Labelings for Decreasing Diagrams

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Quiz

Wikimedia

Overview

- Preliminaries
- Decreasing Diagrams
- Labelings for Decreasing Diagrams
- Experiments
- Conclusion
- Announcement

Preliminaries

Definition (ARS)

$\mathcal{A} = (A, \rightarrow)$

Definition (confluence)

${}^{*}\!\leftarrow \cdot \rightarrow^{*} \subseteq \rightarrow^{*} \cdot {}^{*}\!\leftarrow$

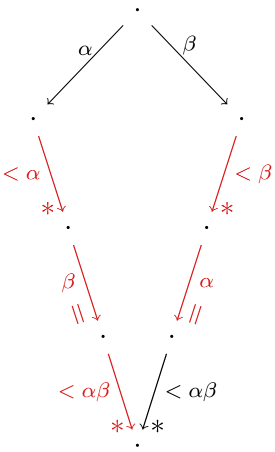
Definition (local confluence)

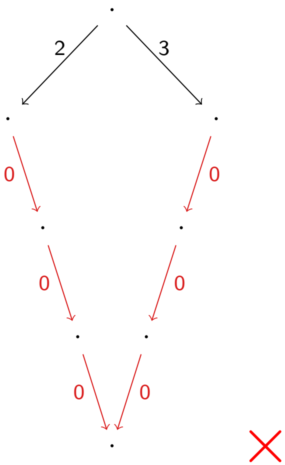
$\leftarrow \cdot \rightarrow \subseteq \rightarrow^{*} \cdot {}^{*}\!\leftarrow$

Theorem (van Oostrom, 1994)

every locally decreasing ARS is confluent

Decreasing Diagrams – Examples





Term Rewriting

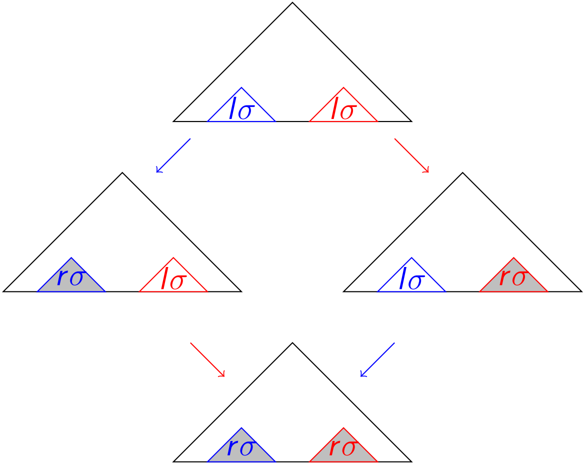
$t \leftarrow s \rightarrow u$

Three possibilities (modulo symmetry)

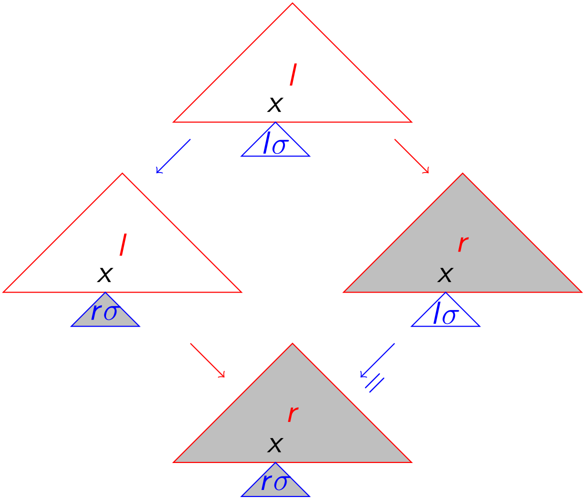
$t \;{}_{p,l\to r}\!\!\leftarrow s[l\sigma]_p = s = s[l\sigma]_q = l\!\!\rightarrow_{q,l\to r} u$

- $p \parallel q$ (parallel overlap)
- $p < q$ and $q \geqslant \mathcal{P}\mathrm{os}_{\mathcal{V}}(s[l]_p)$ (variable overlap)
- $p \leqslant q$ and $q \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(s[l]_p)$ (critical overlap)

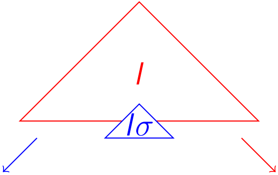
parallel



variable (linear)

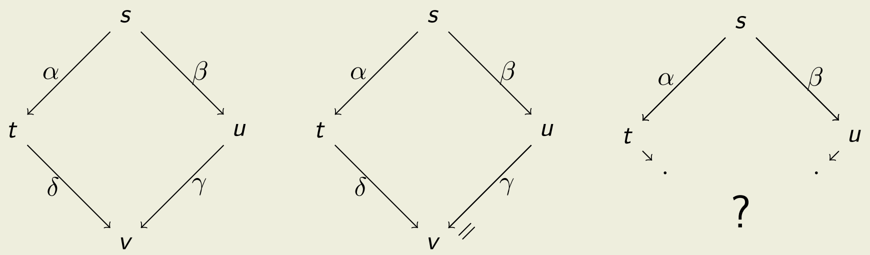


critical



?

linear TRSs (3 kinds of local peaks)



(a) parallel (b) variable linear (c) critical

Definition (labeling)

$\Gamma = s \rightarrow_{p,l\rightarrow r} t \qquad \Delta = u \rightarrow_{q,l'\rightarrow r'} v$

$(\ell, \geqslant, >)$ labeling if $\geqslant \cdot > \cdot \geqslant \subseteq >$

- $\ell(\Gamma) \geqslant \ell(\Delta) \longrightarrow \ell(C[\Gamma\sigma]) \geqslant \ell(C[\Delta\sigma])$
- $\ell(\Gamma) > \ell(\Delta) \longrightarrow \ell(C[\Gamma\sigma]) > \ell(C[\Delta\sigma])$

L-labeling

Definition (L-labeling)

labeling ℓ is L-labeling if $\alpha \geqslant \gamma$, $\beta \geqslant \delta$ (parallel, variable linear)

Theorem

$\mathcal{R}$ linear, critical diagrams decreasing for L-labeling $\longrightarrow$ $\mathcal{R}$ confluent

Example

$(\ell_{\mathsf{rl}}, \geqslant_{\mathbb{N}}, >_{\mathbb{N}})$ is L-labeling $(\ell_{\mathsf{rl}} : \mathcal{R} \to \mathbb{N})$
$(\ell_{\mathsf{sn}}, \to_{\mathcal{R}}^{*}, \to_{\mathcal{S}/\mathcal{R}}^{+})$ is L-labeling $(\ell_{\mathsf{sn}}(s \to t) = s)$
if $\to_{\mathcal{S}} \subseteq \to_{\mathcal{R}}$ and $\mathcal{S}/\mathcal{R}$ is terminating

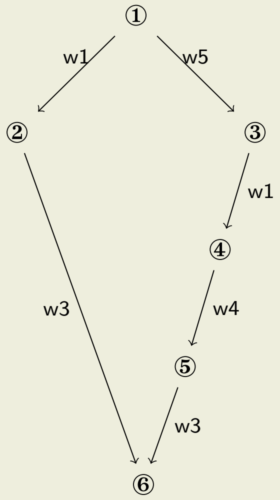
Lemma

ℓ_1, ℓ_2 *L-labelings* $\longrightarrow$ $\ell_1 \times \ell_2$ *L-labeling*

Example (van Oostrom, 2008)

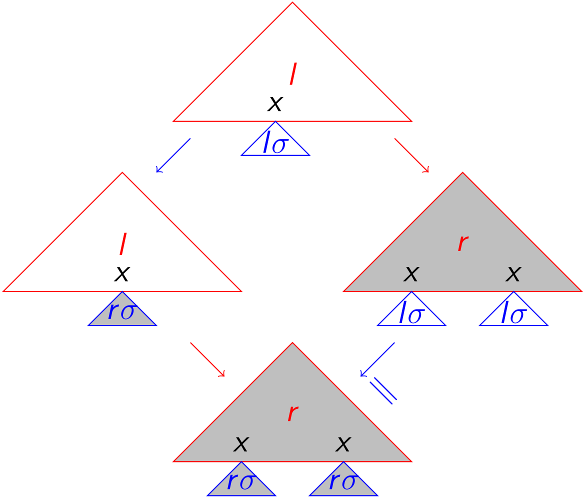
$$
\begin{aligned}
1:&\quad \textsf{nats} \to 0 : \textsf{inc}(\textsf{nats})\\
2:&\quad \textsf{hd}(x : y) \to x\\
3:&\quad \textsf{tl}(x : y) \to y\\
4:&\quad \textsf{inc}(x : y) \to \textsf{s}(x) : \textsf{inc}(y)\\
5:&\quad \textsf{inc}(\textsf{tl}(\textsf{nats})) \to \textsf{tl}(\textsf{inc}(\textsf{nats}))
\end{aligned}
$$

① inc(tl(nats))
② inc(tl(0 : inc(nats)))
③ tl(inc(nats))
④ tl(inc(0 : inc(nats)))
⑤ tl(s(0) : inc(inc(nats)))
⑥ inc(inc(nats))



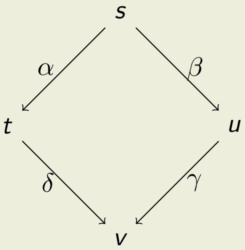
$w1 >_{\mathbb{N}} w3, w4$ shows confluence

variable (left-linear)

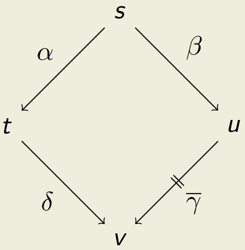


LL-labeling

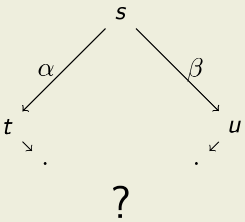
left-linear TRSs (3 kinds of local peaks)



(a) parallel



(b) variable left-linear



(c) critical

Definition (LL-labeling)

L-labeling ℓ is LL-labeling if $\alpha > \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

Theorem

$\mathcal{R}$ left-linear, critical diagrams decreasing for LL-labeling $\longrightarrow$ $\mathcal{R}$ confluent

Example

ℓ_{rl} is not LL-labeling

ℓ_{sn} is LL-labeling if $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ terminating

Definition (LL-labeling)

L-labeling ℓ is weak LL-labeling if $\alpha \geqslant \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

Example

ℓ_{rl} is weak LL-labeling

every LL-labeling is weak LL-labeling

Lemma

LL-labeling ℓ_1, weak LL-labeling ℓ_2 $\longrightarrow$ $\ell_1 \times \ell_2$, $\ell_2 \times \ell_1$ LL-labelings

First result

Corollary

$\mathcal{R}$ left-linear, $\mathcal{R}_d/\mathcal{R}_{nd}$ terminating, critical diagrams decreasing for weak LL-labeling $\longrightarrow$ *$\mathcal{R}$ confluent*

Example (OO03 cont'd)

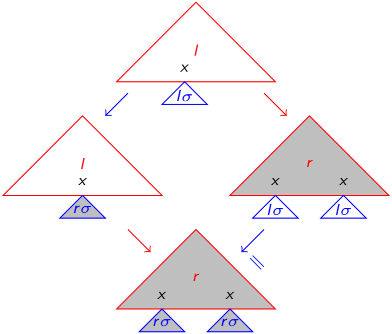
$$\begin{array}{llll}
1: & x + (y + z) \to (x + y) + z & 2: & (x + y) + z \to x + (y + z) \\
3: & \mathsf{s}(x) + y \to x + \mathsf{s}(y) & 4: & x + \mathsf{s}(y) \to \mathsf{s}(x) + y \\
5: & x \times \mathsf{s}(y) \to x + (x \times y) & 6: & \mathsf{s}(x) \times y \to (x \times y) + y \\
7: & x + y \to y + x & 8: & x \times y \to y \times x \\
9: & \mathsf{sq}(x) \to x \times x & 10: & \mathsf{sq}(\mathsf{s}(x)) \to (x \times x) + \mathsf{s}(x + x)
\end{array}$$

$\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ terminating

$+_{\mathbb{N}}(x, y) = x + y \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad \times_{\mathbb{N}}(x, y) = x^2 + xy + y^2 \quad \mathsf{sq}_{\mathbb{N}}(x) = 3x^2 + 1$

weak LL-labeling $\ell_{\mathsf{rl}}(8) = \ell_{\mathsf{rl}}(9) = 2$, $\ell_{\mathsf{rl}}(6) = \ell_{\mathsf{rl}}(10) = 1$, $\ell_{\mathsf{rl}}(\cdot) = 0$

Towards a second result



Example

$$1 : \mathsf{f}(\mathsf{g}(x, \mathsf{a})) \to \mathsf{g}(\mathsf{f}(x), \mathsf{f}(x)) \quad 2 : \mathsf{a} \to \mathsf{b} \quad 3 : \mathsf{b} \to \mathsf{a} \quad 4 : \mathsf{h}(x) \to \mathsf{h}(x)$$

$\mathcal{R}^\star_> = \{\mathsf{f}_1(\mathsf{g}_1(x)) \to \mathsf{g}_1(\mathsf{f}_1(x)), \mathsf{f}_1(\mathsf{g}_1(x)) \to \mathsf{g}_2(\mathsf{f}_1(x))\}$
$\mathcal{R}^\star_= = \{\mathsf{h}_1(x) \to \mathsf{h}_1(x)\}$

Definition

$\ell_\star(s \to_{p,l\to r} t) = \star(s,p)$

Lemma

$\mathcal{R}^\star_>/\mathcal{R}^\star_=$ *terminating* $\longrightarrow$ $(\ell_\star, \to^*_{\mathcal{R}^\star_=}, \to^+_{\mathcal{R}^\star_>/\mathcal{R}^\star_=})$ *LL-labeling*

Corollary

$\mathcal{R}$ *left-linear,* $\mathcal{R}^\star_>/\mathcal{R}^\star_=$ *terminating,* ℓ *weak LL-labeling, critical peaks decreasing for* $\ell_\star \times \ell$ $\longrightarrow$ $\mathcal{R}$ *confluent*

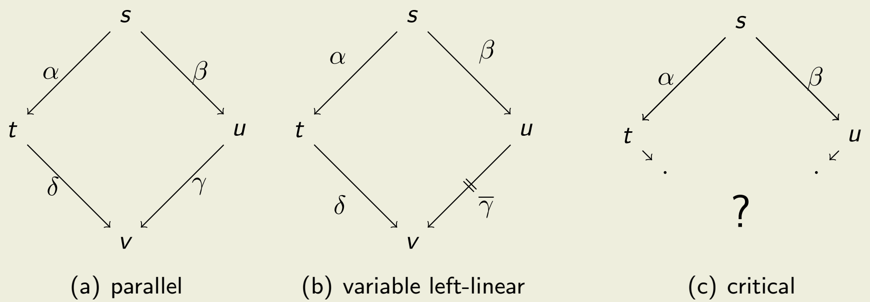
Example (OO03)

$$
\begin{array}{llll}
1: & x + (y + z) \to (x + y) + z & 2: & (x + y) + z \to x + (y + z) \\
3: & \mathsf{s}(x) + y \to x + \mathsf{s}(y) & 4: & x + \mathsf{s}(y) \to \mathsf{s}(x) + y \\
5: & x \times \mathsf{s}(y) \to x + (x \times y) & & \dots
\end{array}
$$

$\mathcal{R}^\star_>$ contains nonterminating $\times_1(x) \to +_2(\times_1(x))$

LL-labeling

left-linear TRSs (3 kinds of local peaks)



Definition (LL-labeling)

L-labeling ℓ is LL-labeling if $\alpha > \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

$\alpha \geqslant \gamma_1$, $\alpha > \gamma_i$ $(i > 1)$

Example (OO03)

$\mathcal{R}^\star_>$		$\mathcal{R}^\star_=$
$\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_1(\times_1(x))$	$\times_1(x) \to \times_2(x)$	$\times_1(\mathsf{s}_1(x)) \to +_1(\times_1(x))$
$\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_1(\times_2(x))$	$\times_2(y) \to \times_1(y)$	$\times_2(\mathsf{s}_1(y)) \to +_2(\times_2(y))$
$\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_2(\mathsf{s}_1(+_1(x)))$	$+_1(x) \to +_2(x)$	$+_1(\mathsf{s}_1(x)) \to +_1(x)$
$\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_2(\mathsf{s}_1(+_2(x)))$	$+_2(y) \to +_1(y)$	$+_1(+_1(x)) \to +_1(x)$
$\mathsf{sq}_1(x) \to \times_1(x)$	$+_1(x) \to +_1(\mathsf{s}_1(x))$	$+_1(+_2(y)) \to +_2(+_1(y))$
$\mathsf{sq}_1(x) \to \times_2(x)$	$+_1(x) \to +_1(+_1(x))$	$+_2(+_2(z)) \to +_2(z)$
$\times_2(y) \to +_1(\times_2(y))$	$+_2(z) \to +_2(+_2(z))$	$+_2(\mathsf{s}_1(y)) \to +_2(y)$
$\times_2(y) \to +_2(y)$	$+_2(y) \to +_2(\mathsf{s}_1(y))$	$+_2(+_1(y)) \to +_1(+_2(y))$
$\times_1(x) \to +_1(x)$	$\times_1(x) \to +_2(\times_1(x))$	$\times_2(y) \to +_1(\times_2(y))$
$\times_1(x) \to +_2(\times_1(x))$		

$\mathsf{sq}_{1\mathbb{N}}(x) = x + 2 \quad \times_{1\mathbb{N}}(x) = \times_{2\mathbb{N}}(x) = x + 1 \quad +_{1\mathbb{N}}(x) = +_{2\mathbb{N}}(x) = \mathsf{s}_1(x) = x$

Quiz

Google Maps

C S I

Experiments

53 left-linear TRSs (30 years of confluence literature)

method	pre	CR(ℓ_{rl})	CR(ℓ_{sn})	CR
rule labeling	40	35	–	35
SN($\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$)	45	40	37	42
SN($\mathcal{R}^{\star}_{>}/\mathcal{R}^{\star}_{=}$)	45	42	34	42
SN($\mathcal{R}^{\star\star}_{>}/\mathcal{R}^{\star\star}_{=}$)	48	45	36	45
ACP				48
CSI				49

Conclusion

Summary

- decreasing diagrams
- (linear) ℓ_{rl}, ℓ_{sn}, $\ell_{\star}$
- (left-linear) $\mathsf{SN}(\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}})$, $\mathsf{SN}(\mathcal{R}^{\star}_{>}/\mathcal{R}^{\star}_{=})$, $\mathsf{SN}(\mathcal{R}^{\star\star}_{>}/\mathcal{R}^{\star\star}_{=})$

Future Work

- study parallel reduction

Implementation

Label with $\mathbb{N}$ (for lexicographic combination)

ℓ_{rl} ✓ ℓ_{sn} ✓ $\ell_\star$ ✓

ℓ_{rl}

(Aoto, 2010), (Hirokawa & Middeldorp, 2010)

ℓ_{sn}

peak $t \leftarrow s \rightarrow u$

partial termination proof of $\mathsf{CDS}(\mathcal{R})/\mathcal{R} \longrightarrow \mathcal{S}/\mathcal{R}'$

$\ell_{\mathsf{sn}}(t_i \rightarrow t_{i+1}) = 1$ if $s \rightarrow_{\mathcal{S}}^{*} t_i$

$\ell_{\mathsf{sn}}(t_i \rightarrow t_{i+1}) = 0$ otherwise

$\ell_\star$

prove termination of $\mathcal{R}_{>}^{\star\star}/\mathcal{R}_{=}^{\star\star}$ with matrix interpretations

Confluence Competition 2012

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`http://coco.nue.riec.tohoku.ac.jp`