# Satisfying KBO Constraints

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- Motivation
- Knuth-Bendix Order
- Encodings
- Experimental Results
- Concluding Remarks

## Why Encode Termination Problems as Satisfiability Problems?

- execution speed
- ease of implementation
- developments in SAT community are directly available

## Why KBO?

- more challenging than LPO

  *Kurihara & Kondo* *1997 2004* *LPO*

- existing implementations (in T$_\mathsf{T}$T and AProVE) based on polynomial time algorithm

  *Korovin & Voronkov* *2003*

  are slow

## Definition

- quasi-precedence $\gtrsim$ is quasi-order on signature $\mathcal{F}$
- weight function $(w, w_0)$ consists of mapping $w\colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for all constants $c \in \mathcal{F}$
- weight of term $t$ is

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- weight function $(w, w_0)$ is admissible for quasi-precedence $\gtrsim$ if

$$f \gtrsim g \quad \forall\, g \in \mathcal{F}$$

whenever $f$ is unary function symbol in $\mathcal{F}$ with $w(f) = 0$

## Definition

Knuth-Bendix order $>_{\text{kbo}}$ on terms:
$s >_{\text{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either

1. $w(s) > w(t)$
2. $w(s) = w(t)$ and either
   1. $\exists\, n > 0\ \exists\, x \in \mathcal{V}$ such that $s = f_n(\ldots(f_1(t))\ldots)$ and $t \in \mathcal{V}$
   2. $s = f(s_1, \ldots, s_n)$ and $t = g(t_1, \ldots, t_m)$ and $f \sim g$ and $\exists\, i$
      $$\forall\, j < i \quad s_j = t_j \qquad s_i >_{\text{kbo}} t_i$$
   3. $s = f(s_1, \ldots, s_n)$ and $t = g(t_1, \ldots, t_m)$ and $f > g$

## Theorem

*TRS $\mathcal{R}$ is KBO terminating if*

$\exists$ *quasi-precedence* $\gtrsim$ $\quad \exists$ *admissible weight function* $(w, w_0)$

*such that $l >_{\text{kbo}} r$ for all $l \to r \in \mathcal{R}$*

# Example TRS/SK90_2.42

$$\begin{aligned}
\mathsf{flat}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{flat}(\mathsf{unit}(x)) &\to \mathsf{flat}(x) \\
\mathsf{flat}(x \mathbin{+\!\!+} y) &\to \mathsf{flat}(x) \mathbin{+\!\!+} \mathsf{flat}(y) \\
\mathsf{flat}(\mathsf{unit}(x) \mathbin{+\!\!+} y) &\to \mathsf{flat}(x) \mathbin{+\!\!+} \mathsf{flat}(y) \\
\mathsf{flat}(\mathsf{flat}(x)) &\to \mathsf{flat}(x) \\
x \mathbin{+\!\!+} \mathsf{nil} &\to x
\end{aligned}
\qquad
\begin{aligned}
\mathsf{rev}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{rev}(\mathsf{unit}(x)) &\to \mathsf{unit}(x) \\
\mathsf{rev}(x \mathbin{+\!\!+} y) &\to \mathsf{rev}(y) \mathbin{+\!\!+} \mathsf{rev}(x) \\
\mathsf{rev}(\mathsf{rev}(x)) &\to x \\
(x \mathbin{+\!\!+} y) \mathbin{+\!\!+} z &\to x \mathbin{+\!\!+} (y \mathbin{+\!\!+} z) \\
\mathsf{nil} \mathbin{+\!\!+} y &\to y
\end{aligned}$$

$$w(\mathsf{flat}) = w(\mathsf{rev}) = w(\mathbin{+\!\!+}) = 0$$

$$w(\mathsf{unit}) = w(\mathsf{nil}) = w_0 = 1$$

$$\mathsf{flat} \sim \mathsf{rev} > \mathsf{unit} > \mathbin{+\!\!+} > \mathsf{nil}$$

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- **Encodings**
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## Aim

define propositional formula SAT($\mathcal{R}$) depending on TRS $\mathcal{R}$ such that

$$\models \text{SAT}(\mathcal{R}) \implies \mathcal{R} \text{ is KBO terminating}$$
$$\not\models \text{SAT}(\mathcal{R}) \implies \mathcal{R} \text{ is not KBO terminating}$$

## Problem

weight function $w \colon \mathcal{F} \to \mathbb{N}$

## Solution

restrict range of weight function to $\{\, 0, \ldots, 2^k - 1 \,\}$ ($k$ bits)

## Revised Aim

define propositional formula $\text{SAT}_k(\mathcal{R})$ such that

$$\models \text{SAT}_k(\mathcal{R}) \implies \mathcal{R} \text{ is KBO terminating}$$
$$\not\models \text{SAT}_k(\mathcal{R}) \implies \mathcal{R} \text{ is not KBO terminating in } k \text{ bits}$$

## Definition (Codish, Lagoon & Stuckey 2006)

- $\mathbf{a} = \langle a_k, \ldots, a_1 \rangle$

- $\ulcorner \mathbf{a} >_j \mathbf{b} \urcorner = \begin{cases} a_1 \wedge \neg b_1 & \text{if } j = 1 \\ a_j \wedge \neg b_j \ \vee \ (a_j \leftrightarrow b_j) \wedge \ulcorner \mathbf{a} >_{j-1} \mathbf{b} \urcorner & \text{if } j > 1 \end{cases}$

- $\ulcorner \mathbf{a} > \mathbf{b} \urcorner = \ulcorner \mathbf{a} >_k \mathbf{b} \urcorner$

- $\ulcorner \mathbf{a} = \mathbf{b} \urcorner = \bigwedge_{i=1}^{k} (a_i \leftrightarrow b_i)$

- $\ulcorner \mathbf{a} \geqslant \mathbf{b} \urcorner = \ulcorner \mathbf{a} > \mathbf{b} \urcorner \vee \ulcorner \mathbf{a} = \mathbf{b} \urcorner$

## Definition

$$\ulcorner (\mathbf{a}, \varphi) + (\mathbf{b}, \psi) \urcorner = (\mathbf{s}, \varphi \wedge \psi \wedge \gamma \wedge \sigma)$$

with

$$\gamma = \neg c_k \wedge \neg c_0 \wedge \bigwedge_{i=1}^{k} \big(c_i \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_{i-1}) \vee (b_i \wedge c_{i-1}))\big)$$

and

$$\sigma = \bigwedge_{i=1}^{k} \big(s_i \leftrightarrow (a_i \oplus b_i \oplus c_{i-1})\big)$$

- fresh variables $c_i$ $(0 \leqslant i \leqslant k)$ for carry and $s_i$ $(1 \leqslant i \leqslant k)$ for sum
- $\neg c_k$ prevents overflow
- $\oplus$ denotes exclusive or

## Definition

- weight of term $t$

$$\mathsf{w}(t) \;=\; \begin{cases} (\mathbf{w_0}, \top) & \text{if } t \in \mathcal{V} \\ \ulcorner (\mathbf{f}, \top) + \sum_{i=1}^{n} \mathsf{w}(t_i) \urcorner & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

- comparing weights

$$\ulcorner (\mathbf{f}, \varphi) > (\mathbf{g}, \psi) \urcorner \;=\; \ulcorner \mathbf{f} > \mathbf{g} \urcorner \wedge \varphi \wedge \psi$$

- admissibility condition

$$\mathsf{ADM}(\mathcal{F}) \;=\; \ulcorner \mathbf{w_0} > \mathbf{0} \urcorner \;\wedge \bigwedge_{c \,\in\, \mathcal{F}^{(0)}} \ulcorner \mathbf{c} \geqslant \mathbf{w_0} \urcorner \;\wedge$$

$$\bigwedge_{f \,\in\, \mathcal{F}^{(1)}} \big(\ulcorner \mathbf{f} = \mathbf{0} \urcorner \;\rightarrow\; \bigwedge_{g \,\in\, \mathcal{F}} (\mathsf{X}_{fg} \vee \mathsf{Y}_{fg})\big)$$

# Quasi-precedence

symbol-based approach Codish, Lagoon & Stuckey 2006

interpret function symbols in $\{ 0, \ldots, |\mathcal{F}| - 1 \}$

## Propositionally ($l := \lceil log_2(|\mathcal{F}|) \rceil$)

interpret function symbols in $\{\langle\overbrace{0, \ldots, 0}^{l}\rangle, \ldots, \langle\overbrace{1, \ldots, 1}^{l}\rangle\}$

$$\text{PREC}(\mathcal{F}) = \bigwedge_{f,\, g \in \mathcal{F}} ((X_{fg} \rightarrow \ulcorner \mathbf{f}' >_l \mathbf{g}' \urcorner) \wedge (Y_{fg} \rightarrow \ulcorner \mathbf{f}' =_l \mathbf{g}' \urcorner))$$

## Definition

$$\mathrm{SAT}(s >_{\mathrm{kbo}} t) = \begin{cases} \bot & \text{if } s \in \mathcal{V} \text{ or } s = t \text{ or } \exists\, x\ |s|_x < |t|_x \\ \ulcorner \mathrm{w}(s) > \mathrm{w}(t) \urcorner \ \vee \ \ulcorner \mathrm{w}(s) = \mathrm{w}(t) \urcorner \wedge \mathrm{SAT}(s >'_{\mathrm{kbo}} t) & \\ & \text{otherwise} \end{cases}$$

with

$$\mathrm{SAT}(s >'_{\mathrm{kbo}} t) = \begin{cases} \top & \text{if } s = f_n(\ldots(f_1(t))\ldots),\ t \in \mathcal{V},\ n > 0 \\ \mathrm{SAT}(s_i >_{\mathrm{kbo}} t_i) & \text{if } s = f(s_1,\ldots,s_n),\ t = f(t_1,\ldots,t_n) \\ \mathrm{X}_{fg} \ \vee \ \mathrm{Y}_{fg} \wedge \mathrm{SAT}(s_i >_{\mathrm{kbo}} t_i) & \\ & \text{if } s = f(s_1,\ldots,s_n),\ t = g(t_1,\ldots,t_m) \end{cases}$$

where $i$ is least $1 \leqslant j \leqslant \min\{m, n\}$ with $s_j \neq t_j$

## Theorem

*TRS $\mathcal{R}$ is KBO terminating if*

$$\textsf{SAT}_k(\mathcal{R}) \;=\; \textsf{ADM}(\mathcal{F}) \;\wedge\; \textsf{PREC}(\mathcal{F}) \;\wedge \bigwedge_{l \to r \,\in\, \mathcal{R}} \textsf{SAT}(\, l >_{\textsf{kbo}} r \,)$$

*is satisfiable*

## Remark

*satisfying assignment encodes quasi-precedence and weight function*

# Pseudo Boolean Constraints (PBC)

## What are PBC?

- $min : 1 * x_2 - 1 * x_3;$ (goal function)
- $-1 * x_1 + 4 * x_2 - 2 * x_5 \geqslant 3;$ (constraint)
- $12345678901234567890 * x_4 + 4 * x_3 \geqslant 10;$ (constraint)
- $2 * x_2 + 3 * x_4 = 5;$ (constraint)

## Why PBC?

- addition and comparisons for free
- less implementation work
- deals with arbitrary large natural numbers
- faster ?

# Encoding KBO

## Admissibility

$$\overline{w}_0 \geqslant 1;$$

$$\overline{w}(c) - \overline{w}_0 \geqslant 0; \qquad \forall \text{ constants } c \in \mathcal{F}$$

$$n * \overline{w}(f) + \sum_{g \in \mathcal{F}} (X_{fg} + Y_{fg}) \geqslant n; \qquad \forall \text{ unary } f \in \mathcal{F}, n := |\mathcal{F}|$$

where $\overline{w}(f) = 2^{k-1} * f_k + \cdots + 2^0 * f_1$ and $\overline{w}_0 = 2^{k-1} * w_{0k} + \cdots + 2^0 * w_{01}$

## Precedence

$$2 * X_{fg} + Y_{fg} + Y_{gf} + 2 * Z_{fg} = 2$$

$$-X_{fg} + i(f) - i(g) \geqslant 0$$

$$2^l * X_{fg} + Y_{fg} + 2^l * Z_{fg} + i(f) - i(g) \geqslant 1$$

where $l := \lceil log_2(|\mathcal{F}|) \rceil$ and $i(f) = 2^{l-1} * f'_l + \cdots + 2^0 * f'_1$

# Encoding KBO (cont'd)

$m := 2^k * |t|$

$$\mathsf{PBC}(s >_{\mathsf{kbo}} t) = \begin{cases} KBO_{st} = 0 & \text{if } \exists x : |s|_x < |t|_x \\ -(m+1) * KBO_{st} + \overline{w}(s) - \overline{w}(t) + KBO'_{st} \geqslant -m; \\ \mathsf{PBC}(s >'_{kbo} t) & \text{otherwise} \end{cases}$$

with $\mathsf{PBC}(s >'_{\mathsf{kbo}} t) = \mathsf{PBC}(s_i >_{\mathsf{kbo}} t_i)$;

$$\begin{cases} KBO_{st} = 1 & \text{if } s = f_n(\ldots(f_1(t))\ldots) \\ -KBO'_{st} + KBO_{s_i t_i} \geqslant 0 & \text{if } f = g \\ -2 * KBO'_{st} + 2 * X_{fg} + Y_{fg} + KBO_{s_i t_i} \geqslant 0 & \text{if } f \neq g \\ KBO_{st} = 0 & \text{otherwise} \end{cases}$$

where $i$ is least $1 \leqslant j \leqslant \min\{m, n\}$ with $s_j \neq t_j$

# Strict Precedence

865 TRSs in version 3.2 of TPDB

| method(# bits) | total time | # successes | # timeouts (60$s$) |
|---|---|---|---|
| sat/pbc(2) | 19.2/16.4 | 72/76 | 0/0 |
| sat/pbc(3) | 20.2/16.3 | 77/77 | 0/0 |
| sat/pbc(4) | 21.9/16.1 | 78/78 | 0/0 |
| sat/pbc(10) | 86.1/16.7 | 78/78 | 1/0 |
| $\mathsf{T_TT}$ | 169.5 | 77 | 1 |

# Example

TRS/various_21

$$
\begin{aligned}
\mathsf{p1} + \mathsf{p1} &\to \mathsf{p2} & \mathsf{p1} + (\mathsf{p1} + x) &\to \mathsf{p2} + x \\
\mathsf{p2} + \mathsf{p1} &\to \mathsf{p1} + \mathsf{p2} & \mathsf{p2} + (\mathsf{p1} + x) &\to \mathsf{p1} + (\mathsf{p2} + x) \\
\mathsf{p5} + \mathsf{p1} &\to \mathsf{p1} + \mathsf{p5} & \mathsf{p5} + (\mathsf{p1} + x) &\to \mathsf{p1} + (\mathsf{p5} + x) \\
\mathsf{p5} + \mathsf{p2} &\to \mathsf{p2} + \mathsf{p5} & \mathsf{p5} + (\mathsf{p2} + x) &\to \mathsf{p2} + (\mathsf{p5} + x) \\
\mathsf{p5} + \mathsf{p5} &\to \mathsf{p10} & \mathsf{p5} + (\mathsf{p5} + x) &\to \mathsf{p10} + x \\
\mathsf{p10} + \mathsf{p1} &\to \mathsf{p1} + \mathsf{p10} & \mathsf{p10} + (\mathsf{p1} + x) &\to \mathsf{p1} + (\mathsf{p10} + x) \\
\mathsf{p10} + \mathsf{p2} &\to \mathsf{p2} + \mathsf{p10} & \mathsf{p10} + (\mathsf{p2} + x) &\to \mathsf{p2} + (\mathsf{p10} + x) \\
\mathsf{p10} + \mathsf{p5} &\to \mathsf{p5} + \mathsf{p10} & \mathsf{p10} + (\mathsf{p5} + x) &\to \mathsf{p5} + (\mathsf{p10} + x) \\
\mathsf{p1} + (\mathsf{p2} + \mathsf{p2}) &\to \mathsf{p5} & \mathsf{p1} + (\mathsf{p2} + (\mathsf{p2} + x)) &\to \mathsf{p5} + x \\
\mathsf{p2} + (\mathsf{p2} + \mathsf{p2}) &\to \mathsf{p1} + \mathsf{p5} & \mathsf{p2} + (\mathsf{p2} + (\mathsf{p2} + x)) &\to \mathsf{p1} + (\mathsf{p5} + x) \\
& & (x + y) + z &\to x + (y + z)
\end{aligned}
$$

$$w(\mathsf{p1}) = w(\mathsf{p2}) = 4 \qquad w(\mathsf{p5}) = 6 \qquad w(\mathsf{p10}) = 11 \qquad w(+) = 0$$

$$\mathsf{p2} > \mathsf{p1} \qquad \mathsf{p5} > \mathsf{p1} \qquad \mathsf{p10} > \mathsf{p1}$$

| | |
|---|---|
| sat(4) | 0.19 |
| TTT | 4016.23 |

# Example    TRS/HM_t000

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
\vdots & \vdots & & \vdots \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 \\
\multicolumn{2}{l}{x+(y:z) \to y:(x+z)} & \multicolumn{2}{c}{0:x \to x} \\
\multicolumn{2}{l}{(x:y)+z \to x:(y+z)} & \multicolumn{2}{c}{x:(y:z) \to (x+y):z}
\end{array}
$$

$$w(0) = w(1) = w(2) = w(3) = w(4) = 1 \qquad w(+) = 7$$

$$w(5) = w(6) = w(7) = w(8) = w(9) = 2 \qquad w(:) = 8 \qquad + > :$$

| | |
|---|---|
| sat(4) | 1.77 |
| T$_\text{T}$T | ?? |

# Quasi-Precedence

865 TRSs in version 3.2 of TPDB

| method(# bits) | total time | # successes | # timeouts (60$s$) |
|---|---|---|---|
| sat/pbc(2) | 20.9/16.8 | 73/77 | 0/0 |
| sat/pbc(3) | 21.9/16.9 | 78/78 | 0/0 |
| sat/pbc(4) | 22.8/17.0 | 79/79 | 0/0 |
| sat/pbc(10) | 90.2/17.2 | 79/79 | 1/0 |

# String Rewrite Systems

322 SRSs in version 3.2 of TPDB

## strict precedence

| method(# bits) | total time | # successes | # timeouts (60$s$) |
|---|---|---|---|
| sat/pbc(2) | 9.1/5.9 | 8/19 | 0 |
| sat/pbc(3) | 12.1/5.9 | 17/24 | 0 |
| sat/pbc(4) | 15.1/6.0 | 24/30 | 0 |
| sat/pbc(7) | 17.0/6.1 | 33/33 | 0 |
| sat/pbc(10) | 21.6/6.3 | 33/33 | 0 |
| T$_\mathrm{T}$T | 72.4 | 29 | 1 |

# Example SRS/Zantema_z113

$$11 \to 43 \qquad 33 \to 56 \qquad 55 \to 62$$
$$12 \to 21 \qquad 34 \to 11 \qquad 56 \to 12$$
$$22 \to 111 \qquad 44 \to 3 \qquad 66 \to 21$$

TTT

$$w(1) = 32471712256 \qquad w(4) = 21696293888$$
$$w(2) = 48725750528 \qquad w(5) = 44731872512$$
$$w(3) = 43247130624 \qquad w(6) = 40598731520$$
$$3 > 1 > 2 \qquad 1 > 4$$

sat(7)

$$w(1) = 31 \qquad w(2) = 47 \qquad w(3) = 41$$
$$w(4) = 21 \qquad w(5) = 43 \qquad w(6) = 39$$
$$3 > 5 > 6 > 1 > 4 \qquad 1 > 2$$

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- $\forall\, k > 0\ \exists$ TRS $\mathcal{R}_k$ s.t. $\mathcal{R}_k$ is terminating in $k$ bits but not in $k - 1$ bits

$$\mathsf{f}(\mathsf{g}(x, y)) \to \mathsf{g}(\mathsf{f}(x), \mathsf{f}(y)) \qquad \mathsf{h}(x) \to \mathsf{f}(\mathsf{f}(x)) \qquad \mathsf{i}(x) \to \mathsf{h}^{2^k}(x)$$

- why is PBC better?
  - encoding → less implementation work
  - performance → simulating adders in SAT destroys structure
- much related work
  - LPO+AF
  - KBO+AF
  - RPO+AF
  - matrix interpretation
  - (linear) polynomial interpretations
  - semantic labeling
  - nontermination
  - ...