Satisfiability of Non-Linear Arithmetic over Algebraic Numbers

Harald Zankl René Thiemann Aart Middeldorp

Institute of Computer Science
University of Innsbruck
Austria

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SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$b + b$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$b + b \qquad 3 \times b$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$(b + b) \quad (3 \times b)$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$(b + b) > (3 \times b)$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$(\mathsf{b} + \mathsf{b}) > (3 \times \mathsf{b}) \qquad \neg x$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$((\mathsf{b} + \mathsf{b}) > (3 \times \mathsf{b})) \vee (\neg x)$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$((\mathsf{b} + \mathsf{b}) > (3 \times \mathsf{b})) \vee (\neg x)$ $\quad$ $\mathsf{a} \times \mathsf{a} = 2$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$$(((\mathsf{b} + \mathsf{b}) > (3 \times \mathsf{b})) \vee (\neg x)) \wedge (\mathsf{a} \times \mathsf{a} = 2)$$

SMT Solving ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)

Example

$(((b + b) > (3 \times b)) \vee (\neg x)) \wedge (a \times a = 2)$

Facts

undecidable over $\mathbb{N}$, $\mathbb{Z}$ (Hilbert's 10th problem, Matiyasevich 1970)

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until 2009 no fast solvers

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Remark

such constraints appear in hard-/software verification

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SMT-COMP

- competition since 2005
- non-linear integer arithmetic (QF_NIA) since 2009
- non-linear real arithmetic (QF_NRA) since 2010

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State-of-the-Art Solvers ($\mathbb{N}$, $\mathbb{Z}$)

no solver handles non-linear arithmetic directly

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- CVC3 : linear arithmetic (Fourier-Motzkin)

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- MiniSmt : bit-blasts to SAT and SMT modulo bit-vectors

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- AProVE-NIA: bit-blasts to SAT

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Overview

- Introduction
- Non-Linear (Ir)rational Arithmetic
  - Extension by $\sqrt{2}$
  - Extension by $\sqrt[n]{m}$
- Evaluation
- Conclusion

Arithmetic over $\mathbb{R}$ ($\mathbb{Q} \cup \{\sqrt{2}\}$)

$(\mathbf{c}, \mathbf{d})$ in **R** with $\mathbf{c}, \mathbf{d}$ from **Q** meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

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Example

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \quad = \quad (\mathbf{5}, \mathbf{11})$

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Example

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} =$ $(\mathbf{5}, \mathbf{11})$

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Example

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2} \equiv (\mathbf{5}, \mathbf{11})$

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$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \quad = \quad (\mathbf{66}, \mathbf{28})$

Arithmetic over $\mathbb{R}$ ($\mathbb{Q} \cup \{\sqrt{2}\}$)

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$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = \qquad (\mathbf{66}, \mathbf{28})$

Arithmetic over $\mathbb{R}$ $(\mathbb{Q} \cup \{\sqrt{2}\})$

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$(\mathbf{3}, \mathbf{5}) \times_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv (3 + 5\sqrt{2}) \times (2 + 6\sqrt{2}) = 66 + 28\sqrt{2} \equiv (\mathbf{66}, \mathbf{28})$

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Definition

$$(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) \; := \; (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f})$$

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\begin{aligned}
(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &:= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &:= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e})
\end{aligned}
$$

Arithmetic over $\mathbb{R}$ ($\mathbb{Q} \cup \{\sqrt{2}\}$)

$(\mathbf{c}, \mathbf{d})$ in $\mathbf{R}$ with $\mathbf{c}, \mathbf{d}$ from $\mathbf{Q}$ meaning $(\mathbf{3}, \mathbf{5}) \equiv 3 + 5\sqrt{2}$

Example

$(\mathbf{3}, \mathbf{5}) +_{\mathbf{R}} (\mathbf{2}, \mathbf{6}) \equiv 3 + 5\sqrt{2} + 2 + 6\sqrt{2} = 5 + 11\sqrt{2} \equiv (\mathbf{5}, \mathbf{11})$

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(\mathbf{c}, \mathbf{d}) +_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &:= (\mathbf{c} +_{\mathbf{Q}} \mathbf{e}, \mathbf{d} +_{\mathbf{Q}} \mathbf{f}) \\
(\mathbf{c}, \mathbf{d}) \times_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &:= (\mathbf{c} \times_{\mathbf{Q}} \mathbf{e} +_{\mathbf{Q}} \mathbf{2} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{f}, \mathbf{c} \times_{\mathbf{Q}} \mathbf{f} +_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{e}) \\
(\mathbf{c}, \mathbf{d}) =_{\mathbf{R}} (\mathbf{e}, \mathbf{f}) &:= \mathbf{c} =_{\mathbf{Q}} \mathbf{e} \wedge \mathbf{d} =_{\mathbf{Q}} \mathbf{f}
\end{aligned}
$$

Arithmetic over $\mathbb{R}$ (cont'd)

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

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$6 + 3\sqrt{2} >_{\mathbb{R}} 2 + 5\sqrt{2}$

Arithmetic over $\mathbb{R}$ (cont'd)

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3\sqrt{2} >_{\mathbb{R}} 2 + 5\sqrt{2}$ $\quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

Arithmetic over $\mathbb{R}$ (cont'd)

Problem

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3 \times \frac{5}{4} >_{\mathbb{Q}} 2 + 5 \times \frac{3}{2}$ $\quad \frac{5}{4} <_{\mathbb{R}} \sqrt{2} <_{\mathbb{R}} \frac{3}{2}$

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Approximating $m\sqrt{2}$

$$\text{under}(m\sqrt{2}) = \left((m \geqslant 0)\,?\,\frac{5}{4} : \frac{3}{2}\right) \times m$$

Arithmetic over $\mathbb{R}$ (cont'd)

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How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$?

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Definition (LPAR-16)

$$(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} (\mathbf{e},\mathbf{f}) \;:=\; \mathbf{c} +_{\mathbf{Q}} \text{under}(\mathbf{d}) \;>_{\mathbf{Q}}\; \mathbf{e} +_{\mathbf{Q}} \text{over}(\mathbf{f})$$

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3\sqrt{2} > 2 + 5\sqrt{2}$

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$?

$6+3\sqrt{2} > 2+5\sqrt{2} \leftrightarrow 4-2\sqrt{2} > 0$

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \leftrightarrow 4 - 2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2}$

Arithmetic over $\mathbb{R}$ (cont'd)

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How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \leftrightarrow 4 - 2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2} \leftrightarrow 16 > 4 \times 2$

Arithmetic over $\mathbb{R}$ (cont'd)

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How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?

$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \leftrightarrow 4 - 2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2} \leftrightarrow 16 > 4 \times 2 \leftrightarrow 16 > 8$

Arithmetic over $\mathbb{R}$ (cont'd)

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Definition (SCSS 2010)

$$(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} 0 \;:=\; (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0)$$

Arithmetic over $\mathbb{R}$ (cont'd)

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How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?
$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \leftrightarrow 4 - 2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2} \leftrightarrow 16 > 4 \times 2 \leftrightarrow 16 > 8 \leftrightarrow \top$

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$$(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} 0 \;:= (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \land \mathbf{d} >_{\mathbf{Q}} 0) \lor (\mathbf{c} >_{\mathbf{Q}} 0 \land \mathbf{d} \geqslant_{\mathbf{Q}} 0)$$

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$?

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with $\varphi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?
$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \leftrightarrow 4 - 2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2} \leftrightarrow 16 > 4 \times 2 \leftrightarrow 16 > 8 \leftrightarrow \top$

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$$
\begin{aligned}
(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} 0 \; := \; & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0) \vee (\mathbf{c} >_{\mathbf{Q}} 0 \wedge \mathbf{d} \geqslant_{\mathbf{Q}} 0) \vee \\
& (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} <_{\mathbf{Q}} 0 \wedge \varphi) \vee (\mathbf{c} \leqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0 \wedge \chi)
\end{aligned}
$$

with $\varphi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$
and $\chi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} <_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$?
$6+3\sqrt{2} > 2+5\sqrt{2} \leftrightarrow 4-2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2} \leftrightarrow 16 > 4\times 2 \leftrightarrow 16 > 8 \leftrightarrow \top$

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with $\varphi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$
and $\chi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} <_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$

Pros

no approximation

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6}, \mathbf{3}) >_{\mathbf{R}} (\mathbf{2}, \mathbf{5})$?
$6 + 3\sqrt{2} > 2 + 5\sqrt{2} \leftrightarrow 4 - 2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2} \leftrightarrow 16 > 4 \times 2 \leftrightarrow 16 > 8 \leftrightarrow \top$

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$$\begin{aligned}(\mathbf{c}, \mathbf{d}) >_{\mathbf{R}} 0 \ := \ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0) \vee (\mathbf{c} >_{\mathbf{Q}} 0 \wedge \mathbf{d} \geqslant_{\mathbf{Q}} 0) \vee \\ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} <_{\mathbf{Q}} 0 \wedge \varphi) \vee (\mathbf{c} \leqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0 \wedge \chi)\end{aligned}$$

with $\varphi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$
and $\chi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} <_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$

Pros

no approximation, arbitrary base

Arithmetic over $\mathbb{R}$ (cont'd)

Alternative

How to test $(\mathbf{6},\mathbf{3}) >_{\mathbf{R}} (\mathbf{2},\mathbf{5})$?
$6+3\sqrt{2} > 2+5\sqrt{2} \leftrightarrow 4-2\sqrt{2} > 0 \leftrightarrow 4 > 2\sqrt{2} \leftrightarrow 16 > 4\times 2 \leftrightarrow 16 > 8 \leftrightarrow \top$

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$$\begin{aligned}(\mathbf{c},\mathbf{d}) >_{\mathbf{R}} 0 \;:=\; & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0) \vee (\mathbf{c} >_{\mathbf{Q}} 0 \wedge \mathbf{d} \geqslant_{\mathbf{Q}} 0) \vee \\ & (\mathbf{c} \geqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} <_{\mathbf{Q}} 0 \wedge \varphi) \vee (\mathbf{c} \leqslant_{\mathbf{Q}} 0 \wedge \mathbf{d} >_{\mathbf{Q}} 0 \wedge \chi)\end{aligned}$$

with $\varphi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} >_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$
and $\chi = \mathbf{c} \times_{\mathbf{Q}} \mathbf{c} <_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} \mathbf{d} \times_{\mathbf{Q}} 2$

Pros & Cons

no approximation, arbitrary base, auxiliary multiplications

Some Examples

Example

$((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a = 2)$

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$((b + b > 3 \times b) \vee (b > -a)) \wedge (a \times a \times a = 2)$ ✗

Solution

extend by $\sqrt[n]{m}$

$(\mathbf{c}_n, \mathbf{m})$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m})$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \dots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Example

$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3})$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Example

$$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2}$$

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Example

$$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$$

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$$([\mathbf{0}, \mathbf{1}], \mathbf{4})$$

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$$([\mathbf{0}, \mathbf{1}], \mathbf{4}) \equiv 0\sqrt[2]{4^0} + 1\sqrt[2]{4^1}$$

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$$([\mathbf{0}, \mathbf{1}], \mathbf{4}) \equiv 0\sqrt[2]{4^0} + 1\sqrt[2]{4^1} = 2$$

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Remark

no canonical representation

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \dots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

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Remark

no canonical representation

Example (Addition)

$$([\mathbf{1}, \mathbf{2}], \mathbf{3}) +_{\mathbf{R}} ([\mathbf{5}, \mathbf{3}], \mathbf{3}) \quad = \quad ([\mathbf{6}, \mathbf{5}], \mathbf{3})$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Example

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Remark

no canonical representation

Example (Addition)

$$([\mathbf{1}, \mathbf{2}], \mathbf{3}) +_{\mathbf{R}} ([\mathbf{5}, \mathbf{3}], \mathbf{3}) \equiv (1 + 2\sqrt[2]{3}) + (5 + 3\sqrt[2]{3}) = \qquad ([\mathbf{6}, \mathbf{5}], \mathbf{3})$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \ldots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

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$$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$$

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$$([\mathbf{1}, \mathbf{2}], \mathbf{3}) +_{\mathbf{R}} ([\mathbf{5}, \mathbf{3}], \mathbf{3}) \equiv (1 + 2\sqrt[2]{3}) + (5 + 3\sqrt[2]{3}) = 6 + 5\sqrt[2]{3} \equiv ([\mathbf{6}, \mathbf{5}], \mathbf{3})$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \dots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

Example

$$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$$

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$$([\mathbf{1}, \mathbf{2}], \mathbf{3}) +_{\mathbf{R}} ([\mathbf{5}, \mathbf{3}], \mathbf{3}) \equiv (1 + 2\sqrt[2]{3}) + (5 + 3\sqrt[2]{3}) = 6 + 5\sqrt[2]{3} \equiv ([\mathbf{6}, \mathbf{5}], \mathbf{3})$$

Definition (Addition)

$$(\mathbf{c}_n, m) +_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; ([c_1 +_{\mathbf{Q}} d_1, \dots, c_n +_{\mathbf{Q}} d_n], m)$$

$$(\mathbf{c}_n, \mathbf{m}) = ([\mathbf{c_1}, \dots, \mathbf{c_n}], \mathbf{m}) \equiv c_1 \sqrt[n]{m^0} + \cdots + c_n \sqrt[n]{m^{n-1}}$$

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$([\mathbf{2}, \mathbf{5}, \mathbf{4}], \mathbf{3}) \equiv 2\sqrt[3]{3^0} + 5\sqrt[3]{3^1} + 4\sqrt[3]{3^2} = 2 + 5\sqrt[3]{3} + 4\sqrt[3]{9}$

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Remark

no canonical representation

Example (Addition)

$([\mathbf{1}, \mathbf{2}], \mathbf{3}) +_{\mathbf{R}} ([\mathbf{5}, \mathbf{3}], \mathbf{3}) \equiv (1 + 2\sqrt[2]{3}) + (5 + 3\sqrt[2]{3}) = 6 + 5\sqrt[2]{3} \equiv ([\mathbf{6}, \mathbf{5}], \mathbf{3})$

Definition (Addition)

$$(\mathbf{c}_n, m) +_{\mathbf{R}} (\mathbf{d}_n, m) := ([c_1 +_{\mathbf{Q}} d_1, \dots, c_n +_{\mathbf{Q}} d_n], m)$$

$$(\mathbf{c}_n, m) -_{\mathbf{R}} (\mathbf{d}_n, m) := ([c_1 -_{\mathbf{Q}} d_1, \dots, c_n -_{\mathbf{Q}} d_n], m)$$

Example (Multiplication)

Example (Multiplication)

$([1,2],2) \times_{\mathsf{R}} ([5,3],2)$

Example (Multiplication)

$$([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) = ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1)$$

Example (Multiplication)

$$
\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= (([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} (([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1)
\end{aligned}
$$

Example (Multiplication)

$$
\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= (([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} (([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\
&= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2})
\end{aligned}
$$

Example (Multiplication)

$$\begin{aligned}(([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\ &= ((([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} ((([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\ &= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})\end{aligned}$$

Example (Multiplication)

$$\begin{aligned}([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\ &= ((([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} ((([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\ &= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})\end{aligned}$$

Definition (Multiplication)

$$(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; ((\mathbf{c}_n, m) \cdot d_1) \gg 0$$

Example (Multiplication)

$$\begin{aligned}(\mathbf{[1,2]},\mathbf{2}) \times_{\mathbf{R}} (\mathbf{[5,3]},\mathbf{2}) &= (((\mathbf{[1,2]},\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} (((\mathbf{[1,2]},\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\ &= ((\mathbf{[5,10]},\mathbf{2}) \gg 0) +_{\mathbf{R}} ((\mathbf{[3,6]},\mathbf{2}) \gg 1) \\ &= (\mathbf{[5,10]},\mathbf{2}) +_{\mathbf{R}} (\mathbf{[12,3]},\mathbf{2}) = (\mathbf{[17,13]},\mathbf{2})\end{aligned}$$

Definition (Multiplication)

$$(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) \ := \ ((\mathbf{c}_n, m) \cdot d_1) \gg 0 +_{\mathbf{R}} \cdots +_{\mathbf{R}} ((\mathbf{c}_n, m) \cdot d_n) \gg (n-1)$$

Example (Multiplication)

$$\begin{aligned}(([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} (((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\ &= ((([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} ((([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\ &= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})\end{aligned}$$

Definition (Multiplication)

$$\begin{aligned}(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) &:= ((\mathbf{c}_n, m) \cdot d_1) \gg 0 +_{\mathbf{R}} \cdots +_{\mathbf{R}} ((\mathbf{c}_n, m) \cdot d_n) \gg (n-1) \\ (\mathbf{c}_n, m) \cdot d &:= ([c_1 \times_{\mathbf{Q}} d, \ldots, c_n \times_{\mathbf{Q}} d], m) \\ (\mathbf{c}_n, m) \gg 0 &:= (\mathbf{c}_n, m) \\ (\mathbf{c}_n, m) \gg (i+1) &:= ([m \times_{\mathbf{Q}} c_n, c_1, \ldots, c_{n-1}], m) \gg i\end{aligned}$$

Example (Multiplication)

$$
\begin{aligned}
([\mathbf{1},\mathbf{2}],\mathbf{2}) \times_{\mathbf{R}} ([\mathbf{5},\mathbf{3}],\mathbf{2}) &= ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{5}) \gg 0) +_{\mathbf{R}} ((([\mathbf{1},\mathbf{2}],\mathbf{2}) \cdot \mathbf{3}) \gg 1) \\
&= (([\mathbf{5},\mathbf{10}],\mathbf{2}) \gg 0) +_{\mathbf{R}} (([\mathbf{3},\mathbf{6}],\mathbf{2}) \gg 1) \\
&= ([\mathbf{5},\mathbf{10}],\mathbf{2}) +_{\mathbf{R}} ([\mathbf{12},\mathbf{3}],\mathbf{2}) = ([\mathbf{17},\mathbf{13}],\mathbf{2})
\end{aligned}
$$

Definition (Multiplication)

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\begin{aligned}
(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) &:= ((\mathbf{c}_n, m) \cdot d_1) \gg 0 +_{\mathbf{R}} \cdots +_{\mathbf{R}} ((\mathbf{c}_n, m) \cdot d_n) \gg (n-1) \\
(\mathbf{c}_n, m) \cdot d &:= ([c_1 \times_{\mathbf{Q}} d, \ldots, c_n \times_{\mathbf{Q}} d], m) \\
(\mathbf{c}_n, m) \gg 0 &:= (\mathbf{c}_n, m) \\
(\mathbf{c}_n, m) \gg (i+1) &:= ([m \times_{\mathbf{Q}} c_n, c_1, \ldots, c_{n-1}], m) \gg i
\end{aligned}
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Example (Multiplication)

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(\mathbf{c}_n, m) \times_{\mathbf{R}} (\mathbf{d}_n, m) &:= ((\mathbf{c}_n, m) \cdot d_1) \gg 0 +_{\mathbf{R}} \cdots +_{\mathbf{R}} ((\mathbf{c}_n, m) \cdot d_n) \gg (n-1) \\
(\mathbf{c}_n, m) \cdot d &:= ([c_1 \times_{\mathbf{Q}} d, \ldots, c_n \times_{\mathbf{Q}} d], m) \\
(\mathbf{c}_n, m) \gg 0 &:= (\mathbf{c}_n, m) \\
(\mathbf{c}_n, m) \gg (i+1) &:= ([m \times_{\mathbf{Q}} c_n, c_1, \ldots, c_{n-1}], m) \gg i
\end{aligned}
$$

Definition

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

Definition

$$(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n)$$

$$(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) \;:=\; (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m)$$

Definition

$$
\begin{aligned}
(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) &:= (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n) \\
(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) &:= (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m) \\
(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) &:= a >_{\mathbf{Q}} 0
\end{aligned}
$$

Definition

$$\begin{aligned}
(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) &:= (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n) \\
(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) &:= (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m) \\
(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) &:= a >_{\mathbf{Q}} 0 \\
(\mathbf{c}_{k+1}, m) >^a_{\mathbf{R}} (\mathbf{d}_{k+1}, m) &:= c_{k+1} +_{\mathbf{Q}} a \geqslant_{\mathbf{Q}} d_{k+1}
\end{aligned}$$

Definition

$$
\begin{aligned}
(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) &:= (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n) \\
(\mathbf{c}_n, m) >_{\mathbf{R}} (\mathbf{d}_n, m) &:= (\mathbf{c}_n, m) >^0_{\mathbf{R}} (\mathbf{d}_n, m) \\
(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) &:= a >_{\mathbf{Q}} 0 \\
(\mathbf{c}_{k+1}, m) >^a_{\mathbf{R}} (\mathbf{d}_{k+1}, m) &:= c_{k+1} +_{\mathbf{Q}} a \geqslant_{\mathbf{Q}} d_{k+1} \\
&\ \wedge\ (\mathbf{c}_k, m) >^{a+(c_{k+1}-d_{k+1})}_{\mathbf{R}} (\mathbf{d}_k, m)
\end{aligned}
$$

Definition

$$
\begin{aligned}
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&\quad \wedge\ (\mathbf{c}_k, m) >^{a+(c_{k+1}-d_{k+1})}_{\mathbf{R}} (\mathbf{d}_k, m)
\end{aligned}
$$

Example

$([2, 0], 4) \neq_{\mathbf{R}} ([0, 1], 4)$

Definition

$$
\begin{aligned}
(\mathbf{c}_n, m) =_{\mathbf{R}} (\mathbf{d}_n, m) &:= (c_1 =_{\mathbf{Q}} d_1) \wedge \cdots \wedge (c_n =_{\mathbf{Q}} d_n) \\
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\end{aligned}
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Example

$([\mathbf{2}, \mathbf{0}], \mathbf{4}) \neq_{\mathbf{R}} ([\mathbf{0}, \mathbf{1}], \mathbf{4})$ but $([\mathbf{2}, \mathbf{0}], \mathbf{4}) \equiv ([\mathbf{0}, \mathbf{1}], \mathbf{4})$

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\end{aligned}
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Example

$([\mathbf{2}, \mathbf{0}], \mathbf{4}) \neq_{\mathbf{R}} ([\mathbf{0}, \mathbf{1}], \mathbf{4})$ but $([\mathbf{2}, \mathbf{0}], \mathbf{4}) \equiv ([\mathbf{0}, \mathbf{1}], \mathbf{4})$

$([\mathbf{2}, \mathbf{4}], \mathbf{2}) >_{\mathbf{R}} ([\mathbf{3}, \mathbf{2}], \mathbf{2})$

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(\mathbf{c}_0, m) >^a_{\mathbf{R}} (\mathbf{d}_0, m) &:= a >_{\mathbf{Q}} 0 \\
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$([\mathbf{2}, \mathbf{4}], \mathbf{2}) >_{\mathbf{R}} ([\mathbf{3}, \mathbf{2}], \mathbf{2}) \equiv 2 + 4\sqrt{2} > 3 + 2\sqrt{2}$

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$([\mathbf{5}, \mathbf{1}], \mathbf{2}) \not>_{\mathbf{R}} ([\mathbf{2}, \mathbf{3}], \mathbf{2})$

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$([\mathbf{5}, \mathbf{1}], \mathbf{2}) \not>_{\mathbf{R}} ([\mathbf{2}, \mathbf{3}], \mathbf{2})$ but $5 + \sqrt{2} > 6.41 > 6.24 > 2 + 3\sqrt{2}$

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Problem

$=_{\mathbf{R}}$ and $>_{\mathbf{R}}$ may not appear at “negative” positions

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Problem

$=_{\mathbf{R}}$ and $>_{\mathbf{R}}$ may not appear at "negative" positions

Solution

ensure canonical representation

Definition

$$
\begin{aligned}
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Example

$([2,0],4) \neq_{\mathbf{R}} ([0,1],4)$ but $([2,0],4) \equiv ([0,1],4)$
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$([5,1],2) \not>_{\mathbf{R}} ([2,3],2)$ but $5+\sqrt{2} > 6.41 > 6.24 > 2+3\sqrt{2}$

Problem

$=_{\mathbf{R}}$ and $>_{\mathbf{R}}$ may not appear at "negative" positions

Solution for $=_{\mathbf{R}}$

ensure canonical representation $\longrightarrow$ $\sqrt[n]{m^p} \notin \mathbb{Z} \quad (1 \leqslant p < n)$

Definition

$$\begin{aligned}
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\end{aligned}$$

Example

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$([2,4],2) >_{\mathbf{R}} ([3,2],2) \equiv 2+4\sqrt{2} > 3+2\sqrt{2} \leftrightarrow \top$ since $2\sqrt{2} > 1$
$([5,1],2) \not>_{\mathbf{R}} ([2,3],2)$ but $5+\sqrt{2} > 6.41 > 6.24 > 2+3\sqrt{2}$

Problem

$=_{\mathbf{R}}$ and $>_{\mathbf{R}}$ may not appear at "negative" positions

Solution for $>_{\mathbf{R}}$

replace $>_{\mathbf{R}}$ at negative positions by approximation of $\leqslant_{\mathbf{R}}$

Evaluation

Implemented in

MiniSmt (http://cl-informatik.uibk.ac.at/software/minismt)

Evaluation

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Comparison (2×1391 problems from termination analysis)

Evaluation

Implemented in

MiniSmt (`http://cl-informatik.uibk.ac.at/software/minismt`)

Comparison (2×1391 problems from termination analysis)

	$(2,4)$		$(3,4)$		$(3,5)$	
	lpar	scss	lpar	scss	lpar	scss

Evaluation

Implemented in

MiniSmt (`http://cl-informatik.uibk.ac.at/software/minismt`)

Comparison (2×1391 problems from termination analysis)

	$(2,4)$		$(3,4)$		$(3,5)$	
	lpar	scss	lpar	scss	lpar	scss
	sat	sat	sat	sat	sat	sat
matrix 1	308	308	306	303	311	294

Evaluation

Implemented in

MiniSmt (`http://cl-informatik.uibk.ac.at/software/minismt`)

Comparison (2 × 1391 problems from termination analysis)

	(2, 4)				(3, 4)				(3, 5)			
	lpar		scss		lpar		scss		lpar		scss	
	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg
matrix 1	308	1.5	308	3.0	306	2.0	303	4.3	311	3.8	294	5.5

Evaluation

Implemented in

MiniSmt (`http://cl-informatik.uibk.ac.at/software/minismt`)

Comparison (2×1391 problems from termination analysis)

	$(2,4)$				$(3,4)$				$(3,5)$			
	lpar		scss		lpar		scss		lpar		scss	
	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg
matrix 1	308	1.5	308	3.0	306	2.0	303	4.3	311	3.8	294	5.5
matrix 2	296		276		285		264		237		221	

Evaluation

Implemented in

MiniSmt (`http://cl-informatik.uibk.ac.at/software/minismt`)

Comparison (2×1391 problems from termination analysis)

	$(2,4)$				$(3,4)$				$(3,5)$			
	lpar		scss		lpar		scss		lpar		scss	
	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg	sat	avg
matrix 1	308	1.5	308	3.0	306	2.0	303	4.3	311	3.8	294	5.5
matrix 2	296	7.0	276	8.4	285	7.8	264	11.6	237	11.0	221	13.7

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time
AProVE-NIA	118	2949
CVC3	65	7
MiniSmt	136	1322

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time
AProVE-NIA	118	2949
CVC3	65	7
MiniSmt	136	1322
Barcelogic (2009)	197	3563

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

QF_NRA category (60 problems, $\mathbb{Q}$, $\mathbb{R}$)

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

QF_NRA category (60 problems, $\mathbb{Q}$, $\mathbb{R}$)

tool	score	time (sec)
CVC3	3	1
MiniSmt	44	52

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

QF_NRA category (60 problems, $\mathbb{Q}$, $\mathbb{R}$)

tool	score	time (sec)	sat	time (sec)
CVC3	3	1	-	-
MiniSmt	44	52	44	52

SMT-COMP 2010

QF_NIA category (204 problems, $\mathbb{N}$, $\mathbb{Z}$)

tool	score	time	sat	time
AProVE-NIA	118	2949	118	2949
CVC3	65	7	5	2
MiniSmt	136	1322	136	1322
Barcelogic (2009)	197	3563	140	2460

QF_NRA category (60 problems, $\mathbb{Q}$, $\mathbb{R}$)

tool	score	time (sec)	sat	time (sec)
CVC3	3	1	-	-
MiniSmt	44	52	44	52

Details

http://www.smtcomp.org/2010/

Exemplary Constraints

A "hard" SMT problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((a Real))
:formula (= (* a a) 2)
)
```

Exemplary Constraints

A "hard" SMT problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((a Real))
:formula (= (* a a) 2)
)
```

As SAT problem (5 bits for variables)

1,207 clauses   417 variables < 0.05 seconds solving time

Exemplary Constraints (cont'd)

A "hard" termination problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((x17 Real) ... (x0 Real))
:formula (and (and (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3
x7))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11
1))) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0
(+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11)))))) (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3 x7
))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11 1)
)) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0 (
+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11))))))) (and (and (>= x2 1) (>= x8 1)) (>= x14 1))))
```

Exemplary Constraints (cont'd)

A "hard" termination problem

```
(benchmark ttt2
:logic QF_NRA
:status unknown
:extrafuns ((x17 Real) ... (x0 Real))
:formula (and (and (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3
x7))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11
1))) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0
(+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11)))))) (and (and (and (> (+ x0 (+ (* x2 x6) (* x3 x7))) 0) (and (>= (+ x0 (+ (* x2 x6) (* x3 x7
))) 0) (>= (+ x1 (+ (* x4 x6) (* x5 x7))) 0))) (and (and (and (>= x8 1) (>= x9 0)) (>= x10 0)) (>= x11 1)
)) (and (and (and (> (+ x0 (+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (and (>= (+ x0 (
+ (* x2 x12) (* x3 x13))) (+ x12 (+ (* x14 x0) (* x15 x1)))) (>= (+ x1 (+ (* x4 x12) (* x5 x13))) (+ x13
(+ (* x16 x0) (* x17 x1)))))) (and (and (and (>= (+ (* x2 x14) (* x3 x16)) (+ (* x14 x2) (* x15 x4))) (>=
(+ (* x2 x15) (* x3 x17)) (+ (* x14 x3) (* x15 x5)))) (>= (+ (* x4 x14) (* x5 x16)) (+ (* x16 x2) (* x17
x4)))) (>= (+ (* x4 x15) (* x5 x17)) (+ (* x16 x3) (* x17 x5))))) (and (and (and (>= x8 (+ (* x14 x8) (*
x15 x10))) (>= x9 (+ (* x14 x9) (* x15 x11)))) (>= x10 (+ (* x16 x8) (* x17 x10)))) (>= x11 (+ (* x16 x9)
(* x17 x11))))))) (and (and (>= x2 1) (>= x8 1)) (>= x14 1))))
```

As SAT problem (5 bits for variables)

65,420 clauses   149,755 variables < 1 second solving time

Conclusion

SMT solving is suitable for termination analysis

Arctic Interpretations
Multiset Path Order
Knuth-Bendix Order
predictive labeling
usable rules
Finding Loops
dependency pairs
dependency graphs
semantic labeling
recursive
argument filterings
usable rules
SCC
below zero
Matrix Interpretations
Lexicographic Path Order
match-bounds
Polynomial Interpretations
Increasing Interpretations
Monotonic Semantic Path Order

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Future Work

- preprocessing

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Future Work

- preprocessing
- unsatisfiability

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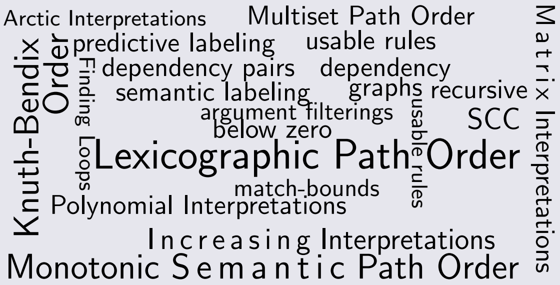

Future Work

- preprocessing
- unsatisfiability
- certification