Satisfiability of Non-Linear (Ir)rational Arithmetic

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Overview

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$$((\mathsf{b} + \mathsf{b} > 3 \times \mathsf{b}) \vee (\mathsf{b} > -a)) \wedge (\mathsf{a} \times \mathsf{a} = 2)$$

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Facts

undecidable over $\mathbb{N}$, $\mathbb{Z}$ (Hilbert's 10th problem, Matiyasevich 1970)

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Remark

such constraints appear in hard software verification

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- nlsol : $\approx$ Barcelogic

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- Z3 : ?

Arithmetic over $\mathbb{Q}$

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$$\left(\frac{1}{2} \times \frac{4}{2}\right) \times \frac{3}{2} + \frac{1}{2} =$$

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Fast Arithmetic for $\mathbf{Q}$

Solution: cancel if denominator exceeds limit (here: 2)

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Remarks

- cancel fractions by side-constraints
- formula might get unsatisfiable

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$(\mathbf{c}, \mathbf{d})$ in **R** with $\mathbf{c}, \mathbf{d}$ from **Q** meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

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Example

$(\mathbf{3},\mathbf{5}) +_{\mathbf{R}} (\mathbf{2},\mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2} \equiv (\mathbf{5},\mathbf{11})$

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$(\mathbf{3}, \mathbf{5}) \times_\mathbf{R} (\mathbf{2}, \mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = 66 + 28\sqrt{2} \equiv (\mathbf{66}, \mathbf{28})$

Arithmetic over $\mathbb{R}$ (cont'd)

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

Arithmetic over $\mathbb{R}$ (cont'd)

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Approximating $m\sqrt{2}$

$$\mathrm{under}(m\sqrt{2}) = \left((m \geqslant 0)\,?\,\frac{5}{4} : \frac{3}{2}\right) \times m$$

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Arithmetic over $\mathbb{R}$ (cont'd)

Problem

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$? $\quad 6 + 3 \times \frac{5}{4} \;>_{\mathbb{Q}}\; 2 + 5 \times \frac{3}{2} \quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

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$$\mathrm{over}(m\sqrt{2}) = \left((m \geqslant 0)\,?\,\frac{3}{2} : \frac{5}{4}\right) \times m$$

Definition

$$(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) \quad = \quad \mathbf{c} +_{\mathbf{Q}} \mathrm{under}(\mathbf{d}) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \mathrm{over}(\mathbf{f})$$

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

	MiniSmt(bv)	Barcelogic	CVC3
	2	1	3
yes	268	266	113
no	-	189	139
time	3190	1188	13169

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

	MiniSmt(bv)	Barcelogic	CVC3
rank	2	1	3
yes	268	266	113
no	-	189	139
time	3190	1188	13169

termination analysis (1391 problems, $\mathbb{N}$)

	MiniSmt(bv)	Barcelogic	CVC3
rank	2	1	3

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

	MiniSmt(bv)	Barcelogic	CVC3
	2	1	3
yes	268	266	113
no	-	189	139
time	3190	1188	13169

termination analysis (1391 problems, $\mathbb{N}$)

	MiniSmt(bv)	Barcelogic	CVC3
	2	1	3
yes	405	408	117
no	-	3	130
time	5,248	40,525	17,747

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

	MiniSmt(bv)	Barcelogic	CVC3
	2	1	3
yes	268	266	113
no	-	189	139
time	3190	1188	13169

termination analysis (1391 problems, $\mathbb{Q}$)

	nlsol	MiniSmt(sat)	CVC3
	2	1	3
yes			
no			
time			

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

	MiniSmt(bv)	Barcelogic	CVC3
	2	1	3
yes	268	266	113
no	-	189	139
time	3190	1188	13169

termination analysis (1391 problems, $\mathbb{Q}$)

	nlsol	MiniSmt(sat)	CVC3
	2	1	3
yes	289	407	58
no	-	-	125
time	19,523	6,505	69,494

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

	MiniSmt(bv)	Barcelogic	CVC3
	2	1	3
yes	268	266	113
no	-	189	139
time	3190	1188	13169

termination analysis (1391 problems, $\mathbb{R}$)

		MiniSmt(sat)	
	2	1	3
yes			
no			
time			

Evaluation

QF_NIA category (470 problems, $\mathbb{Z}$)

	MiniSmt(bv)	Barcelogic	CVC3
	2	1	3
yes	268	266	113
no	-	189	139
time	3190	1188	13169

termination analysis (1391 problems, $\mathbb{R}$)

		MiniSmt(sat)	
	2	1	3
yes		354	
no		-	
time		19,892	

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0)))$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0)))$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0)))))$$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0)))))$$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0))))$$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(0)))$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$
$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0)))))$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$
$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(0)))$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$$
$$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$$

Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}}$
$\mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$

Definition (Termination)

TRS $\mathcal{R}$ terminating iff no reduction $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

Theorem

$\exists$ *reduction order* $>$ *s.t.* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}$ *terminating*

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &> 0 \\ \mathsf{half}(\mathsf{s}(0)) &> 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &> \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &> 0 \\ \mathsf{bits}(\mathsf{s}(x)) &> \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$$

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$\exists$ *reduction order* $>$ *s.t.* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}$ *terminating*

Term Rewrite Systems

Example

$$\begin{aligned} \mathsf{half}(0) &\to 0 & \mathsf{bits}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 & \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned}$$

$$\mathsf{bits}(\mathsf{s}(\mathsf{s}(0))) > \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(\mathsf{s}(0))))) > \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(0)))) > \mathsf{s}(\mathsf{bits}(\mathsf{s}(0))) > \mathsf{s}(\mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(0))))) > \mathsf{s}(\mathsf{s}(\mathsf{bits}(0))) > \mathsf{s}(\mathsf{s}(0))$$

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Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^\delta_{\mathbb{R}^d})$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^\delta_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^{\delta}_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f$

$$\mathsf{add}(0, y) \to 0 \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^{\delta}_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f$

$$\mathsf{add}(0, y) \to 0 \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

$$\mathsf{add}_{\mathbb{R}}(\vec{x}, \vec{y}) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} \quad \mathsf{s}_{\mathbb{R}}(\vec{x}) = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad 0_{\mathbb{R}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^{\delta}_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f$ $\quad (f_{i(1,1)} \geqslant 1 \longrightarrow \text{monotone})$

$$\mathsf{add}(0, y) \to 0 \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

$$\mathsf{add}_{\mathbb{R}}(\vec{x}, \vec{y}) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} \quad \mathsf{s}_{\mathbb{R}}(\vec{x}) = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad 0_{\mathbb{R}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^\delta_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f \qquad (f_{i(1,1)} \geqslant 1 \longrightarrow \text{monotone})$

$\mathsf{add}(0, y) \to 0 \qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$

$\mathsf{add}_\mathbb{R}(\vec{x}, \vec{y}) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} \quad \mathsf{s}_\mathbb{R}(\vec{x}) = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad 0_\mathbb{R} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} >^\delta_{\mathbb{R}^2} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^{\delta}_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f \qquad (f_{i(1,1)} \geqslant 1 \longrightarrow \text{monotone})$

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$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} >^{\delta}_{\mathbb{R}^2} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{y}$$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^\delta_{\mathbb{R}^d})$

$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f \qquad (f_{i(1,1)} \geqslant 1 \longrightarrow \text{monotone})$

$a >^\delta_{\mathbb{R}^d} b \quad :\Leftrightarrow \quad a_1 - b_1 \geqslant \delta \text{ and } a_i \geqslant_{\mathbb{R}} b_i \qquad (\delta > 0 \longrightarrow \text{well-founded})$

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$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} >^\delta_{\mathbb{R}^2} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{y}$$

$$2 > \sqrt{2} \;\wedge\; 0 \geqslant 0 \;\wedge\; 1 \geqslant 0 \;\wedge\; 0 \geqslant 0 \;\wedge\; 0 \geqslant 0 \;\wedge\; 0 \geqslant 0$$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^\delta_{\mathbb{R}^d})$

$$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f \qquad (f_{i(1,1)} \geqslant 1 \longrightarrow \text{monotone})$$

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$$\mathsf{add}(0, y) \to 0 \qquad\qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

$$\mathsf{add}_{\mathbb{R}}(\vec{x}, \vec{y}) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} \quad \mathsf{s}_{\mathbb{R}}(\vec{x}) = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad 0_{\mathbb{R}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{y} >^\delta_{\mathbb{R}^2} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{y}$$

$$2 > \sqrt{2} \;\wedge\; 0 \geqslant 0 \;\wedge\; 1 \geqslant 0 \;\wedge\; 0 \geqslant 0 \;\wedge\; 0 \geqslant 0 \;\wedge\; 0 \geqslant 0$$

Matrix Interpretations

Example $(\mathbb{R}^d_+, [\cdot], >^\delta_{\mathbb{R}^d})$

$$f_A(x_1, \ldots, x_n) = f_1 x_1 + \cdots + f_n x_n + c_f \qquad (f_{i(1,1)} \geqslant 1 \longrightarrow \text{monotone})$$

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$$\mathsf{add}(0, y) \to 0 \qquad\qquad \mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

$$\mathsf{add}_{\mathbb{R}}(\vec{x}, \vec{y}) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \vec{x} + \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \vec{y} \quad \mathsf{s}_{\mathbb{R}}(\vec{x}) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} x + \begin{pmatrix} ? \\ ? \end{pmatrix} \quad 0_{\mathbb{R}} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

$$? > ? \ \wedge \ ? \geqslant ? \ \wedge \ ? \geqslant ? \ \wedge \ ? \geqslant ? \ \wedge \ ? \geqslant ? \ \wedge \ ? \geqslant ?$$

Exemplary Constraints

As SMT problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((x17 Real) ... (x0 Real))
:formula (and (and (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3
x7))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11
1))) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0
(+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11)))))) (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3 x7
))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11 1)
)) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0 (
+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11))))))) (and (and (>= x2 1) (>= x8 1)) (>= x14 1))))
```

Exemplary Constraints

As SMT problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((x17 Real) ... (x0 Real))
:formula (and (and (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3
x7))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11
1))) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0
(+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11)))))) (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3 x7
))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11 1)
)) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0 (
+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11))))))) (and (and (>= x2 1) (>= x8 1)) (>= x14 1))))
```

As SAT problem (5 bits for variables)

65,420 clauses   149,755 variables $<$ 1 second solving time

Evaluation within $\mathsf{T_TT_2}$

Matrix Interpretations for 1391 TRSs

	1×1		
	yes	time	t/o
$\mathbb{N}$	545	8885	83
$\mathbb{Q}$	599	8574	67
$\mathbb{Q}_1$	606	5906	46
$\mathbb{Q}_2$	627	10109	93
$\mathbb{R}$	535	17029	198

Evaluation within $\mathsf{T_TT_2}$

Matrix Interpretations for 1391 TRSs

	1×1			2×2		
	yes	time	t/o	yes	time	t/o
$\mathbb{N}$	545	8885	83	618	23820	326
$\mathbb{Q}$	599	8574	67	597	20238	261
$\mathbb{Q}_1$	606	5906	46	655	15279	173
$\mathbb{Q}_2$	627	10109	93	651	23102	308
$\mathbb{R}$	535	17029	198	630	16517	200

Evaluation within $\mathsf{T_TT_2}$

Matrix Interpretations for 1391 TRSs

	1 × 1			2 × 2			3 × 3		
	yes	time	t/o	yes	time	t/o	yes	time	t/o
$\mathbb{N}$	545	8885	83	618	23820	326	627	25055	349
$\mathbb{Q}$	599	8574	67	597	20238	261	496	19490	252
$\mathbb{Q}_1$	606	5906	46	655	15279	173	643	14062	164
$\mathbb{Q}_2$	627	10109	93	651	23102	308	619	23806	330
$\mathbb{R}$	535	17029	198	630	16517	200	599	29346	415

Evaluation within $\mathsf{T_TT_2}$

Matrix Interpretations for 1391 TRSs

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$\mathbb{R}$	535	17029	198	630	16517	200	599	29346	415

For the experts ...

$\mathbb{N}, \mathbb{Q}, \mathbb{Q}_1, \mathbb{Q}_2$ ... $\max\{5 - d, 2\}$ bits for coefficients
$\mathbb{R}$ ... $\max\{3 - d, 1\}$ bits for coefficients
one additional bit for computations
$\mathbb{Q}, \mathbb{Q}_1, \mathbb{Q}_2$ ... all coefficients have denominator 2

Evaluation within $\mathsf{T_TT_2}$ (cont'd)

TRS from [Lucas 06]

$$k(x,x,b_1) \to k(g(x),b_2,b_2) \qquad g(c(x)) \to f(c(f(x)))$$
$$k(x,a_2,b_1) \to k(a_1,x,b_1) \qquad f(f(x)) \to g(x)$$
$$k(a_4,x,b_1) \to k(x,a_3,b_1) \qquad f(f(f(f(x)))) \to k(x,x,x)$$
$$k(g(x),b_3,b_3) \to k(x,x,b_4)$$

$\mathsf{T_TT_2}$ finds the interpretation that orients all rules strictly

$$a_{1\mathbb{R}} = 0 \qquad b_{1\mathbb{R}} = 2+\sqrt{2} \qquad f_{\mathbb{R}}(x) = \sqrt{2}x+\sqrt{2}$$
$$a_{2\mathbb{R}} = 1+2\sqrt{2} \qquad b_{2\mathbb{R}} = 0 \qquad g_{\mathbb{R}}(x) = 2x+1+\sqrt{2}$$
$$a_{3\mathbb{R}} = 0 \qquad b_{3\mathbb{R}} = 1+\sqrt{2} \qquad c_{\mathbb{R}}(x) = x+1+2\sqrt{2}$$
$$a_{4\mathbb{R}} = 1+\sqrt{2} \qquad b_{4\mathbb{R}} = \sqrt{2} \qquad k_{\mathbb{R}}(x,y,z) = x+y+\sqrt{2}z+3\sqrt{2}$$

within a fraction of a second

Conclusion

SMT solving is suitable for termination analysis

Arctic Interpretations
Multiset Path Order
Matrix Interpretations
Knuth-Bendix Order
predictive labeling
usable rules
Finding Loops
dependency pairs
dependency graphs
semantic labeling
recursive SCC
argument filterings
below zero
usable rules
Lexicographic Path Order
match-bounds
Polynomial Interpretations
Increasing Interpretations
Monotonic Semantic Path Order

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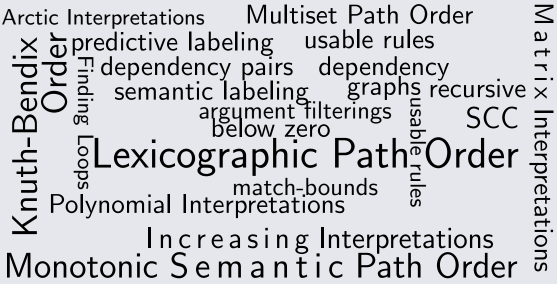
Future Work

- preprocessing

Conclusion

SMT solving is suitable for termination analysis



Future Work

- preprocessing
- unsatisfiability

Conclusion

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Future Work

- preprocessing
- unsatisfiability
- certification