Real Matrix Interpretations

Harald Zankl

Institute of Computer Science
University of Innsbruck
Austria

JAIST 19 August 2009

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0)))$

Example (Term Rewrite System)

$$\mathsf{half}(0) \to 0 \qquad \mathsf{bits}(0) \to 0$$
$$\mathsf{half}(\mathsf{s}(0)) \to 0 \qquad \mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$$
$$\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x))$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0)))))$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$
$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}}$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$
$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}}$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(\mathsf{0})))) \to_{\mathcal{R}}$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{0}))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{0}))) \to_{\mathcal{R}}$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} & \mathsf{bits}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(\mathsf{0})))) \to_{\mathcal{R}}$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{0}))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{0}))$$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} & \mathsf{bits}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(\mathsf{0})))) \to_{\mathcal{R}}$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{0}))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{0}))$$

Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$$

Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

Theorem

$\exists$ *reduction order* $>$ *s.t.* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}$ *terminating*

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &> 0 \\ \mathsf{half}(\mathsf{s}(0)) &> 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &> \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &> 0 \\ \mathsf{bits}(\mathsf{s}(x)) &> \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$$

Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

Theorem

$\exists$ *reduction order* $>$ *s.t.* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}$ *terminating*

Example (Term Rewrite System)

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) > \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) > \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) >$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) > \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) > \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) > \mathsf{s}(\mathsf{s}(0))$$

Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

Theorem

$\exists$ *reduction order* $>$ *s.t.* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}$ *terminating*

Definition (Defined Symbol)

$\mathcal{D}(\mathcal{R}) = \{root(l) \mid l \to r \in \mathcal{R}\}$

Definition (Defined Symbol)

$\mathcal{D}(\mathcal{R}) = \{root(l) \mid l \to r \in \mathcal{R}\}$

Definition (Dependency Pairs [AG00])

$\mathrm{DP}(\mathcal{R}) = \{l^\sharp \to u^\sharp \mid l \to r \in \mathcal{R}, r \trianglerighteq u, root(u) \text{ defined}\}$

Definition (Defined Symbol)

$\mathcal{D}(\mathcal{R}) = \{root(l) \mid l \to r \in \mathcal{R}\}$

Definition (Dependency Pairs [AG00])

$\mathrm{DP}(\mathcal{R}) = \{l^\sharp \to u^\sharp \mid l \to r \in \mathcal{R}, r \trianglerighteq u, root(u) \text{ defined}\}$

Example

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\begin{aligned} \mathsf{bits}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

Definition (Defined Symbol)

$\mathcal{D}(\mathcal{R}) = \{root(l) \mid l \to r \in \mathcal{R}\}$

Definition (Dependency Pairs [AG00])

$\mathrm{DP}(\mathcal{R}) = \{l^\sharp \to u^\sharp \mid l \to r \in \mathcal{R}, r \trianglerighteq u, root(u) \text{ defined}\}$

Example

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\begin{aligned} \mathsf{bits}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

Definition (Defined Symbol)

$\mathcal{D}(\mathcal{R}) = \{root(l) \mid l \to r \in \mathcal{R}\}$

Definition (Dependency Pairs [AG00])

$\mathrm{DP}(\mathcal{R}) = \{l^\sharp \to u^\sharp \mid l \to r \in \mathcal{R}, r \trianglerighteq u, root(u) \text{ defined}\}$

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

with DPs

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{half}^\sharp(x) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{half}^\sharp(\mathsf{s}(x)) \end{aligned}$$

Definition (Defined Symbol)

$\mathcal{D}(\mathcal{R}) = \{root(l) \mid l \to r \in \mathcal{R}\}$

Definition (Dependency Pairs [AG00])

$\mathsf{DP}(\mathcal{R}) = \{l^\sharp \to u^\sharp \mid l \to r \in \mathcal{R}, r \trianglerighteq u, root(u) \text{ defined}\}$

Example

$$\begin{aligned} \mathsf{half}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{0})) &\to \mathsf{0} \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

with DPs

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{half}^\sharp(x) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{half}^\sharp(\mathsf{s}(x)) \end{aligned}$$

Theorem (Termination)

$\mathcal{R}$ *terminating iff no sequence* $s_0 \to_{\mathsf{DP}(\mathcal{R})} t_0 \to^*_{\mathcal{R}} s_1 \to_{\mathsf{DP}(\mathcal{R})} t_1 \to^*_{\mathcal{R}} \cdots$

Definition

$(\gtrsim, >)$ is reduction pair if

- $>$ is rewrite-order, stable, well-founded
- $\gtrsim$ is rewrite-preorder, monotone, stable
- $\gtrsim \cdot > \cdot \gtrsim \subseteq >$

Theorem

if $\exists$ *reduction pair* $(\gtrsim, >)$ *with* $\mathcal{R} \subseteq \gtrsim$ *and* $\mathsf{DP}(\mathcal{R}) \subseteq >$ *then* $\mathcal{R}$ *terminating*

Definition

$(\gtrsim, >)$ is reduction pair if

- $>$ is rewrite-order, stable, well-founded
- $\gtrsim$ is rewrite-preorder, monotone, stable
- $\gtrsim \cdot > \cdot \gtrsim \subseteq >$

Theorem

if $\exists$ reduction pair $(\gtrsim, >)$ with $\mathcal{R} \subseteq \gtrsim$ and $\mathsf{DP}(\mathcal{R}) \subseteq >$ then $\mathcal{R}$ terminating

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

with DPs

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{half}^\sharp(x) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{half}^\sharp(\mathsf{s}(x)) \end{aligned}$$

Definition

$(\gtrsim, >)$ is reduction pair if

- $>$ is rewrite-order, stable, well-founded
- $\gtrsim$ is rewrite-preorder, monotone, stable
- $\gtrsim \cdot > \cdot \gtrsim \subseteq >$

Theorem

if $\exists$ reduction pair $(\gtrsim, >)$ with $\mathcal{R} \subseteq \gtrsim$ and $\mathsf{DP}(\mathcal{R}) \subseteq >$ then $\mathcal{R}$ terminating

Example

$$\begin{aligned} \mathsf{half}(0) &\gtrsim 0 \\ \mathsf{half}(\mathsf{s}(0)) &\gtrsim 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\gtrsim \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\gtrsim 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\gtrsim \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

with DPs

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &> \mathsf{half}^\sharp(x) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &> \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &> \mathsf{half}^\sharp(\mathsf{s}(x)) \end{aligned}$$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall\alpha\, [s, \alpha]_A \geqslant [t, \alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t$
- $\forall\alpha\, [s, \alpha]_A > [t, \alpha]_A \rightsquigarrow s >_{\mathcal{A}} t$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall \alpha\, [s, \alpha]_A \geqslant [t, \alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t$
- $\forall \alpha\, [s, \alpha]_A > [t, \alpha]_A \rightsquigarrow s >_{\mathcal{A}} t$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)\ \forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall\alpha\, [s,\alpha]_A \geqslant [t,\alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t$
- $\forall\alpha\, [s,\alpha]_A > [t,\alpha]_A \rightsquigarrow s >_{\mathcal{A}} t$

Theorem

weakly-monotone algebra $(A, [\cdot], \geqslant, >) \longrightarrow (\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ *reduction pair*

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall \alpha \, [s, \alpha]_A \geqslant [t, \alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t$
- $\forall \alpha \, [s, \alpha]_A > [t, \alpha]_A \rightsquigarrow s >_{\mathcal{A}} t$

Theorem

weakly-monotone algebra $(A, [\cdot], \geqslant, >) \longrightarrow (\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ *reduction pair*

Linear Interpretations

- $f_A(x_0, \ldots, x_n) = f_0 x_0 + \cdots + f_n x_n + f_{n+1}$ linear polynomial $\forall f$
- $[s, \alpha]_A = s_0 \alpha(x_0) + \cdots + s_k \alpha(x_k) + s_{k+1}$
- $[t, \alpha]_A = t_0 \alpha(x_0) + \cdots + t_k \alpha(x_k) + t_{k+1}$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall \alpha\, [s, \alpha]_A \geqslant [t, \alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t$
- $\forall \alpha\, [s, \alpha]_A > [t, \alpha]_A \rightsquigarrow s >_{\mathcal{A}} t$

Theorem

weakly-monotone algebra $(A, [\cdot], \geqslant, >) \longrightarrow (\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ *reduction pair*

Linear Interpretations

- $f_A(x_0, \ldots, x_n) = f_0 x_0 + \cdots + f_n x_n + f_{n+1}$ linear polynomial $\forall f$
- $[s, \alpha]_A = s_0 \alpha(x_0) + \cdots + s_k \alpha(x_k) + s_{k+1}$
- $[t, \alpha]_A = t_0 \alpha(x_0) + \cdots + t_k \alpha(x_k) + t_{k+1}$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall \alpha\, [s, \alpha]_A \geqslant [t, \alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t \leftsquigarrow s_i \geqslant_? t_i$ $(1 \leqslant i \leqslant k+1)$
- $\forall \alpha\, [s, \alpha]_A > [t, \alpha]_A \rightsquigarrow s >_{\mathcal{A}} t$

Theorem

weakly-monotone algebra $(A, [\cdot], \geqslant, >) \longrightarrow (\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ *reduction pair*

Linear Interpretations

- $f_A(x_0, \ldots, x_n) = f_0 x_0 + \cdots + f_n x_n + f_{n+1}$ linear polynomial $\forall f$
- $[s, \alpha]_A = s_0 \alpha(x_0) + \cdots + s_k \alpha(x_k) + s_{k+1}$
- $[t, \alpha]_A = t_0 \alpha(x_0) + \cdots + t_k \alpha(x_k) + t_{k+1}$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall \alpha\, [s, \alpha]_A \geqslant [t, \alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t \leftsquigarrow s_i \geqslant_? t_i$ $(1 \leqslant i \leqslant k+1)$
- $\forall \alpha\, [s, \alpha]_A > [t, \alpha]_A \rightsquigarrow s >_{\mathcal{A}} t \leftsquigarrow s_i \geqslant_? t_i$ $(1 \leqslant i \leqslant k)$, $s_{k+1} >_? t_{k+1}$

Theorem

weakly-monotone algebra $(A, [\cdot], \geqslant, >) \longrightarrow (\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ *reduction pair*

Linear Interpretations

- $f_A(x_0, \ldots, x_n) = f_0 x_0 + \cdots + f_n x_n + f_{n+1}$ linear polynomial $\forall f$
- $[s, \alpha]_A = s_0 \alpha(x_0) + \cdots + s_k \alpha(x_k) + s_{k+1}$
- $[t, \alpha]_A = t_0 \alpha(x_0) + \cdots + t_k \alpha(x_k) + t_{k+1}$

Weakly-Monotone Algebra

$(A, [\cdot], \geqslant, >)$ with non-empty algebra $(A, [\cdot])$ and

- if $a_i \geqslant b$ then $f_A(a_1, \ldots, a_i, \ldots, a_n) \geqslant f_A(a_1, \ldots, b, \ldots, a_n)$ $\forall f$
- $>$ well-founded
- $\geqslant \cdot > \subseteq >$

Definition

- $\forall \alpha\, [s, \alpha]_A \geqslant [t, \alpha]_A \rightsquigarrow s \geqslant_{\mathcal{A}} t \leftsquigarrow s_i \geqslant_? t_i \ (1 \leqslant i \leqslant k+1)$
- $\forall \alpha\, [s, \alpha]_A > [t, \alpha]_A \rightsquigarrow s >_{\mathcal{A}} t \leftsquigarrow s_i \geqslant_? t_i \ (1 \leqslant i \leqslant k)$, $s_{k+1} >_? t_{k+1}$

Theorem

weakly-monotone algebra $(A, [\cdot], \geqslant, >) \longrightarrow (\geqslant_{\mathcal{A}}, >_{\mathcal{A}})$ *reduction pair*

Linear Interpretations

- $f_A(x_0, \ldots, x_n) = f_0 x_0 + \cdots + f_n x_n + f_{n+1}$ linear polynomial $\forall f$
- $[s, \alpha]_A = s_0 \alpha(x_0) + \cdots + s_k \alpha(x_k) + s_{k+1}$
- $[t, \alpha]_A = t_0 \alpha(x_0) + \cdots + t_k \alpha(x_k) + t_{k+1}$

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{half}^\sharp(x) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{half}^\sharp(\mathsf{s}(x)) \end{aligned}$$

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{half}^\sharp(x) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{half}^\sharp(\mathsf{s}(x)) \end{aligned}$$

(almost weakly monotone) algebra $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\begin{aligned} \mathsf{half}^\sharp(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{half}^\sharp(x) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x))) \\ \mathsf{bits}^\sharp(\mathsf{s}(x)) &\to \mathsf{half}^\sharp(\mathsf{s}(x)) \end{aligned}$$

(almost weakly monotone) algebra $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ with (weakly monotone) interpretations

$$0_{\mathbb{R}} = 0 \qquad \mathsf{s}_{\mathbb{R}}(x) = x + 1 \qquad \mathsf{half}_{\mathbb{R}}(x) = 1/2x \qquad \mathsf{bits}_{\mathbb{R}}(x) = 4x$$

$$\mathsf{bits}^\sharp_{\mathbb{R}}(x) = 2x + 1 \qquad \mathsf{half}^\sharp_{\mathbb{R}}(x) = x$$

Example

$$\begin{aligned} 0 &\geqslant_{\mathbb{R}} 0 \\ 1/2 &\geqslant_{\mathbb{R}} 0 \\ 1/2x+1 &\geqslant_{\mathbb{R}} 1/2x+1 \quad \forall x \end{aligned} \qquad \begin{aligned} 0 &\geqslant_{\mathbb{R}} 0 \\ 4x+4 &\geqslant_{\mathbb{R}} 2x+3 \quad \forall x \end{aligned}$$

$$\begin{aligned} x+2 &>_{\mathbb{R}} x && \forall x \\ 2x+4 &>_{\mathbb{R}} x+2 && \forall x \\ 2x+3 &>_{\mathbb{R}} x+1 && \forall x \end{aligned}$$

(almost weakly monotone) algebra $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ with (weakly monotone) interpretations

$$0_{\mathbb{R}} = 0 \qquad \mathsf{s}_{\mathbb{R}}(x) = x+1 \qquad \mathsf{half}_{\mathbb{R}}(x) = 1/2x \qquad \mathsf{bits}_{\mathbb{R}}(x) = 4x$$

$$\mathsf{bits}^{\sharp}_{\mathbb{R}}(x) = 2x+1 \qquad \mathsf{half}^{\sharp}_{\mathbb{R}}(x) = x$$

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Problem

$>_{\mathbb{R}}$ is not well-founded, e.g., $\frac{1}{2} >_{\mathbb{R}} \frac{1}{3} >_{\mathbb{R}} \frac{1}{4} >_{\mathbb{R}} \cdots$

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Problem

$>_{\mathbb{R}}$ is not well-founded, e.g., $\frac{1}{2} >_{\mathbb{R}} \frac{1}{3} >_{\mathbb{R}} \frac{1}{4} >_{\mathbb{R}} \cdots$

Solution

$a >^{\delta}_{\mathbb{R}} b :\Longleftrightarrow a - b \geqslant_{\mathbb{R}} \delta$

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Problem

$>_{\mathbb{R}}$ is not well-founded, e.g., $\frac{1}{2} >_{\mathbb{R}} \frac{1}{3} >_{\mathbb{R}} \frac{1}{4} >_{\mathbb{R}} \cdots$

Solution

$a >_{\mathbb{R}}^{\delta} b :\Longleftrightarrow a - b \geqslant_{\mathbb{R}} \delta$

Lemma

for any $\delta \in \mathbb{R}^{>0}$ the order $>_{\mathbb{R}}^{\delta}$ is well-founded (on $\mathbb{R}^{\geqslant 0}$)

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Problem

$>_{\mathbb{R}}$ is not well-founded, e.g., $\frac{1}{2} >_{\mathbb{R}} \frac{1}{3} >_{\mathbb{R}} \frac{1}{4} >_{\mathbb{R}} \cdots$

Solution

$a >_{\mathbb{R}}^{\delta} b :\Longleftrightarrow a - b \geqslant_{\mathbb{R}} \delta$

Lemma

for any $\delta \in \mathbb{R}^{>0}$ the order $>_{\mathbb{R}}^{\delta}$ is well-founded (on $\mathbb{R}^{\geqslant 0}$)

Problem: how to find *suitable* δ

$\frac{3}{8} \not>_{\mathbb{R}}^{\frac{1}{2}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \not\geqslant_{\mathbb{R}} \frac{1}{2}$

$\frac{3}{8} >_{\mathbb{R}}^{\frac{1}{8}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \geqslant_{\mathbb{R}} \frac{1}{8}$

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Problem

$>_{\mathbb{R}}$ is not well-founded, e.g., $\frac{1}{2} >_{\mathbb{R}} \frac{1}{3} >_{\mathbb{R}} \frac{1}{4} >_{\mathbb{R}} \cdots$

Solution

$a >^{\delta}_{\mathbb{R}} b :\Longleftrightarrow a - b \geqslant_{\mathbb{R}} \delta$

Lemma

for any $\delta \in \mathbb{R}^{>0}$ the order $>^{\delta}_{\mathbb{R}}$ is well-founded (on $\mathbb{R}^{\geqslant 0}$)

Problem: how to find *suitable* δ

$\frac{3}{8} \not>^{\frac{1}{2}}_{\mathbb{R}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \not\geqslant_{\mathbb{R}} \frac{1}{2}$

$\frac{3}{8} >^{\frac{1}{8}}_{\mathbb{R}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \geqslant_{\mathbb{R}} \frac{1}{8}$

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Problem

$>_\mathbb{R}$ is not well-founded, e.g., $\frac{1}{2} >_\mathbb{R} \frac{1}{3} >_\mathbb{R} \frac{1}{4} >_\mathbb{R} \cdots$

Solution

$a >_\mathbb{R}^\delta b :\Longleftrightarrow a - b \geqslant_\mathbb{R} \delta$

Lemma

for any $\delta \in \mathbb{R}^{>0}$ the order $>_\mathbb{R}^\delta$ is well-founded (on $\mathbb{R}^{\geqslant 0}$)

Problem: how to find *suitable* δ

$\frac{3}{8} \not>_\mathbb{R}^{\frac{1}{2}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \not\geqslant_\mathbb{R} \frac{1}{2}$

$\frac{3}{8} >_\mathbb{R}^{\frac{1}{8}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \geqslant_\mathbb{R} \frac{1}{8}$

Remarks

- finding δ is no problem in practice (finitely many comparisons)
- in sequel $>_\mathbb{R}$ is $>_\mathbb{R}^\delta$ for suitable δ

Convention

in the sequel $\mathbb{R}$ means $\mathbb{R}^{\geqslant 0}$

Problem

$>_{\mathbb{R}}$ is not well-founded, e.g., $\frac{1}{2} >_{\mathbb{R}} \frac{1}{3} >_{\mathbb{R}} \frac{1}{4} >_{\mathbb{R}} \cdots$

Solution

$a >^{\delta}_{\mathbb{R}} b :\Longleftrightarrow a - b \geqslant_{\mathbb{R}} \delta$

Lemma

for any $\delta \in \mathbb{R}^{>0}$ the order $>^{\delta}_{\mathbb{R}}$ is well-founded (on $\mathbb{R}^{\geqslant 0}$)

Problem: how to find *suitable* δ

$\frac{3}{8} \not>^{\frac{1}{2}}_{\mathbb{R}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \not\geqslant_{\mathbb{R}} \frac{1}{2}$

$\frac{3}{8} >^{\frac{1}{8}}_{\mathbb{R}} \frac{1}{8}$ since $\frac{3}{8} - \frac{1}{8} = \frac{2}{8} \geqslant_{\mathbb{R}} \frac{1}{8}$

Remarks

- finding δ is no problem in practice (finitely many comparisons)
- in sequel $>_{\mathbb{R}}$ is $>^{\delta}_{\mathbb{R}}$ for suitable δ

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \dots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Lemma

$[s, \alpha]_{\mathbb{R}^d}$ $>_{\mathbb{R}^d}$ $[t, \alpha]_{\mathbb{R}^d}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Lemma

$[s, \alpha]_{\mathbb{R}^d} = S_1\alpha(x_1) + \cdots + S_k\alpha(x_k) + \vec{s} >_{\mathbb{R}^d}$ $[t, \alpha]_{\mathbb{R}^d}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Lemma

$[s, \alpha]_{\mathbb{R}^d} = S_1\alpha(x_1) + \cdots + S_k\alpha(x_k) + \vec{s} >_{\mathbb{R}^d} T_1\alpha(x_1) + \cdots + T_k\alpha(x_k) + \vec{t} = [t, \alpha]_{\mathbb{R}^d}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Lemma

$[s, \alpha]_{\mathbb{R}^d} = S_1\alpha(x_1) + \cdots + S_k\alpha(x_k) + \vec{s} >_{\mathbb{R}^d} T_1\alpha(x_1) + \cdots + T_k\alpha(x_k) + \vec{t} = [t, \alpha]_{\mathbb{R}^d}$
iff $S_i \geqslant_? T_i$ $(1 \leqslant i \leqslant k)$ *and* $\vec{s} >_? \vec{t}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Definition

for matrices (and vectors) $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$

$$A \geqslant_? B :\Longleftrightarrow a_{ij} \geqslant_{\mathbb{R}} b_{ij} \; (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m)$$

$$A >_? B :\Longleftrightarrow a_{11} >_{\mathbb{R}} b_{11} \text{ and } A \geqslant_? B$$

Lemma

$[s, \alpha]_{\mathbb{R}^d} = S_1\alpha(x_1) + \cdots + S_k\alpha(x_k) + \vec{s} >_{\mathbb{R}^d} T_1\alpha(x_1) + \cdots + T_k\alpha(x_k) + \vec{t} = [t, \alpha]_{\mathbb{R}^d}$
iff $S_i \geqslant_? T_i$ $(1 \leqslant i \leqslant k)$ *and* $\vec{s} >_? \vec{t}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Definition

for matrices (and vectors) $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$

$$A \geqslant_? B :\Longleftrightarrow a_{ij} \geqslant_{\mathbb{R}} b_{ij} \; (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m)$$

$$A >_? B :\Longleftrightarrow a_{11} >_{\mathbb{R}} b_{11} \text{ and } A \geqslant_? B$$

Lemma

$[s, \alpha]_{\mathbb{R}^d} = S_1\alpha(x_1) + \cdots + S_k\alpha(x_k) + \vec{s} >_{\mathbb{R}^d} T_1\alpha(x_1) + \cdots + T_k\alpha(x_k) + \vec{t} = [t, \alpha]_{\mathbb{R}^d}$
iff $S_i \geqslant_? T_i \; (1 \leqslant i \leqslant k)$ *and* $\vec{s} >_? \vec{t}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Definition

for matrices (and vectors) $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$

$$A \geqslant_{\mathbb{R}^d} B :\Longleftrightarrow a_{ij} \geqslant_{\mathbb{R}} b_{ij} \ (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m)$$
$$A >_{\mathbb{R}^d} B :\Longleftrightarrow a_{11} >_{\mathbb{R}} b_{11} \text{ and } A \geqslant_{?} B$$

Lemma

$[s, \alpha]_{\mathbb{R}^d} = S_1\alpha(x_1) + \cdots + S_k\alpha(x_k) + \vec{s} >_{\mathbb{R}^d} T_1\alpha(x_1) + \cdots + T_k\alpha(x_k) + \vec{t} = [t, \alpha]_{\mathbb{R}^d}$
iff $S_i \geqslant_{?} T_i$ $(1 \leqslant i \leqslant k)$ *and* $\vec{s} >_{?} \vec{t}$

Matrix Interpretations [HW06, EWZ08]

before: $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$ wish: $(\mathbb{R}^d, [\cdot], \geqslant_{\mathbb{R}^d}, >_{\mathbb{R}^d})$ for $d \in \mathbb{N}^{>0}$

Interpretations

$f_{\mathbb{R}^d}(\vec{x}_1, \ldots, \vec{x}_n) = F_1\vec{x}_1 + \cdots + F_n\vec{x}_n + \vec{f}$ with $F_i \in \mathbb{R}^{d \times d}$ and $f \in \mathbb{R}^d$

Definition

for matrices (and vectors) $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$

$$A \geqslant_{\mathbb{R}^d} B :\Longleftrightarrow a_{ij} \geqslant_{\mathbb{R}} b_{ij} \ (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m)$$
$$A >_{\mathbb{R}^d} B :\Longleftrightarrow a_{11} >_{\mathbb{R}} b_{11} \text{ and } A \geqslant_{?} B$$

Lemma

$[s, \alpha]_{\mathbb{R}^d} = S_1\alpha(x_1) + \cdots + S_k\alpha(x_k) + \vec{s} >_{\mathbb{R}^d} T_1\alpha(x_1) + \cdots + T_k\alpha(x_k) + \vec{t} = [t, \alpha]_{\mathbb{R}^d}$
iff $S_i \geqslant_? T_i$ $(1 \leqslant i \leqslant k)$ *and* $\vec{s} >_? \vec{t}$

Theorem

if $A = \mathbb{R}^{\geqslant 0}$, $d \in \mathbb{N}^{>0}$, $\delta \in \mathbb{R}^{>0}$, *and every interpretation as above then* $(A^d, [\cdot], \geqslant_{A^d}, >^{\delta}_{A^d})$ *is a weakly monotone algebra*

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

yields $2x + 2 >_\mathbb{R} x + 1$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

yields $2x + 2 >_\mathbb{R} x + 1$

Dimension 2

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}^2, [\cdot], \geqslant_{\mathbb{R}^2}, >_{\mathbb{R}^2})$

$$\mathsf{bits}^\sharp_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{half}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{s}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_{\mathbb{R}}, >_{\mathbb{R}})$

$$\mathsf{bits}^\sharp_{\mathbb{R}}(x) = 2x \qquad \mathsf{half}_{\mathbb{R}}(x) = 1/2x \qquad \mathsf{s}_{\mathbb{R}}(x) = x + 1$$

yields $2x + 2 >_{\mathbb{R}} x + 1$

Dimension 2

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}^2, [\cdot], \geqslant_{\mathbb{R}^2}, >_{\mathbb{R}^2})$

$$\mathsf{bits}^\sharp_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{half}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{s}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) > \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) \right)$$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

yields $2x + 2 >_\mathbb{R} x + 1$

Dimension 2

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}^2, [\cdot], \geqslant_{\mathbb{R}^2}, >_{\mathbb{R}^2})$

$$\mathsf{bits}^\sharp_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{half}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{s}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

yields

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad > \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

yields $2x + 2 >_\mathbb{R} x + 1$

Dimension 2

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}^2, [\cdot], \geqslant_{\mathbb{R}^2}, >_{\mathbb{R}^2})$

$$\mathsf{bits}^\sharp_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{half}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{s}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

yields

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad > \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

yields $2x + 2 >_\mathbb{R} x + 1$

Dimension 2

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}^2, [\cdot], \geqslant_{\mathbb{R}^2}, >_{\mathbb{R}^2})$

$$\mathsf{bits}^\sharp_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{half}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{s}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

yields

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad > \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e.

$$0 \geqslant_\mathbb{R} 0 \;\wedge\; 1 \geqslant_\mathbb{R} 1 \;\wedge\; 0 \geqslant_\mathbb{R} 0 \;\wedge\; 1 \geqslant_\mathbb{R} 1 \;\wedge\; 2 >_\mathbb{R} 1 \;\wedge\; 1 \geqslant_\mathbb{R} 1$$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

yields $2x + 2 >_\mathbb{R} x + 1$

Dimension 2

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}^2, [\cdot], \geqslant_{\mathbb{R}^2}, >_{\mathbb{R}^2})$

$$\mathsf{bits}^\sharp_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{half}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} \quad \mathsf{s}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

yields

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad > \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e.

$$0 \geqslant_\mathbb{R} 0 \ \wedge \ 1 \geqslant_\mathbb{R} 1 \ \wedge \ 0 \geqslant_\mathbb{R} 0 \ \wedge \ 1 \geqslant_\mathbb{R} 1 \ \wedge \ 2 >_\mathbb{R} 1 \ \wedge \ 1 \geqslant_\mathbb{R} 1$$

Dimension 1

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}, [\cdot], \geqslant_\mathbb{R}, >_\mathbb{R})$

$$\mathsf{bits}^\sharp_\mathbb{R}(x) = 2x \qquad \mathsf{half}_\mathbb{R}(x) = 1/2x \qquad \mathsf{s}_\mathbb{R}(x) = x + 1$$

yields $2x + 2 >_\mathbb{R} x + 1$

Dimension 2

$\mathsf{bits}^\sharp(\mathsf{s}(x)) \to \mathsf{bits}^\sharp(\mathsf{half}(\mathsf{s}(x)))$ with algebra $(\mathbb{R}^2, [\cdot], \geqslant_{\mathbb{R}^2}, >_{\mathbb{R}^2})$

$$\mathsf{bits}^\sharp_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \vec{x} \quad \mathsf{half}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \vec{x} \quad \mathsf{s}_{\mathbb{R}^2}(\vec{x}) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \vec{x} + \begin{pmatrix} ? \\ ? \end{pmatrix}$$

yields

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \vec{x} + \begin{pmatrix} ? \\ ? \end{pmatrix} \quad > \quad \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \vec{x} + \begin{pmatrix} ? \\ ? \end{pmatrix}$$

i.e.

$$? \geqslant_\mathbb{R} ? \;\wedge\; ? \geqslant_\mathbb{R} ? \;\wedge\; ? \geqslant_\mathbb{R} ? \;\wedge\; ? \geqslant_\mathbb{R} ? \;\wedge\; ? >_\mathbb{R} ? \;\wedge\; ? \geqslant_\mathbb{R} ?$$

Automation

Problem

find coefficients such that constraints satisfied

Automation

Problem

find coefficients such that constraints satisfied

Solution

- fix dimension d
- fix maximal coefficients
- finite search space $\longrightarrow$ SAT

Automation

Problem

find coefficients such that constraints satisfied

Solution

- fix dimension d
- fix maximal coefficients
- finite search space $\longrightarrow$ SAT

Automation

Problem

find coefficients such that constraints satisfied

Solution

- fix dimension d
- fix maximal coefficients
- finite search space $\longrightarrow$ SAT

Automation

Problem

find coefficients such that constraints satisfied

Solution

- fix dimension d
- fix maximal coefficients
- finite search space $\longrightarrow$ SAT

Goal

How to represent arithmetic in SAT for coefficients in

- $\mathbb{N}$
- $\mathbb{Q}$
- $\mathbb{R}$

Arithmetic over $\mathbb{N}$

$\vec{a}_k = \langle a_k, \ldots, a_1 \rangle$ is binary representation of $a < 2^k$, i.e.,
$\langle \top, \bot, \top, \top \rangle \equiv 2^3 + 2^1 + 2^0 = 11$

Arithmetic over $\mathbb{N}$

$\vec{a}_k = \langle a_k, \ldots, a_1 \rangle$ is binary representation of $a < 2^k$, i.e.,
$\langle \top, \bot, \top, \top \rangle \equiv 2^3 + 2^1 + 2^0 = 11$

Definition

$$\vec{a}_k >_{\mathbf{N}} \vec{b}_k = \begin{cases} a_1 \wedge \neg b_1 & \text{if } k = 1 \\ (a_k \wedge \neg b_k) \vee \big((a_k \leftrightarrow b_k) \wedge \vec{a}_{k-1} >_{\mathbf{N}} \vec{b}_{k-1}\big) & \text{if } k > 1 \end{cases}$$

$$\vec{a}_k =_{\mathbf{N}} \vec{b}_k = \bigwedge_{i=1}^{k} (a_i \leftrightarrow b_i)$$

$$\vec{a}_k +_{\mathbf{N}} \vec{b}_k = \langle c_k, s_k, \ldots, s_1 \rangle$$

$$\text{for } 1 \leqslant i \leqslant k \text{ with } \begin{aligned} c_0 &= \bot \\ s_i &= a_i \oplus b_i \oplus c_{i-1} \\ c_i &= (a_i \wedge b_i) \vee (a_i \wedge c_{i-1}) \vee (b_i \wedge c_{i-1}) \end{aligned}$$

$\oplus$ is exclusive or

Arithmetic over $\mathbb{N}$

$\vec{a}_k = \langle a_k, \ldots, a_1 \rangle$ is binary representation of $a < 2^k$, i.e.,
$\langle \top, \bot, \top, \top \rangle \equiv 2^3 + 2^1 + 2^0 = 11$

Definition

$$\vec{a}_k >_{\mathbf{N}} \vec{b}_k = \begin{cases} a_1 \wedge \neg b_1 & \text{if } k = 1 \\ (a_k \wedge \neg b_k) \vee \big((a_k \leftrightarrow b_k) \wedge \vec{a}_{k-1} >_{\mathbf{N}} \vec{b}_{k-1}\big) & \text{if } k > 1 \end{cases}$$

$$\vec{a}_k =_{\mathbf{N}} \vec{b}_k = \bigwedge_{i=1}^{k} (a_i \leftrightarrow b_i)$$

$$\vec{a}_k +_{\mathbf{N}} \vec{b}_k = \langle c_k, s_k, \ldots, s_1 \rangle$$

$$\text{for } 1 \leqslant i \leqslant k \text{ with } \begin{aligned} c_0 &= \bot \\ s_i &= a_i \oplus b_i \oplus c_{i-1} \\ c_i &= (a_i \wedge b_i) \vee (a_i \wedge c_{i-1}) \vee (b_i \wedge c_{i-1}) \end{aligned}$$

$\oplus$ is exclusive or

Arithmetic over $\mathbb{N}$

$\vec{a}_k = \langle a_k, \ldots, a_1 \rangle$ is binary representation of $a < 2^k$, i.e.,
$\langle \top, \bot, \top, \top \rangle \equiv 2^3 + 2^1 + 2^0 = 11$

Definition

$$\vec{a}_k >_{\mathbf{N}} \vec{b}_k = \begin{cases} a_1 \wedge \neg b_1 & \text{if } k = 1 \\ (a_k \wedge \neg b_k) \vee \big((a_k \leftrightarrow b_k) \wedge \vec{a}_{k-1} >_{\mathbf{N}} \vec{b}_{k-1}\big) & \text{if } k > 1 \end{cases}$$

$$\vec{a}_k =_{\mathbf{N}} \vec{b}_k = \bigwedge_{i=1}^{k} (a_i \leftrightarrow b_i)$$

$$\vec{a}_k +_{\mathbf{N}} \vec{b}_k = \langle c_k, s_k, \ldots, s_1 \rangle$$

$$\text{for } 1 \leqslant i \leqslant k \text{ with } \quad \begin{aligned} c_0 &= \bot \\ s_i &= a_i \oplus b_i \oplus c_{i-1} \\ c_i &= (a_i \wedge b_i) \vee (a_i \wedge c_{i-1}) \vee (b_i \wedge c_{i-1}) \end{aligned}$$

$\oplus$ is exclusive or

Arithmetic over $\mathbb{N}$ (2)

Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1 \rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1 \rangle$$

$$\langle a_3, a_2, a_1 \rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot \rangle$$

Arithmetic over $\mathbb{N}$ (2)

Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1 \rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1 \rangle$$

$$\langle a_3, a_2, a_1 \rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot \rangle$$

Arithmetic over $\mathbb{N}$ (2)

Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1 \rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1 \rangle$$
$$\langle a_3, a_2, a_1 \rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot \rangle$$

Definition

$$\vec{a}_m \times_{\mathbf{N}} \vec{b}_n \quad = \quad \big((\vec{a}_m \cdot b_1 \ll 0) +_{\mathbf{N}} \cdots +_{\mathbf{N}} (\vec{a}_m \cdot b_n \ll (n-1))\big)_{m+n}$$

Arithmetic over $\mathbb{N}$ (2)

Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1 \rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1 \rangle$$
$$\langle a_3, a_2, a_1 \rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot \rangle$$

Definition

$$\vec{a}_m \times_{\mathbf{N}} \vec{b}_n \quad = \quad \left((\vec{a}_m \cdot b_1 \ll 0) +_{\mathbf{N}} \cdots +_{\mathbf{N}} (\vec{a}_m \cdot b_n \ll (n-1))\right)_{m+n}$$

Example

$$\langle \top, \top \rangle \ +_{\mathbf{N}} \ \langle \top, \bot, \top \rangle = \langle \top, \bot, \bot, \bot \rangle \qquad 3 + 5 = 8$$
$$\langle \top, \top \rangle \ \times_{\mathbf{N}} \ \langle \top, \bot, \top \rangle = \langle \bot, \top, \top, \top, \top \rangle \qquad 3 \times 5 = 15$$
$$\langle \bot, \top, \top, \top, \top \rangle \ \times_{\mathbf{N}} \ \langle \bot, \bot, \top \rangle = \langle \bot, \bot, \bot, \bot, \top, \top, \top, \top \rangle \qquad 15 \times 1 = 15$$

Arithmetic over $\mathbb{N}$ (2)

Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1 \rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1 \rangle$$
$$\langle a_3, a_2, a_1 \rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot \rangle$$

Definition

$$\vec{a}_m \times_{\mathbf{N}} \vec{b}_n \quad = \quad \left((\vec{a}_m \cdot b_1 \ll 0) +_{\mathbf{N}} \cdots +_{\mathbf{N}} (\vec{a}_m \cdot b_n \ll (n-1))\right)_{m+n}$$

Example

$$\langle \top, \top \rangle \;+_{\mathbf{N}}\; \langle \top, \bot, \top \rangle = \langle \top, \bot, \bot, \bot \rangle \qquad 3 + 5 = 8$$
$$\langle \top, \top \rangle \;\times_{\mathbf{N}}\; \langle \top, \bot, \top \rangle = \langle \bot, \top, \top, \top, \top \rangle \qquad 3 \times 5 = 15$$
$$\langle \bot, \top, \top, \top, \top \rangle \;\times_{\mathbf{N}}\; \langle \bot, \bot, \top \rangle = \langle \bot, \bot, \bot, \bot, \top, \top, \top, \top \rangle \qquad 15 \times 1 = 15$$

Arithmetic over $\mathbb{N}$ (2)

Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1 \rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1 \rangle$$
$$\langle a_3, a_2, a_1 \rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot \rangle$$

Definition

$$\vec{a}_m \times_{\mathbf{N}} \vec{b}_n \quad = \quad \left((\vec{a}_m \cdot b_1 \ll 0) +_{\mathbf{N}} \cdots +_{\mathbf{N}} (\vec{a}_m \cdot b_n \ll (n-1))\right)_{m+n}$$

Example

$$\langle \top, \top \rangle \;+_{\mathbf{N}}\; \langle \top, \bot, \top \rangle = \langle \top, \bot, \bot, \bot \rangle \qquad 3 + 5 = 8$$
$$\langle \top, \top \rangle \;\times_{\mathbf{N}}\; \langle \top, \bot, \top \rangle = \langle \bot, \top, \top, \top, \top \rangle \qquad 3 \times 5 = 15$$
$$\langle \bot, \top, \top, \top, \top \rangle \;\times_{\mathbf{N}}\; \langle \bot, \bot, \top \rangle = \langle \bot, \bot, \bot, \bot, \top, \top, \top, \top \rangle \qquad 15 \times 1 = 15$$

Arithmetic over $\mathbb{N}$ (2)

Scalar Multiplication and Shifting

$$\langle a_3, a_2, a_1\rangle \cdot x = \langle x \wedge a_3, x \wedge a_2, x \wedge a_1\rangle$$
$$\langle a_3, a_2, a_1\rangle \ll 3 = \langle a_3, a_2, a_1, \bot, \bot, \bot\rangle$$

Definition

$$\vec{a}_m \times_{\mathbf{N}} \vec{b}_n \quad = \quad \left((\vec{a}_m \cdot b_1 \ll 0) +_{\mathbf{N}} \cdots +_{\mathbf{N}} (\vec{a}_m \cdot b_n \ll (n-1))\right)_{m+n}$$

Example

$$\langle \top, \top\rangle \ +_{\mathbf{N}} \ \langle \top, \bot, \top\rangle = \langle \top, \bot, \bot, \bot\rangle \qquad 3 + 5 = 8$$
$$\langle \top, \top\rangle \ \times_{\mathbf{N}} \ \langle \top, \bot, \top\rangle = \langle \bot, \top, \top, \top, \top\rangle \qquad 3 \times 5 = 15$$
$$\langle \bot, \top, \top, \top, \top\rangle \ \times_{\mathbf{N}} \ \langle \bot, \bot, \top\rangle = \langle \bot, \bot, \bot, \bot, \top, \top, \top, \top\rangle \qquad 15 \times 1 = 15$$

Remark

restrict bit-width using side-constraints

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in **Q** with $\vec{a}$ from **N** and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in **Q** with $\vec{a}$ from **N** and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Definition

$$\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &= (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}$$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in **Q** with $\vec{a}$ from **N** and $d \in \mathbb{N}^{>0}$, e.g., $(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Definition

$$
\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &= (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}
$$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in **Q** with $\vec{a}$ from **N** and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Definition

$$
\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &= (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}
$$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in **Q** with $\vec{a}$ from **N** and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle\top, \bot, \top, \top\rangle, 2) \equiv \frac{11}{2}$

Definition

$$\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &\quad=\quad \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &\quad=\quad \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &\quad=\quad (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &\quad=\quad (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}$$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in **Q** with $\vec{a}$ from **N** and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Definition

$$\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &\quad = \quad (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}$$

Problem

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} =$$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in $\mathbf{Q}$ with $\vec{a}$ from $\mathbf{N}$ and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Definition

$$
\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &= \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &= (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &= (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}
$$

Problem

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} =$$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in $\mathbf{Q}$ with $\vec{a}$ from $\mathbf{N}$ and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Definition

$$
\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &\quad = \quad (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}
$$

Problem

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} = \frac{12}{8} + \frac{1}{2} =$$

Arithmetic over $\mathbb{Q}$

$(\vec{a}, d)$ in $\mathbf{Q}$ with $\vec{a}$ from $\mathbf{N}$ and $d \in \mathbb{N}^{>0}$, e.g.,
$(\langle \top, \bot, \top, \top \rangle, 2) \equiv \frac{11}{2}$

Definition

$$\begin{aligned}
(\vec{a}, d) >_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad \vec{a} >_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) =_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad \vec{a} =_{\mathbf{N}} \vec{b} \\
(\vec{a}, d) +_{\mathbf{Q}} (\vec{b}, d) &\quad = \quad (\vec{a} +_{\mathbf{N}} \vec{b}, d) \\
(\vec{a}, d) \times_{\mathbf{Q}} (\vec{b}, d') &\quad = \quad (\vec{a} \times_{\mathbf{N}} \vec{b}, d \times d')
\end{aligned}$$

Problem

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} = \frac{12}{8} + \frac{1}{2} = \frac{16}{8}$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = {\color{red}\frac{6}{4}} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{4}{4} \times \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for **Q**

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{?}{1} + \frac{1}{2} =$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{?}{1} + \frac{1}{2} = \frac{?}{1}$$

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{?}{1} + \frac{1}{2} = \frac{?}{1}$$

Remarks

- cancel fractions by side-constraints
- formula might get unsatisfiable

Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{2}{2} \times \frac{3}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

Problem: if denominator chosen too small (here: 1)

$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} = \frac{1}{1} \times \frac{3}{2} + \frac{1}{2} = \frac{?}{1} + \frac{1}{2} = \frac{?}{1}$$

Remarks

- cancel fractions by side-constraints
- formula might get unsatisfiable

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in **R** with $\mathbf{c}, \mathbf{d}$ from **Q** meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in **R** with $\mathbf{c}, \mathbf{d}$ from **Q** meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3\sqrt{2} >_{\mathbb{R}} 2 + 5\sqrt{2}$$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in **R** with $\mathbf{c}, \mathbf{d}$ from **Q** meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3\sqrt{2} >_{\mathbb{R}} 2 + 5\sqrt{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in **R** with $\mathbf{c}, \mathbf{d}$ from **Q** meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in $\mathbf{R}$ with $\mathbf{c}, \mathbf{d}$ from $\mathbf{Q}$ meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Definition

$$\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \;\wedge\; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}$$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in $\mathbf{R}$ with $\mathbf{c}, \mathbf{d}$ from $\mathbf{Q}$ meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Definition

$$\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \ >_{\mathbf{Q}} \ \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \ \wedge \ \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}$$

Arithmetic over $\mathbb{R}$

$(\mathbf{c},\mathbf{d})$ in $\mathbf{R}$ with $\mathbf{c},\mathbf{d}$ from $\mathbf{Q}$ meaning $(\mathbf{3},\mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$?

$$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Definition

$$\begin{aligned}
(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5},4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3},2) \\
(\mathbf{c},\mathbf{d}) =_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \;\wedge\; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c},\mathbf{d}) +_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c},\mathbf{d}) \times_{\mathbf{R}} (\mathbf{e},\mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}$$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in $\mathbf{R}$ with $\mathbf{c}, \mathbf{d}$ from $\mathbf{Q}$ meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Definition

$$\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \; >_{\mathbf{Q}} \; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \; \wedge \; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}$$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in $\mathbf{R}$ with $\mathbf{c}, \mathbf{d}$ from $\mathbf{Q}$ meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Definition

$$\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \;\wedge\; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}$$

Example

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2} \equiv (\mathbf{5}, \mathbf{11})$

$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = 66 + 28\sqrt{2} \equiv (\mathbf{66}, \mathbf{28})$

Arithmetic over $\mathbb{R}$

$(\mathbf{c}, \mathbf{d})$ in $\mathbf{R}$ with $\mathbf{c}, \mathbf{d}$ from $\mathbf{Q}$ meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2} \qquad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$$

Definition

$$\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} (\mathbf{5}, 4) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathbf{f} \times_{\mathbf{Q}} (\mathbf{3}, 2) \\
(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \;\wedge\; \mathbf{d} =_{\mathbf{Q}} \mathbf{f} \\
(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}$$

Example

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2} \equiv (\mathbf{5}, \mathbf{11})$

$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = 66 + 28\sqrt{2} \equiv (\mathbf{66}, \mathbf{28})$

Experimental Results

using $\mathsf{T_TT_2}$ `(dp;edg;(sccs;ur;matrix -dp -ur □)*)`

Table: Matrices with dependency pairs for 1391 TRSs

	1 × 1			2 × 2			3 × 3		
	yes	time	t/o	yes	time	t/o	yes	time	t/o
$\mathbb{N}$	545	8885	83	618	23820	326	627	25055	349
$\mathbb{Q}$	599	8574	67	597	20238	261	496	19490	252
$\mathbb{Q}_1$	606	5906	46	655	15279	173	643	14062	164
$\mathbb{Q}_2$	627	10109	93	651	23102	308	619	23806	330
$\mathbb{R}$	535	17029	198	630	16517	200	599	29346	415

Experimental Results

using $\mathsf{T_TT_2}$ `(dp;edg;(sccs;ur;matrix -dp -ur □)*)`

Table: Matrices with dependency pairs for 1391 TRSs

	1×1			2×2			3×3		
	yes	time	t/o	yes	time	t/o	yes	time	t/o
$\mathbb{N}$	545	8885	83	618	23820	326	627	25055	349
$\mathbb{Q}$	599	8574	67	597	20238	261	496	19490	252
$\mathbb{Q}_1$	606	5906	46	655	15279	173	643	14062	164
$\mathbb{Q}_2$	627	10109	93	651	23102	308	619	23806	330
$\mathbb{R}$	535	17029	198	630	16517	200	599	29346	415

$\mathbb{N}, \mathbb{Q}, \mathbb{Q}_1, \mathbb{Q}_2$ . . . $\max\{5 - d, 2\}$ bits for coefficients
$\mathbb{R}$ . . . $\max\{3 - d, 1\}$ bits for coefficients
one additional bit for computations

$\mathbb{Q}, \mathbb{Q}_1, \mathbb{Q}_2$ . . . all coefficients have denominator 2

Experimental Results (2)

TRS from [Luc06]

$$\begin{aligned} \mathsf{k}(x,x,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{g}(x),\mathsf{b}_2,\mathsf{b}_2) & \mathsf{g}(\mathsf{c}(x)) &\to \mathsf{f}(\mathsf{c}(\mathsf{f}(x))) \\ \mathsf{k}(x,\mathsf{a}_2,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_1,x,\mathsf{b}_1) & \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{k}(\mathsf{a}_4,x,\mathsf{b}_1) &\to \mathsf{k}(x,\mathsf{a}_3,\mathsf{b}_1) & \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) &\to \mathsf{k}(x,x,x) \\ \mathsf{k}(\mathsf{g}(x),\mathsf{b}_3,\mathsf{b}_3) &\to \mathsf{k}(x,x,\mathsf{b}_4) \end{aligned}$$

Experimental Results (2)

TRS from [Luc06]

$$
\begin{aligned}
\mathsf{k}(x,x,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{g}(x),\mathsf{b}_2,\mathsf{b}_2) & \mathsf{g}(\mathsf{c}(x)) &\to \mathsf{f}(\mathsf{c}(\mathsf{f}(x)))\\
\mathsf{k}(x,\mathsf{a}_2,\mathsf{b}_1) &\to \mathsf{k}(\mathsf{a}_1,x,\mathsf{b}_1) & \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x)\\
\mathsf{k}(\mathsf{a}_4,x,\mathsf{b}_1) &\to \mathsf{k}(x,\mathsf{a}_3,\mathsf{b}_1) & \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))) &\to \mathsf{k}(x,x,x)\\
\mathsf{k}(\mathsf{g}(x),\mathsf{b}_3,\mathsf{b}_3) &\to \mathsf{k}(x,x,\mathsf{b}_4)
\end{aligned}
$$

$\mathsf{T_TT_2}$ finds the interpretation that orients all rules strictly

$$
\begin{aligned}
\mathsf{a}_{1\mathbb{R}} &= 0 & \mathsf{b}_{1\mathbb{R}} &= 2+\sqrt{2} & \mathsf{f}_{\mathbb{R}}(x) &= \sqrt{2}x+\sqrt{2}\\
\mathsf{a}_{2\mathbb{R}} &= 1+2\sqrt{2} & \mathsf{b}_{2\mathbb{R}} &= 0 & \mathsf{g}_{\mathbb{R}}(x) &= 2x+1+\sqrt{2}\\
\mathsf{a}_{3\mathbb{R}} &= 0 & \mathsf{b}_{3\mathbb{R}} &= 1+\sqrt{2} & \mathsf{c}_{\mathbb{R}}(x) &= x+1+2\sqrt{2}\\
\mathsf{a}_{4\mathbb{R}} &= 1+\sqrt{2} & \mathsf{b}_{4\mathbb{R}} &= \sqrt{2} & \mathsf{k}_{\mathbb{R}}(x,y,z) &= x+y+\sqrt{2}z+3\sqrt{2}
\end{aligned}
$$

within a fraction of a second

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Related Work

- [Der79], [Gie95], [Hof01], [Luc06], ... (polynomials over $\mathbb{R}$)
- [FNO$^+$08], [BLN$^+$09] (polynomials over $\mathbb{Q}$)
- [GHW07], [ALN09] (matrices over $\mathbb{Q}$, $\mathbb{R}$)

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Related Work

- [Der79], [Gie95], [Hof01], [Luc06], . . . (polynomials over $\mathbb{R}$)
- [FNO+08], [BLN+09] (polynomials over $\mathbb{Q}$)
- [GHW07], [ALN09] (matrices over $\mathbb{Q}$, $\mathbb{R}$)

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Related Work

- [Der79], [Gie95], [Hof01], [Luc06], ... (polynomials over $\mathbb{R}$)
- [FNO$^+$08], [BLN$^+$09] (polynomials over $\mathbb{Q}$)
- [GHW07], [ALN09] (matrices over $\mathbb{Q}$, $\mathbb{R}$)

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Related Work

- [Der79], [Gie95], [Hof01], [Luc06], ... (polynomials over $\mathbb{R}$)
- [FNO+08], [BLN+09] (polynomials over $\mathbb{Q}$)
- [GHW07], [ALN09] (matrices over $\mathbb{Q}$, $\mathbb{R}$)
- this talk

Conclusion

Message

- SAT solving is suitable for matrices over $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$
- allowing $\mathbb{Q}$ pays off (not in naive setting)
- first (efficient) approach capable of $\mathbb{R}$ (in SAT)

Related Work

- [Der79], [Gie95], [Hof01], [Luc06], ... (polynomials over $\mathbb{R}$)
- [FNO+08], [BLN+09] (polynomials over $\mathbb{Q}$)
- [GHW07], [ALN09] (matrices over $\mathbb{Q}$, $\mathbb{R}$)
- this talk

Future Work

- [MSW08] (triangular matrices $\rightarrow$ polynomial derivational complexity)

T. Arts and J. Giesl.
Termination of term rewriting using dependency pairs.
Theoretical Computer Science, 236(1-2):133–178, 2000.

B. Alarcón, S. Lucas, and R. Navarro-Marset.
Proving termination with matrix interpretations over the reals.
In *Proc. of the 10th International Workshop on Termination (WST 2009)*, pages 12–15, 2009.

C. Borralleras, S. Lucas, R. Navarro-Marset, E. Rodriguez-Carbonell, and Albert Rubio.
Solving non-linear polynomial arithmetic via SAT modulo linear arithmetic.
In *Proc. of the 22nd International Conference on Automated Deduction (CADE 2009)*, volume 5663 of *Lecture Notes in Artificial Intelligence*, pages 294–305, 2009.

N. Dershowitz.
A note on simplification orderings.
Information Processing Letters, 9(5):212–215, 1979.

J. Endrullis, J. Waldmann, and H. Zantema.
Matrix interpretations for proving termination of term rewriting.
Journal of Automated Reasoning, 40(2-3):195–220, 2008.

C. Fuhs, R. Navarro-Marset, C. Otto, J. Giesl, S. Lucas, and P. Schneider-Kamp.
Search techniques for rational polynomial orders.
In *Proc. of the 9th International Conference on Artificial Intelligence and Symbolic Computation (AISC 2008)*, volume 5144 of *Lecture Notes in Artificial Intelligence*, pages 109–124, 2008.

A. Gebhardt, D. Hofbauer, and J. Waldmann.
Matrix evolutions.
In *Proc. of the 9th International Workshop on Termination (WST 2007)*, pages 4–8, 2007.

J. Giesl.
Generating polynomial orderings for termination proofs.
In *Proc. of the 6th International Conference on Rewriting Techniques and Applications (RTA 1995)*, volume 914 of *Lecture Notes in Computer Science*, pages 426–431, 1995.

D. Hofbauer.

Termination proofs by context-dependent interpretations.
In *Proc. of the 12th International Conference on Rewriting Techniques and Applications (RTA 2001)*, volume 2051 of *Lecture Notes in Computer Science*, pages 108–121, 2001.

D. Hofbauer and J. Waldmann.

Termination of string rewriting with matrix interpretations.
In *Proc. of the 17th International Conference on Rewriting Techniques and Applications (RTA 2006)*, volume 4098 of *Lecture Notes in Computer Science*, pages 328–342, 2006.

S. Lucas.

On the relative power of polynomials with real, rational, and integer coefficients in proofs of termination of rewriting.
Applicable Algebra in Engineering, Communication and Computing, 17(1):49–73, 2006.

G. Moser, A. Schnabl, and J. Waldmann.

Complexity analysis of term rewriting based on matrix and context dependent interpretations.
In *Proc. of the 28th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2008)*, volume 2 of *Leibniz International Proceedings in Informatics*, pages 304–315, 2008.