Modular Complexity Analysis via Relative Complexity

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Overview

- Derivational Complexity
- Relative Rewriting
- Modular Complexity
- Experiments
- Conclusion

Derivational Complexity

Definition (Derivation Height)

$\mathrm{dh}(t, \to) = \max\{n \mid t = t_0 \to t_1 \to \cdots \to t_n\}$

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Lemma

$>$ *rewrite relation*

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$>$ rewrite relation and $\mathcal{R} \subseteq >$

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Current approaches

TMIs

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TMIs, AMIs, matchbounds

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$\mathsf{a}(x) \to \mathsf{b}(x)$

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$\mathsf{a}(x) > \mathsf{b}(x)$ $\qquad$ $\mathsf{a}_{\mathbb{N}}(x) = x + 2 \quad \mathsf{b}_{\mathbb{N}}(x) = x + 1$

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$$[t]_{\mathbb{N}} \leqslant 2 \cdot |t| \longrightarrow \mathsf{dc}(m, >_{\mathbb{N}}) = \mathcal{O}(m) \longrightarrow \mathsf{dc}(m, \rightarrow_{\mathcal{R}}) = \mathcal{O}(m)$$

Relative Termination

Definition

$\to_{\mathcal{R}/\mathcal{S}} = \to_{\mathcal{S}}^{*} \cdot \to_{\mathcal{R}} \cdot \to_{\mathcal{S}}^{*}$

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$\mathit{SN}(\mathcal{R} \cup \mathcal{S}) \iff \mathit{SN}(\mathcal{R}/\mathcal{S}) \wedge \mathit{SN}(\mathcal{S})$

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Relative Termination for Complexity

Question

$$\mathsf{dc}(m, \to_{\mathcal{R} \cup \mathcal{S}}) \; = \; \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}/\mathcal{S}}) + \mathsf{dc}(m, \to_{\mathcal{S}}))$$

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Counterexample (Hofbauer, 2006)

$$\mathcal{R} = \{\mathsf{cL} \to \mathsf{R}\}$$
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Relative Complexity

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Theorem

$\mathrm{dc}(m, \to_{(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{S}}) = \Theta(\mathrm{dc}(m, \to_{\mathcal{R}_1/(\mathcal{S} \cup \mathcal{R}_2)}) + \mathrm{dc}(m, \to_{\mathcal{R}_2/(\mathcal{S} \cup \mathcal{R}_1)}))$

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Definition (Complexity Pair)

$(>, \geqslant)$ with rewrite relations $>$, $\geqslant$

Modular Complexity

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathsf{dc}(m, \to_{\mathcal{R}/\varnothing}) \qquad \mathsf{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$

Theorem

$\mathsf{dc}(m, \to_{(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{S}}) = \Theta(\mathsf{dc}(m, \to_{\mathcal{R}_1/(\mathcal{S} \cup \mathcal{R}_2)}) + \mathsf{dc}(m, \to_{\mathcal{R}_2/(\mathcal{S} \cup \mathcal{R}_1)}))$

Definition (Complexity Pair)

$(>, \geqslant)$ with rewrite relations $>$, $\geqslant$ and $> \cdot \geqslant \subseteq >$ and $\geqslant \cdot > \subseteq >$

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$(>, \geqslant)$ *complexity pair*

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$(>, \geqslant)$ complexity pair, $\mathcal{R} \subseteq >$, $\mathcal{S} \subseteq \geqslant$

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$(>, \geqslant)$ *complexity pair*, $\mathcal{R} \subseteq >, \mathcal{S} \subseteq \geqslant \quad \longrightarrow \mathsf{dc}(m, >) \geqslant \mathsf{dc}(m, \to_{\mathcal{R}/\mathcal{S}})$

Complexity Pairs

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$f_{\mathcal{M}}(\vec{x}_1, \ldots, \vec{x}_n)$

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$s >_{\mathcal{M}} t \iff [s]_{\mathcal{M}} > [t]_{\mathcal{M}}$

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TMI $\mathcal{M}$ of dimension d

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Lemma

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Convention

TMI of dimension 1 $\longrightarrow$ SLI

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \quad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \quad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \quad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

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Example

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \quad 0_{\mathcal{M}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 1_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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$$\mathsf{f}(1) > \mathsf{f}(\mathsf{g}(1)) \quad \mathsf{f}(\mathsf{f}(x)) > \mathsf{f}(x) \quad \mathsf{g}(0) \geqslant \mathsf{g}(\mathsf{f}(0)) \quad \mathsf{g}(\mathsf{g}(x)) \geqslant \mathsf{g}(x)$$

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Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m^2)$

Complexity pairs cont'd (Matchbounds)

Idea

$\mathsf{ab} \rightarrow \mathsf{baa}$

Complexity pairs cont'd (Matchbounds)

Idea

$$ab \to baa \quad \xrightarrow{match} \quad \begin{array}{ll} a_0b_0 \to b_1a_1a_1 & a_0b_1 \to b_1a_1a_1 \\ a_1b_0 \to b_1a_1a_1 & a_1b_1 \to b_2a_2a_2 \\ a_1b_2 \to b_2a_2a_2 & a_1b_2 \to b_2a_2a_2 \\ \dots & \dots \end{array}$$

Complexity pairs cont'd (Matchbounds)

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$$ab \rightarrow baa \quad \xrightarrow{match} \quad \begin{array}{ll} a_0 b_0 \rightarrow b_1 a_1 a_1 & a_0 b_1 \rightarrow b_1 a_1 a_1 \\ a_1 b_0 \rightarrow b_1 a_1 a_1 & a_1 b_1 \rightarrow b_2 a_2 a_2 \\ a_1 b_2 \rightarrow b_2 a_2 a_2 & a_1 b_2 \rightarrow b_2 a_2 a_2 \\ \dots & \dots \end{array}$$

Complexity pairs cont'd (Matchbounds)

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$$\mathsf{ab} \to \mathsf{baa} \quad \xrightarrow{\textit{match}} \quad \begin{array}{ll} \mathsf{a}_0\mathsf{b}_0 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 & \mathsf{a}_0\mathsf{b}_1 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 \\ \mathsf{a}_1\mathsf{b}_0 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 & \mathsf{a}_1\mathsf{b}_1 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 \\ \mathsf{a}_1\mathsf{b}_2 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 & \mathsf{a}_1\mathsf{b}_2 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 \\ \dots & \dots \end{array}$$

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Termination

are labels in $\mathsf{lift}_0(\mathcal{T}(\mathcal{F})) \rightarrow^*_{match(\mathcal{R})} \cdot$ bounded by some $c \in \mathbb{N}$?

Complexity pairs cont'd (Matchbounds)

Idea

$$\mathsf{ab} \to \mathsf{baa} \quad \xrightarrow{\textit{match}} \quad \begin{array}{lcl} \mathsf{a}_0\mathsf{b}_0 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 & & \mathsf{a}_0\mathsf{b}_1 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 \\ \mathsf{a}_1\mathsf{b}_0 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 & & \mathsf{a}_1\mathsf{b}_1 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 \\ \mathsf{a}_1\mathsf{b}_2 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 & & \mathsf{a}_1\mathsf{b}_2 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 \\ \dots & & \dots \end{array}$$

Termination

labels in $\mathsf{lift}_0(\mathcal{T}(\mathcal{F})) \to^*_{match(\mathcal{R})} \cdot$ bounded by some $c \in \mathbb{N}$

$\longrightarrow \mathsf{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m)$

Complexity pairs cont'd (Matchbounds)

Idea

$$\mathsf{ab} \to \mathsf{baa} \quad \xrightarrow{\mathit{match}} \quad \begin{array}{ll} \mathsf{a}_0\mathsf{b}_0 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 & \mathsf{a}_0\mathsf{b}_1 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 \\ \mathsf{a}_1\mathsf{b}_0 \to \mathsf{b}_1\mathsf{a}_1\mathsf{a}_1 & \mathsf{a}_1\mathsf{b}_1 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 \\ \mathsf{a}_1\mathsf{b}_2 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 & \mathsf{a}_1\mathsf{b}_2 \to \mathsf{b}_2\mathsf{a}_2\mathsf{a}_2 \\ \dots & \dots \end{array}$$

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Matchbounds for $\mathcal{R}/\mathcal{S}$ (Idea)

do not increase labels for non-length-increasing rules in $\mathcal{S}$

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Beyond Complexity Pairs

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$$\mathrm{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2}) + m)$$

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$\mathrm{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2}) + m)$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}_1$ *non-duplicating*

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$\mathrm{dc}(m, \rightarrow_{\mathcal{R}_1 \cup \mathcal{R}_2/\mathcal{S}}) = \mathcal{O}(\mathrm{dc}(m, \rightarrow_{\mathcal{R}_1/\mathcal{R}_2 \cup \mathcal{S}}) + m)$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{M}}$ for SLI $\mathcal{M}$, $\mathcal{R}_1$ non-duplicating

Example ($\mathcal{R}_1$)

$\mathsf{aa} \rightarrow \mathsf{cb}$ $(\mathcal{R}_1)$

$(\mathcal{R}_2)$

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Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$\mathrm{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2}) + m)$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}_1$ *non-duplicating*

Corollary

$\mathrm{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2/\mathcal{S}}) = \mathcal{O}(\mathrm{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2 \cup \mathcal{S}}) + m)$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}_1$ *non-duplicating*

Example ($\mathcal{R}_1 \cup \mathcal{R}_2$)

$\mathsf{aa} \to \mathsf{cb}$		$(\mathcal{R}_1)$
$\mathsf{bb} \to \mathsf{ca}$	$\mathsf{cc} \to \mathsf{ba}$	$(\mathcal{R}_2)$

Beyond Complexity Pairs

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$$\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1/\mathcal{R}_2}) + m)$$

if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}_1$ *non-duplicating*

Corollary

$$\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1 \cup \mathcal{R}_2/\mathcal{S}}) = \mathcal{O}(\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1/\mathcal{R}_2 \cup \mathcal{S}}) + m)$$

if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}_1$ *non-duplicating*

Example ($\mathcal{R}_1 \cup \mathcal{R}_2$ = Zantema/z086)

$$\mathsf{aa} \rightarrow \mathsf{cb} \qquad (\mathcal{R}_1)$$

$$\mathsf{bb} \rightarrow \mathsf{ca} \qquad \mathsf{cc} \rightarrow \mathsf{ba} \qquad (\mathcal{R}_2)$$

Beyond Complexity Pairs

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$$\mathsf{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2}) + m)$$

if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$ for SLI $\mathcal{M}$, $\mathcal{R}_1$ non-duplicating

Corollary

$$\mathsf{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2/\mathcal{S}}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2 \cup \mathcal{S}}) + m)$$

if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{M}}$ for SLI $\mathcal{M}$, $\mathcal{R}_1$ non-duplicating

Example (Advertisement RTALooP ♯105)

$$\mathsf{aa} \to \mathsf{cb} \qquad (\mathcal{R}_1)$$

$$\mathsf{bb} \to \mathsf{ca} \qquad \mathsf{cc} \to \mathsf{ba} \qquad (\mathcal{R}_2)$$

Beyond Complexity Pairs

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$$\mathsf{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2}) + m)$$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$ for SLI $\mathcal{M}$, $\mathcal{R}_1$ non-duplicating

Corollary

$$\mathsf{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2/\mathcal{S}}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2 \cup \mathcal{S}}) + m)$$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{M}}$ for SLI $\mathcal{M}$, $\mathcal{R}_1$ non-duplicating

Example ($\mathcal{R}_1 \cup \mathcal{R}_2$ = Zantema/z086)

$$\mathsf{aa} \not> \mathsf{cb} \qquad (\mathcal{R}_1)$$

$$\mathsf{bb} > \mathsf{ca} \qquad \mathsf{cc} > \mathsf{ba} \qquad (\mathcal{R}_2)$$

$\mathsf{a}_{\mathbb{N}}(x) = x \qquad \mathsf{b}_{\mathbb{N}}(x) = \mathsf{c}_{\mathbb{N}}(x) = x + 1$

Beyond Complexity Pairs

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$$\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1/\mathcal{R}_2}) + m)$$

if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}_1$ *non-duplicating*

Corollary

$$\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1 \cup \mathcal{R}_2/\mathcal{S}}) = \mathcal{O}(\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1/\mathcal{R}_2 \cup \mathcal{S}}) + m)$$

if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{M}}$ *for SLI* $\mathcal{M}$, $\mathcal{R}_1$ *non-duplicating*

Example ($\mathcal{R}_1 \cup \mathcal{R}_2$ = Zantema/z086)

$$\mathsf{aa} \not\geqslant \mathsf{cb} \qquad (\mathcal{R}_1)$$

$$\mathsf{bb} > \mathsf{ca} \qquad \mathsf{cc} > \mathsf{ba} \qquad (\mathcal{R}_2)$$

$$\mathsf{a}_{\mathbb{N}}(x) = x \qquad \mathsf{b}_{\mathbb{N}}(x) = \mathsf{c}_{\mathbb{N}}(x) = x + 1$$

$\mathsf{dc}(m, \rightarrow_{\mathcal{R}_1 \cup \mathcal{R}_2})$

Beyond Complexity Pairs

Lemma (Weight Gap Principle (Hirokawa & Moser 2008))

$$\mathsf{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2}) + m)$$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$ for SLI $\mathcal{M}$, $\mathcal{R}_1$ non-duplicating

Corollary

$$\mathsf{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2/\mathcal{S}}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2 \cup \mathcal{S}}) + m)$$
if $\mathcal{R}_2 \subseteq >_{\mathcal{M}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{M}}$ for SLI $\mathcal{M}$, $\mathcal{R}_1$ non-duplicating

Example ($\mathcal{R}_1 \cup \mathcal{R}_2 =$ Zantema/z086)

$$\mathsf{aa} \not> \mathsf{cb} \qquad (\mathcal{R}_1)$$

$$\mathsf{bb} > \mathsf{ca} \qquad \mathsf{cc} > \mathsf{ba} \qquad (\mathcal{R}_2)$$

$$\mathsf{a}_{\mathbb{N}}(x) = x \qquad \mathsf{b}_{\mathbb{N}}(x) = \mathsf{c}_{\mathbb{N}}(x) = x + 1$$

$$\mathsf{dc}(m, \to_{\mathcal{R}_1 \cup \mathcal{R}_2}) = \mathcal{O}(\mathsf{dc}(m, \to_{\mathcal{R}_1/\mathcal{R}_2}) + m)$$

Assessment

Lemma

modular setting subsumes direct one

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq >$

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq > \quad \longrightarrow \exists$ complexity pairs $(>, \geqslant \quad)$ and $(>, \geqslant \quad)$

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq > \quad \longrightarrow \exists$ complexity pairs $(>, \geqslant\ )$ and $(>, \geqslant\ )$

$\mathcal{R} \subseteq >, \mathcal{S} \subseteq \geqslant \qquad \mathcal{R} \subseteq \geqslant\ , \mathcal{S} \subseteq >$

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq > \;\longrightarrow \exists$ complexity pairs $(>, \geqslant)$ and $(>, \geqslant)$
$\mathcal{R} \subseteq >, \mathcal{S} \subseteq \geqslant \qquad \mathcal{R} \subseteq \geqslant, \mathcal{S} \subseteq >$
$\longrightarrow \mathrm{dc}(m, >) = \Theta(\mathrm{dc}(m, >) + \mathrm{dc}(m, >))$

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq > \quad \longrightarrow \exists$ complexity pairs $(>, >^{=})$ and $(>, >^{=})$

$\mathcal{R} \subseteq >, \mathcal{S} \subseteq >^{=} \qquad\qquad \mathcal{R} \subseteq >^{=}, \mathcal{S} \subseteq >$

$\longrightarrow \mathsf{dc}(m, >) = \Theta(\mathsf{dc}(m, >) + \mathsf{dc}(m, >))$

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq > \quad \longrightarrow \exists$ complexity pairs $(>, >^{=})$ and $(>, >^{=})$
$\mathcal{R} \subseteq >, \mathcal{S} \subseteq >^{=} \qquad \mathcal{R} \subseteq >^{=}, \mathcal{S} \subseteq >$
$\longrightarrow \mathrm{dc}(m, >) = \Theta(\mathrm{dc}(m, >) + \mathrm{dc}(m, >))$

Lemma

equal power for SLIs

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq > \quad \longrightarrow \exists$ complexity pairs $(>, >^=)$ and $(>, >^=)$
$\mathcal{R} \subseteq >, \mathcal{S} \subseteq >^= \qquad \mathcal{R} \subseteq >^=, \mathcal{S} \subseteq >$
$\longrightarrow \mathrm{dc}(m, >) = \Theta(\mathrm{dc}(m, >) + \mathrm{dc}(m, >))$

Lemma

equal power for SLIs

Proof idea

SLI $\mathcal{A} \qquad \mathcal{R} \subseteq >_{\mathcal{A}}, \mathcal{S} \subseteq \geqslant_{\mathcal{A}}$

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq >$ $\longrightarrow$ $\exists$ complexity pairs $(>, >^{=})$ and $(>, >^{=})$

$\mathcal{R} \subseteq >$, $\mathcal{S} \subseteq >^{=}$ $\quad$ $\mathcal{R} \subseteq >^{=}$, $\mathcal{S} \subseteq >$

$\longrightarrow \mathsf{dc}(m, >) = \Theta(\mathsf{dc}(m, >) + \mathsf{dc}(m, >))$

Lemma

equal power for SLIs

Proof idea

SLI $\mathcal{A}$, $\mathcal{A}$ $\quad$ $\mathcal{R} \subseteq >_{\mathcal{A}}$, $\mathcal{S} \subseteq \geqslant_{\mathcal{A}}$ $\quad$ $\mathcal{R} \subseteq \geqslant_{\mathcal{A}}$, $\mathcal{S} \subseteq >_{\mathcal{A}}$

Assessment

Lemma

modular setting subsumes direct one

Proof idea

$(\mathcal{R} \cup \mathcal{S}) \subseteq >$ $\longrightarrow$ $\exists$ complexity pairs $(>, >^{=})$ and $(>, >^{=})$
$\mathcal{R} \subseteq >, \mathcal{S} \subseteq >^{=}$ $\quad$ $\mathcal{R} \subseteq >^{=}, \mathcal{S} \subseteq >$
$\longrightarrow \mathsf{dc}(m, >) = \Theta(\mathsf{dc}(m, >) + \mathsf{dc}(m, >))$

Lemma

equal power for SLIs

Proof idea

SLI $\mathcal{A}$, $\mathcal{A}$ $\quad$ $\mathcal{R} \subseteq >_{\mathcal{A}}, \mathcal{S} \subseteq \geqslant_{\mathcal{A}}$ $\quad$ $\mathcal{R} \subseteq \geqslant_{\mathcal{A}}, \mathcal{S} \subseteq >_{\mathcal{A}}$
$\longrightarrow (\mathcal{R} \cup \mathcal{S}) \subseteq >_{\mathcal{A}+\mathcal{A}}$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \qquad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \qquad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \qquad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \qquad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \qquad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \qquad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

Lemma

$\nexists$ TMI $\mathcal{M}$ of dimension 2 with $\mathcal{R} \subseteq >_{\mathcal{M}}$

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \qquad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \qquad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \qquad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

Lemma

$\nexists$ TMI $\mathcal{M}$ of dimension 2 with $\mathcal{R} \subseteq >_{\mathcal{M}}$

Proof idea

```
(benchmark ttt2 :logic QF_NIA :status unknown :extrafuns ((x13 Int) ... (x0 Int)) :assumption (>= x13 0)
...  :assumption (>= x0 0) :formula (and (and (and (and (and (and (and (>= (+ x0 (+ (* x2 x5) (* x3 x6)))
(+ x0 (+ (* x2 (+ x7 (+ (* x9 x5) (* x10 x6)))) (* x3 (+ x8 (* x11 x6)))))) (>= (+ x1 (* x4 x6)) (+ x1 (*
x4 (+ x8 (* x11 x6)))))) (and (and (>= (+ x0 (+ (* x2 x0) (* x3 x1))) x0) (>= (+ x1 (* x4 x1)) x1)) (and
(and (>= (* x2 x2) x2) (>= (+ (* x2 x3) (* x3 x4)) x3)) (>= (* x4 x4) x4)))) (and (>= (+ x7 (+ (* x9 x12)
(* x10 x13))) (+ x7 (+ (* x9 (+ x0 (+ (* x2 x12) (* x3 x13)))) (* x10 (+ x1 (* x4 x13)))))) (>= (+ x8 (*
x11 x13)) (+ x8 (* x11 (+ x1 (* x4 x13))))))) (and (and (>= (+ x7 (+ (* x9 x7) (* x10 x8))) x7) (>= (+ x8
(* x11 x8)) x8)) (and (and (>= (* x9 x9) x9) (>= (+ (* x9 x10) (* x10 x11)) x10)) (>= (* x11 x11) x11))))
(or (or (or (and (> (+ x0 (+ (* x2 x5) (* x3 x6))) (+ x0 (+ (* x2 (+ x7 (+ (* x9 x5) (* x10 x6)))) (* x3
...
```

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$f(1) \to f(g(1)) \qquad f(f(x)) \to f(x) \qquad g(0) \to g(f(0)) \qquad g(g(x)) \to g(x)$

Lemma

$\nexists$ *TMI* $\mathcal{M}$ *of dimension 2 with* $\mathcal{R} \subseteq >_{\mathcal{M}}$

Proof idea

```
(benchmark ttt2 :logic QF_NIA :status unknown :extrafuns ((x13 Int) ... (x0 Int)) :assumption (>= x13 0)
...  :assumption (>= x0 0) :formula (and (and (and (and (and (and (and (>= (+ x0 (+ (* x2 x5) (* x3 x6)))
(+ x0 (+ (* x2 (+ x7 (+ (* x9 x5) (* x10 x6)))) (* x3 (+ x8 (* x11 x6)))))) (>= (+ x1 (* x4 x6)) (+ x1 (*
x4 (+ x8 (* x11 x6)))))) (and (and (>= (+ x0 (+ (* x2 x0) (* x3 x1))) x0) (>= (+ x1 (* x4 x1)) x1)) (and
(and (>= (* x2 x2) x2) (>= (+ (* x2 x3) (* x3 x4)) x3)) (>= (* x4 x4) x4)))) (and (>= (+ x7 (+ (* x9 x12)
(* x10 x13))) (+ x7 (+ (* x9 (+ x0 (+ (* x2 x12) (* x3 x13)))) (* x10 (+ x1 (* x4 x13)))))) (>= (+ x8 (*
x11 x13)) (+ x8 (* x11 (+ x1 (* x4 x13))))))) (and (and (>= (+ x7 (+ (* x9 x7) (* x10 x8))) x7) (>= (+ x8
(* x11 x8)) x8)) (and (and (>= (* x9 x9) x9) (>= (+ (* x9 x10) (* x10 x11)) x10)) (>= (* x11 x11) x11))))
(or (or (or (and (> (+ x0 (+ (* x2 x5) (* x3 x6))) (+ x0 (+ (* x2 (+ x7 (+ (* x9 x5) (* x10 x6)))) (* x3
...
```

is not satisfiable

Example ($\mathcal{R}$ = Strategy_removed_AG01/#4.21)

$$\mathsf{f}(1) \to \mathsf{f}(\mathsf{g}(1)) \qquad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \qquad \mathsf{g}(0) \to \mathsf{g}(\mathsf{f}(0)) \qquad \mathsf{g}(\mathsf{g}(x)) \to \mathsf{g}(x)$$

Lemma

$\nexists$ TMI $\mathcal{M}$ of dimension 2 with $\mathcal{R} \subseteq >_{\mathcal{M}}$

Proof idea

```
(benchmark ttt2 :logic QF_NIA :status unknown :extrafuns ((x13 Int) ... (x0 Int)) :assumption (>= x13 0)
...  :assumption (>= x0 0) :formula (and (and (and (and (and (and (and (>= (+ x0 (+ (* x2 x5) (* x3 x6)))
(+ x0 (+ (* x2 (+ x7 (+ (* x9 x5) (* x10 x6)))) (* x3 (+ x8 (* x11 x6)))))) (>= (+ x1 (* x4 x6)) (+ x1 (*
x4 (+ x8 (* x11 x6)))))) (and (and (>= (+ x0 (+ (* x2 x0) (* x3 x1))) x0) (>= (+ x1 (* x4 x1)) x1)) (and
(and (>= (* x2 x2) x2) (>= (+ (* x2 x3) (* x3 x4)) x3)) (>= (* x4 x4) x4)))) (and (>= (+ x7 (+ (* x9 x12)
(* x10 x13))) (+ x7 (+ (* x9 (+ x0 (+ (* x2 x12) (* x3 x13)))) (* x10 (+ x1 (* x4 x13)))))) (>= (+ x8 (*
x11 x13)) (+ x8 (* x11 (+ x1 (* x4 x13))))))) (and (and (>= (+ x7 (+ (* x9 x7) (* x10 x8))) x7) (>= (+ x8
(* x11 x8)) x8)) (and (and (>= (* x9 x9) x9) (>= (+ (* x9 x10) (* x10 x11)) x10)) (>= (* x11 x11) x11))))
(or (or (or (and (> (+ x0 (+ (* x2 x5) (* x3 x6))) (+ x0 (+ (* x2 (+ x7 (+ (* x9 x5) (* x10 x6)))) (* x3
...
```

is not satisfiable

Lemma

modular setting strictly more powerful (in theory)

Example ($\mathcal{R}$ = TRCSR/Ex16_Luc06_GM)

$$\mathsf{mark}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(x,y) \to \mathsf{f}(x,y) \qquad \mathsf{c} \to \mathsf{a}$$
$$\mathsf{mark}(\mathsf{b}) \to \mathsf{c} \qquad \mathsf{g}(x,x) \to \mathsf{g}(\mathsf{a},\mathsf{b}) \qquad \mathsf{c} \to \mathsf{b}$$
$$\mathsf{mark}(\mathsf{f}(x,y)) \to \mathsf{g}(\mathsf{mark}(x),y)$$

Example ($\mathcal{R}$ = TRCSR/Ex16_Luc06_GM)

$$\begin{aligned} \mathsf{mark}(\mathsf{a}) &\to \mathsf{a} & \mathsf{g}(x,y) &\to \mathsf{f}(x,y) & \mathsf{c} &\to \mathsf{a} \\ \mathsf{mark}(\mathsf{b}) &\to \mathsf{c} & \mathsf{g}(x,x) &\to \mathsf{g}(\mathsf{a},\mathsf{b}) & \mathsf{c} &\to \mathsf{b} \\ \mathsf{mark}(\mathsf{f}(x,y)) &\to \mathsf{g}(\mathsf{mark}(x),y) \end{aligned}$$

Lemma

$\mathsf{dc}(m, \to) = \mathcal{O}(m^2)$

Example ($\mathcal{R}$ = TRCSR/Ex16_Luc06_GM)

$$\mathsf{mark}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(x,y) \to \mathsf{f}(x,y) \qquad \mathsf{c} \to \mathsf{a}$$
$$\mathsf{mark}(\mathsf{b}) \to \mathsf{c} \qquad \mathsf{g}(x,x) \to \mathsf{g}(\mathsf{a},\mathsf{b}) \qquad \mathsf{c} \to \mathsf{b}$$
$$\mathsf{mark}(\mathsf{f}(x,y)) \to \mathsf{g}(\mathsf{mark}(x),y)$$

Lemma

$\mathsf{dc}(m, \to) = \mathcal{O}(m^2)$

Proof

TMIs of dimension 2

Example ($\mathcal{R}$ = TRCSR/Ex16_Luc06_GM)

$$\mathsf{mark}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(x,y) \to \mathsf{f}(x,y) \qquad \mathsf{c} \to \mathsf{a}$$
$$\mathsf{mark}(\mathsf{b}) \to \mathsf{c} \qquad \mathsf{g}(x,x) \to \mathsf{g}(\mathsf{a},\mathsf{b}) \qquad \mathsf{c} \to \mathsf{b}$$
$$\mathsf{mark}(\mathsf{f}(x,y)) \to \mathsf{g}(\mathsf{mark}(x),y)$$

Lemma

$\mathsf{dc}(m, \to) = \mathcal{O}(m^2)$

Proof

TMIs of dimension 2, match-rt bounded

Example ($\mathcal{R}$ = TRCSR/Ex16_Luc06_GM)

$$\mathsf{mark}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(x,y) \to \mathsf{f}(x,y) \qquad \mathsf{c} \to \mathsf{a}$$
$$\mathsf{mark}(\mathsf{b}) \to \mathsf{c} \qquad \mathsf{g}(x,x) \to \mathsf{g}(\mathsf{a},\mathsf{b}) \qquad \mathsf{c} \to \mathsf{b}$$
$$\mathsf{mark}(\mathsf{f}(x,y)) \to \mathsf{g}(\mathsf{mark}(x),y)$$

Lemma

$\mathsf{dc}(m, \to) = \mathcal{O}(m^2)$

Proof

TMIs of dimension 2, match-rt bounded

Remark

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \Theta(m^2)$

Example (SK90/4.30)

$$\begin{aligned} \mathsf{f}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{g}(\mathsf{nil}) &\to \mathsf{nil} \\ \mathsf{f}(\mathsf{nil} \circ y) &\to \mathsf{nil} \circ \mathsf{f}(y) & \mathsf{g}(x \circ \mathsf{nil}) &\to \mathsf{g}(x) \circ \mathsf{nil} \\ \mathsf{f}((x \circ y) \circ z) &\to \mathsf{f}(x \circ (y \circ z)) & \mathsf{g}(x \circ (y \circ z)) &\to \mathsf{g}((x \circ y) \circ z) \end{aligned}$$

Example (SK90/4.30)

$$\begin{aligned} \mathsf{f}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{g}(\mathsf{nil}) &\to \mathsf{nil} \\ \mathsf{f}(\mathsf{nil} \circ y) &\to \mathsf{nil} \circ \mathsf{f}(y) & \mathsf{g}(x \circ \mathsf{nil}) &\to \mathsf{g}(x) \circ \mathsf{nil} \\ \mathsf{f}((x \circ y) \circ z) &\to \mathsf{f}(x \circ (y \circ z)) & \mathsf{g}(x \circ (y \circ z)) &\to \mathsf{g}((x \circ y) \circ z) \end{aligned}$$

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m^3)$

Example (SK90/4.30)

$$\mathsf{f}(\mathsf{nil}) \to \mathsf{nil} \qquad \mathsf{g}(\mathsf{nil}) \to \mathsf{nil}$$

$$\mathsf{f}(\mathsf{nil} \circ y) \to \mathsf{nil} \circ \mathsf{f}(y) \qquad \mathsf{g}(x \circ \mathsf{nil}) \to \mathsf{g}(x) \circ \mathsf{nil}$$

$$\mathsf{f}((x \circ y) \circ z) \to \mathsf{f}(x \circ (y \circ z)) \qquad \mathsf{g}(x \circ (y \circ z)) \to \mathsf{g}((x \circ y) \circ z)$$

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m^3)$

Proof

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \circ_{\mathcal{M}}(x, y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x + y + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \qquad \mathsf{nil}_{\mathcal{M}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example (SK90/4.30)

$$\begin{aligned} \mathsf{f}(\mathsf{nil}) &> \mathsf{nil} & \mathsf{g}(\mathsf{nil}) &\geqslant \mathsf{nil} \\ \mathsf{f}(\mathsf{nil} \circ y) &> \mathsf{nil} \circ \mathsf{f}(y) & \mathsf{g}(x \circ \mathsf{nil}) &\geqslant \mathsf{g}(x) \circ \mathsf{nil} \\ \mathsf{f}((x \circ y) \circ z) &> \mathsf{f}(x \circ (y \circ z)) & \mathsf{g}(x \circ (y \circ z)) &\geqslant \mathsf{g}((x \circ y) \circ z) \end{aligned}$$

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m^3)$

Proof

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \circ_{\mathcal{M}}(x, y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x + y + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \qquad \mathsf{nil}_{\mathcal{M}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example (SK90/4.30)

$$\begin{aligned} \mathsf{f}(\mathsf{nil}) &\geqslant \mathsf{nil} \\ \mathsf{f}(\mathsf{nil} \circ y) &\geqslant \mathsf{nil} \circ \mathsf{f}(y) \\ \mathsf{f}((x \circ y) \circ z) &\geqslant \mathsf{f}(x \circ (y \circ z)) \end{aligned} \qquad \begin{aligned} \mathsf{g}(\mathsf{nil}) &> \mathsf{nil} \\ \mathsf{g}(x \circ \mathsf{nil}) &> \mathsf{g}(x) \circ \mathsf{nil} \\ \mathsf{g}(x \circ (y \circ z)) &> \mathsf{g}((x \circ y) \circ z) \end{aligned}$$

Lemma

$\mathsf{dc}(m, \rightarrow_{\mathcal{R}}) = \mathcal{O}(m^3)$

Proof

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \circ_{\mathcal{M}}(x, y) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + y + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \qquad \mathsf{nil}_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Example (SK90/4.30)

$$\begin{aligned} \mathsf{f}(\mathsf{nil}) &\to \mathsf{nil} & \mathsf{g}(\mathsf{nil}) &\to \mathsf{nil} \\ \mathsf{f}(\mathsf{nil} \circ y) &\to \mathsf{nil} \circ \mathsf{f}(y) & \mathsf{g}(x \circ \mathsf{nil}) &\to \mathsf{g}(x) \circ \mathsf{nil} \\ \mathsf{f}((x \circ y) \circ z) &\to \mathsf{f}(x \circ (y \circ z)) & \mathsf{g}(x \circ (y \circ z)) &\to \mathsf{g}((x \circ y) \circ z) \end{aligned}$$

Lemma

$\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathcal{O}(m^3)$ $\qquad$ $\mathsf{dc}(m, \to_{\mathcal{R}}) = \Theta(m^2)$

Proof

$$\mathsf{f}_{\mathcal{M}}(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \circ_{\mathcal{M}}(x, y) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + y + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

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Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

direct
modular

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1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$
direct	168
modular	

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Experiments (direct vs modular)

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	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
direct	168	252
modular	193	

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
direct	168	252
modular	193	283

Experiments (direct vs modular)

1172 non-duplicating TRSs from TPDB 7.0.2

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$
direct	168	252	287
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1172 non-duplicating TRSs from TPDB 7.0.2

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Facts

$\approx$ 15% more linear bounds

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$\approx$ 15% ($\approx$ 12%) more linear (quadratic) bounds

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$\approx$ 15% ($\approx$ 12%, $\approx$ 9%) more linear (quadratic, polynomial) bounds

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Facts

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Corollary

modular setting more powerful (in practice)

Experiments (comparison with other tools)

Termination Competition 2009 (derivational complexity)

	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^4)$
1. CaT	84	125	132	137
2. Matchbox	44	96	101	102
3. $\text{T}_\text{C}\text{T}$	32	101	109	109

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Termination Competition 2010

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Termination Competition 2009 (derivational complexity)

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live during IJCAR

Conclusion

modular Complexity via Relative Complexity

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- $\mathsf{dc}(m, \to_{\mathcal{R}}) = \mathsf{dc}(m, \to_{\mathcal{R}/\varnothing}) \qquad \mathsf{dc}(m, \to_{\varnothing/\mathcal{S}}) = \mathcal{O}(1)$
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Complexity Pairs

triangular matrix interpretations, match-rt bounds, arctic matrix interpretations, beyond complexity pairs (weight gap principle)

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Complexity Pairs

triangular matrix interpretations, match-rt bounds, arctic matrix interpretations, beyond complexity pairs (weight gap principle)

Pros

easy to implement, partial proofs, more powerful, any (TMI) refinement directly applicable

Example (Strategy_removed_AG01/#4.28 and TCT_09/bits)

$$\begin{aligned} \mathsf{half}(0) &\to 0 \\ \mathsf{half}(\mathsf{s}(0)) &\to 0 \\ \mathsf{half}(\mathsf{s}(\mathsf{s}(x))) &\to \mathsf{s}(\mathsf{half}(x)) \end{aligned} \qquad \begin{aligned} \mathsf{bits}(0) &\to 0 \\ \\ \mathsf{bits}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x)))) \end{aligned}$$

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Details at WST'10 talk on July 15