avoid monolithic proofs:
  AMIs, SLIs, TMIs, PMIs, PoP*, matchbounds

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\rightarrow$ spectral radius
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Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\rightarrow$ spectral radius
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  AMIs, SLIs, TMIs, PMIs, PoP*, matchbounds $\rightarrow$ framework

Lemma (Modular Complexity)

$$\mathsf{steps}(\rightarrow_{\mathcal{R}\cup\mathcal{S}}) \;=\; \mathcal{O}(\mathsf{steps}(\rightarrow_{\mathcal{R}/\mathcal{S}}) + \mathsf{steps}(\rightarrow_{\mathcal{S}/\mathcal{R}}))$$

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\rightarrow$ spectral radius
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  AMIs, SLIs, TMIs, PMIs, PoP*, matchbounds $\rightarrow$ framework

Lemma (Modular Complexity)

$$\mathsf{steps}(\rightarrow_{\mathcal{R}\cup\mathcal{S}}) \;=\; \mathcal{O}(\mathsf{steps}(\rightarrow_{\mathcal{R}/\mathcal{S}}) + \mathsf{steps}(\rightarrow_{\mathcal{S}/\mathcal{R}}))$$

Proof idea: $\mathcal{R} \subseteq > \; \mathcal{S} \subseteq \geqslant \qquad \mathcal{S} \subseteq > \; \mathcal{R} \subseteq \geqslant$

$$t_1 \;\rightarrow_{\mathcal{R}}\; t_2 \;\rightarrow_{\mathcal{S}}\; t_3 \;\rightarrow_{\mathcal{S}}\; t_4 \;\rightarrow_{\mathcal{R}}\; \cdots$$

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\rightarrow$ spectral radius
- avoid monolithic proofs:
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Lemma (Modular Complexity)

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Proof idea: $\mathcal{R} \subseteq > \; \mathcal{S} \subseteq \geqslant \qquad \mathcal{S} \subseteq > \; \mathcal{R} \subseteq \geqslant$

$$t_1 \;\rightarrow_{\mathcal{R}}\; t_2 \;\rightarrow_{\mathcal{S}}\; t_3 \;\rightarrow_{\mathcal{S}}\; t_4 \;\rightarrow_{\mathcal{R}}\; \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix}$$

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\rightarrow$ spectral radius
- avoid monolithic proofs:
  AMIs, SLIs, TMIs, PMIs, PoP*, matchbounds $\rightarrow$ framework

Lemma (Modular Complexity)

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Proof idea: $\mathcal{R} \subseteq > \; \mathcal{S} \subseteq \geqslant \qquad \mathcal{S} \subseteq > \; \mathcal{R} \subseteq \geqslant$

$$t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{S}} t_3 \rightarrow_{\mathcal{S}} t_4 \rightarrow_{\mathcal{R}} \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix}$$

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\rightarrow$ spectral radius
- avoid monolithic proofs:
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Proof idea: $\mathcal{R} \subseteq > \ \mathcal{S} \subseteq \geqslant \qquad \mathcal{S} \subseteq > \ \mathcal{R} \subseteq \geqslant$

$$t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{S}} t_3 \rightarrow_{\mathcal{S}} t_4 \rightarrow_{\mathcal{R}} \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_3 \\ t_3 \end{pmatrix}$$

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\rightarrow$ spectral radius
- avoid monolithic proofs:
  AMIs, SLIs, TMIs, PMIs, PoP*, matchbounds $\rightarrow$ framework

Lemma (Modular Complexity)

$$\mathsf{steps}(\rightarrow_{\mathcal{R}\cup\mathcal{S}}) \;=\; \mathcal{O}(\mathsf{steps}(\rightarrow_{\mathcal{R}/\mathcal{S}}) + \mathsf{steps}(\rightarrow_{\mathcal{S}/\mathcal{R}}))$$

Proof idea: $\mathcal{R} \subseteq > \; \mathcal{S} \subseteq \geqslant \qquad \mathcal{S} \subseteq > \; \mathcal{R} \subseteq \geqslant$

$$t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{S}} t_3 \rightarrow_{\mathcal{S}} t_4 \rightarrow_{\mathcal{R}} \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_3 \\ t_3 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_4 \\ t_4 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \cdots$$

Complexity (2)

Challenges

- polynomial growth of entries in matrix products $\to$ spectral radius
- avoid monolithic proofs:
  AMIs, SLIs, TMIs, PMIs, PoP*, matchbounds $\to$ framework

Lemma (Modular Complexity)

$$\mathsf{steps}(\to_{\mathcal{R}\cup\mathcal{S}}) \;=\; \Theta(\mathsf{steps}(\to_{\mathcal{R}/\mathcal{S}}) + \mathsf{steps}(\to_{\mathcal{S}/\mathcal{R}}))$$

Proof idea: $\mathcal{R} \subseteq > \ \mathcal{S} \subseteq \geqslant \qquad \mathcal{S} \subseteq > \ \mathcal{R} \subseteq \geqslant$

$$t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{S}} t_3 \to_{\mathcal{S}} t_4 \to_{\mathcal{R}} \cdots$$

$$\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \begin{pmatrix} t_2 \\ t_2 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_3 \\ t_3 \end{pmatrix} \begin{matrix} \geqslant \\ > \end{matrix} \begin{pmatrix} t_4 \\ t_4 \end{pmatrix} \begin{matrix} > \\ \geqslant \end{matrix} \cdots$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof

Complexity (3) – Tighten Bounds

Approach

(1) find proof

Example

$\{1, 2, 3, 4, 5\}$

Complexity (3) – Tighten Bounds

Approach

(1) find proof

Example

$$\begin{array}{c}\{1,2,3,4,5\} \\ \Big\downarrow \mathcal{O}(1) \\ \{1,2,3,4,5\}/\varnothing\end{array}$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof

Example

$$\{1,2,3,4,5\}$$
$$\big\downarrow \mathcal{O}(1)$$
$$\{1,2,3,4,5\}/\varnothing$$
$$\big\downarrow \mathcal{O}(n^3)$$
$$\{2,4\}/\{1,3,5\}$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof

Example

$$\{1,2,3,4,5\}$$
$$\downarrow \mathcal{O}(1)$$
$$\{1,2,3,4,5\}/\varnothing$$
$$\downarrow \mathcal{O}(n^3)$$
$$\{2,4\}/\{1,3,5\}$$
$$\downarrow \mathcal{O}(n)$$
$$\varnothing/\{1,2,3,4,5\}$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof ($\mathcal{O}(n^3)$)

Example

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(n^3)} \{2,4\}/\{1,3,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof ($\mathcal{O}(n^3)$) (2) tighten bound

Example

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(n^3)} \{2,4\}/\{1,3,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

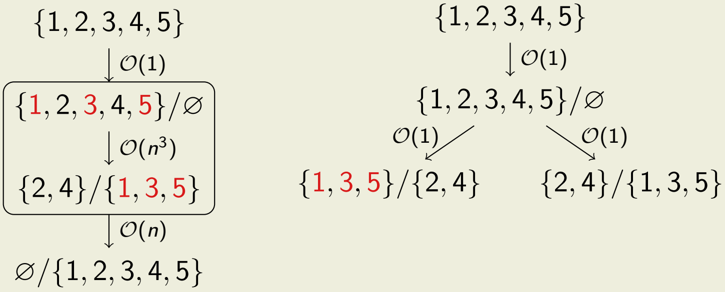
$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof ($\mathcal{O}(n^3)$) (2) tighten bound

Example

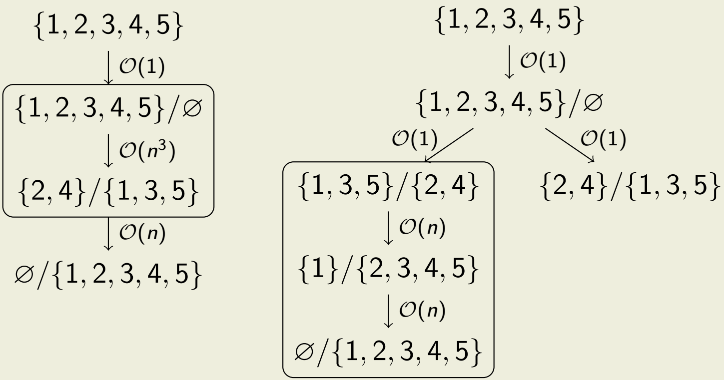


Complexity (3) – Tighten Bounds

Approach

(1) find proof ($\mathcal{O}(n^3)$) (2) tighten bound

Example



Complexity (3) – Tighten Bounds

Approach

(1) find proof ($\mathcal{O}(n^3)$) (2) tighten bound

Example

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(n^3)} \{2,4\}/\{1,3,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing$$

$$\{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(1)} \{1,3,5\}/\{2,4\} \xrightarrow{\mathcal{O}(n)} \{1\}/\{2,3,4,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

$$\{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(1)} \{2,4\}/\{1,3,5\}$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof ($\mathcal{O}(n^3)$)   (2) tighten bound

Example

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(n^3)} \{2,4\}/\{1,3,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing$$

$$\{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(1)} \{1,3,5\}/\{2,4\} \xrightarrow{\mathcal{O}(n)} \{1\}/\{2,3,4,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

$$\{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(1)} \{2,4\}/\{1,3,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

Complexity (3) – Tighten Bounds

Approach

(1) find proof ($\mathcal{O}(n^3)$) (2) tighten bound ($\mathcal{O}(n)$)

Example

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing \xrightarrow{\mathcal{O}(n^3)} \{2,4\}/\{1,3,5\} \xrightarrow{\mathcal{O}(n)} \varnothing/\{1,2,3,4,5\}$$

$$\{1,2,3,4,5\} \xrightarrow{\mathcal{O}(1)} \{1,2,3,4,5\}/\varnothing$$

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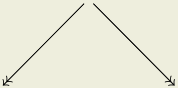
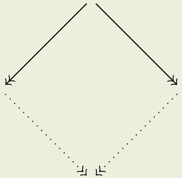
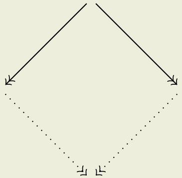
Confluence



Confluence

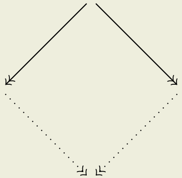


Confluence



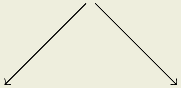
- Applications
  - compiler optimizations
  - equational theorem proving

Confluence

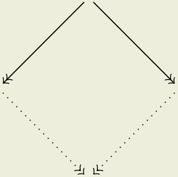


- Applications
  - compiler optimizations
  - equational theorem proving

Decreasing Diagrams

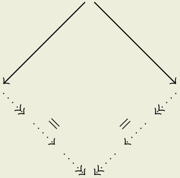


Confluence

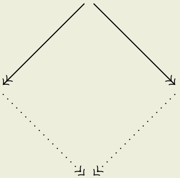


- Applications
  - compiler optimizations
  - equational theorem proving

Decreasing Diagrams

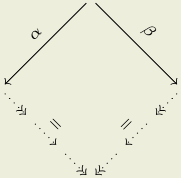


Confluence

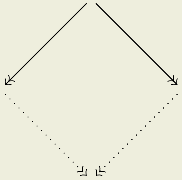


- Applications
  - compiler optimizations
  - equational theorem proving

Decreasing Diagrams

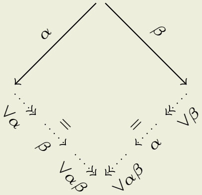


Confluence

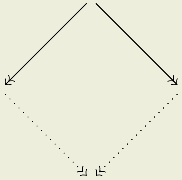


- Applications
  - compiler optimizations
  - equational theorem proving

Decreasing Diagrams

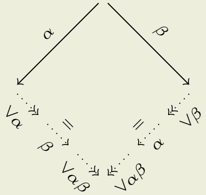


Confluence



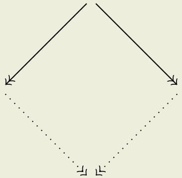
- Applications
  - compiler optimizations
  - equational theorem proving

Decreasing Diagrams



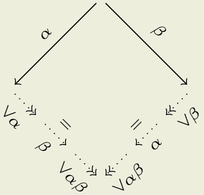
- Status Quo
  - complete method (countable systems)
  - automation: one order

Confluence



- Applications
  - compiler optimizations
  - equational theorem proving

Decreasing Diagrams



- Status Quo
  - complete method (countable systems)
  - automation: one order
- Challenges
  - automation: several orders (combined)
  - formalization: Isabelle

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$x \circ x^{-1} \approx x^{-1} \circ x$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$x \circ x^{-1} \approx x^{-1} \circ x \qquad x \circ x^{-1}$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad \mathsf{e}$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$x \circ x^{-1} \approx x^{-1} \circ x$ $\qquad$ $x^{-1} \circ (x^{-1})^{-1}$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ (x^{-1})^{-1}$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad (x^{-1} \circ \mathsf{e}) \circ (x^{-1})^{-1}$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$x \circ x^{-1} \approx x^{-1} \circ x \qquad (x^{-1} \circ \mathsf{e}) \circ (x^{-1})^{-1}$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ (\mathsf{e} \circ (x^{-1})^{-1})$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ (\mathsf{e} \circ (x^{-1})^{-1})$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ ((x \circ x^{-1}) \circ (x^{-1})^{-1})$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ ((x \circ x^{-1}) \circ (x^{-1})^{-1})$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ (x \circ (x^{-1} \circ (x^{-1})^{-1}))$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ (x \circ (x^{-1} \circ (x^{-1})^{-1}))$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ (x \circ \mathsf{e})$$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$$x \circ x^{-1} \approx x^{-1} \circ x \qquad x^{-1} \circ (x \circ \mathsf{e})$$

Completion

Example (Group Theory)

$(x \circ y) \circ z \approx x \circ (y \circ z)$ $x \circ \mathsf{e} \approx x$ $x \circ x^{-1} \approx \mathsf{e}$

$x \circ x^{-1} \approx x^{-1} \circ x$ $x^{-1} \circ x$

Completion

Example (Group Theory)

$$(x \circ y) \circ z \approx x \circ (y \circ z) \qquad x \circ \mathsf{e} \approx x \qquad x \circ x^{-1} \approx \mathsf{e}$$

$x \circ x^{-1} \approx x^{-1} \circ x$ $\qquad x^{-1} \circ x$

	Equational Logic
proof search	hard
proof checking	easy

Completion

Example (Group Theory)

$$(x \circ y) \circ z \to x \circ (y \circ z) \qquad x \circ \mathsf{e} \to x \qquad x \circ x^{-1} \to \mathsf{e}$$

$x \circ x^{-1} \approx x^{-1} \circ x$

	Equational Logic	Completion
proof search	hard	
proof checking	easy	

Completion

Example (Group Theory)

$$(x \circ y) \circ z \to x \circ (y \circ z) \qquad x \circ \mathsf{e} \to x \qquad x \circ x^{-1} \to \mathsf{e}$$
$$\mathsf{e}^{-1} \to \mathsf{e} \qquad \mathsf{e} \circ x \to x \qquad x^{-1} \circ x \to \mathsf{e}$$
$$(x^{-1})^{-1} \to x \qquad x \circ (x^{-1} \circ z) \to z$$
$$(y \circ x)^{-1} \to x^{-1} \circ y^{-1} \qquad x^{-1} \circ (x \circ z) \to z$$

$$x \circ x^{-1} \approx x^{-1} \circ x$$

	Equational Logic	Completion
proof search	hard	
proof checking	easy	

Completion

Example (Group Theory)

$$(x \circ y) \circ z \to x \circ (y \circ z) \qquad x \circ e \to x \qquad x \circ x^{-1} \to e$$
$$e^{-1} \to e \qquad e \circ x \to x \qquad x^{-1} \circ x \to e$$
$$(x^{-1})^{-1} \to x \qquad x \circ (x^{-1} \circ z) \to z$$
$$(y \circ x)^{-1} \to x^{-1} \circ y^{-1} \qquad x^{-1} \circ (x \circ z) \to z$$

$x \circ x^{-1} \approx x^{-1} \circ x$

	Equational Logic	Completion
proof search	hard	easy
proof checking	easy	

Completion

Example (Group Theory)

$(x \circ y) \circ z \to x \circ (y \circ z)$	$x \circ \mathsf{e} \to x$	$x \circ x^{-1} \to \mathsf{e}$
$\mathsf{e}^{-1} \to \mathsf{e}$	$\mathsf{e} \circ x \to x$	$x^{-1} \circ x \to \mathsf{e}$
$(x^{-1})^{-1} \to x$	$x \circ (x^{-1} \circ z) \to z$	
$(y \circ x)^{-1} \to x^{-1} \circ y^{-1}$	$x^{-1} \circ (x \circ z) \to z$	

$x \circ x^{-1} \approx x^{-1} \circ x$

	Equational Logic	Completion
proof search	hard	easy
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Completion

Example (Group Theory)

$$(x \circ y) \circ z \to x \circ (y \circ z) \qquad x \circ \mathsf{e} \to x \qquad x \circ x^{-1} \to \mathsf{e}$$
$$\mathsf{e}^{-1} \to \mathsf{e} \qquad \mathsf{e} \circ x \to x \qquad x^{-1} \circ x \to \mathsf{e}$$
$$(x^{-1})^{-1} \to x \qquad x \circ (x^{-1} \circ z) \to z$$
$$(y \circ x)^{-1} \to x^{-1} \circ y^{-1} \qquad x^{-1} \circ (x \circ z) \to z$$

$x \circ x^{-1} \approx x^{-1} \circ x$

	Equational Logic	Completion	Recording Completion
proof search	hard	easy	
proof checking	easy	hard	

Completion

Example (Group Theory)

$$(x \circ y) \circ z \to x \circ (y \circ z) \qquad x \circ \mathsf{e} \to x \qquad x \circ x^{-1} \to \mathsf{e}$$

$$\mathsf{e}^{-1} \to \mathsf{e} \qquad \mathsf{e} \circ x \to x \qquad x^{-1} \circ x \to \mathsf{e}$$

$$(x^{-1})^{-1} \to x \qquad x \circ (x^{-1} \circ z) \to z$$

$$(y \circ x)^{-1} \to x^{-1} \circ y^{-1} \qquad x^{-1} \circ (x \circ z) \to z$$

$$x \circ x^{-1} \approx x^{-1} \circ x$$

	Equational Logic	Completion	Recording Completion
proof search	hard	easy	easy
proof checking	easy	hard	

Completion

Example (Group Theory)

$$(x \circ y) \circ z \to x \circ (y \circ z) \qquad x \circ \mathsf{e} \to x \qquad x \circ x^{-1} \to \mathsf{e}$$
$$\mathsf{e}^{-1} \to \mathsf{e} \qquad \mathsf{e} \circ x \to x \qquad x^{-1} \circ x \to \mathsf{e}$$
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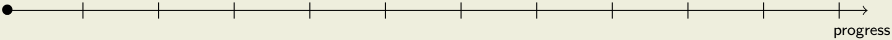
$x \circ x^{-1} \approx x^{-1} \circ x$

	Equational Logic	Completion	Recording Completion
proof search	hard	easy	easy
proof checking	easy	hard	easy

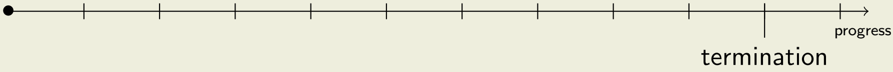
Conclusion

Contributions



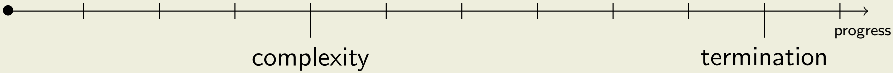
Conclusion

Contributions



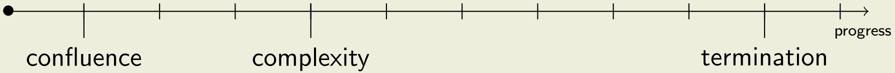
Conclusion

Contributions



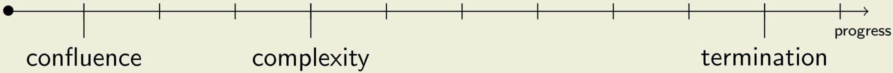
Conclusion

Contributions



Conclusion

Contributions

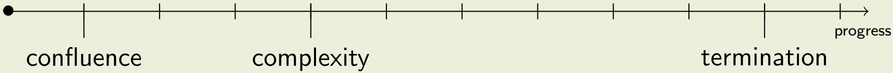


Achievements (FLoC Olympic Games)

- CeTA 1st place (CoCo – certification)
- CSI 2nd place (CoCo – confluence)
- $\mathsf{T_TT_2}$ 2nd place (TermComp – termination)
- CaT 3rd place (TermComp – complexity)

Conclusion

Contributions



Achievements (FLoC Olympic Games)

- CeTA 1st place (CoCo – certification)
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- CaT 3rd place (TermComp – complexity)

Future Directions

- real-world programs (C, Java, Haskell, . . .)
- safety-critical systems (Sternagel and Thiemann, START)

Included Publications

1. Harald Zankl, Bertram Felgenhauer, and Aart Middeldorp, *Labelings for Decreasing Diagrams*, Journal of Automated Reasoning 54(2), 101–133, 2015.
2. Harald Zankl, Confluence by Decreasing Diagrams – Formalized, 24th RTA, LIPIcs 21, pp. 352–367, 2013.
3. Harald Zankl and Martin Korp, *Modular Complexity Analysis for Term Rewriting*, Logical Methods in Computer Science 10(1:19), 33 pages, 2014.
4. Friedrich Neurauter, Harald Zankl, and Aart Middeldorp, *Revisiting Matrix Interpretations for Polynomial Derivational Complexity of Term Rewriting*, LPAR-17, LNCS (ARCoSS) 6397, pp. 550–564, 2010.
5. Nao Hirokawa, Aart Middeldorp, and Harald Zankl, *Uncurrying for Termination and Complexity*, Journal of Automated Reasoning 43(2), 279–315, 2013.
6. Harald Zankl, Sarah Winkler, and Aart Middeldorp, *Beyond Polynomials and Peano Arithmetic – Automation of Elementary and Ordinal Interpretations*, Journal of Symbolic Computation 69, 129–158, 2015.
7. Harald Zankl, Bertram Felgenhauer, and Aart Middeldorp, *CSI – A Confluence Tool*, 23rd CADE, LNAI 6803, pp. 499–505, 2011.
8. Thomas Sternagel and Harald Zankl, *KBCV – Knuth-Bendix Completion Visualizer*, 6th IJCAR, LNAI 7364, pp. 530–536, 2012.