Labelings for Decreasing Diagrams

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Overview

- Preliminaries
- Decreasing Diagrams
- Labelings for Decreasing Diagrams
- Implementation
- Experiments
- Conclusion

Preliminaries

Definition (ARS)

$\mathcal{A} = (A, \to)$

Definition (Confluence)

${}^*\!\leftarrow \cdot \to^* \subseteq \to^* \cdot {}^*\!\leftarrow$

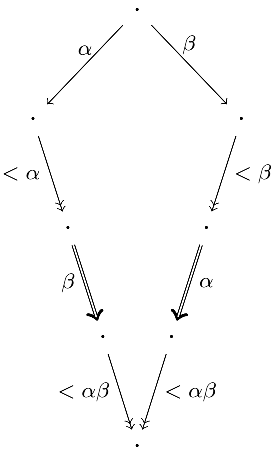
Definition (Local Confluence)

$\leftarrow \cdot \to \subseteq \to^* \cdot {}^*\!\leftarrow$

Theorem (van Oostrom, 94)

every locally decreasing ARS is confluent

Decreasing Diagrams



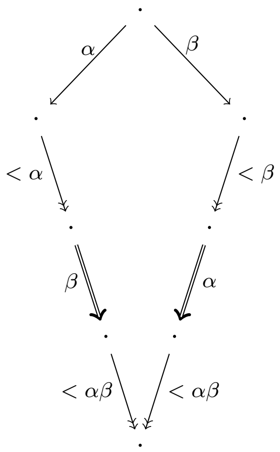
Definition (local diagram)

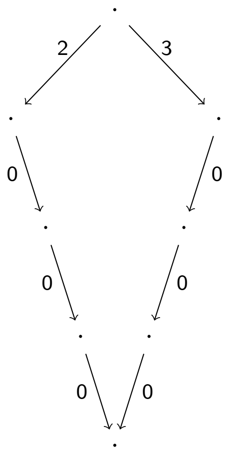
$$\leftarrow \cdot \rightarrow \; \subseteq \; \rightarrow^* \cdot {}^*\!\leftarrow$$

Definition (local decreasing)

$${}_\alpha\!\leftarrow \cdot \rightarrow_\beta \; \subseteq \; \overset{\vee}{\rightarrow}{}^*_\alpha \cdot \rightarrow^=_\beta \cdot \overset{\vee}{\rightarrow}{}^*_{\alpha\beta} \cdot {}^*_{\alpha\beta}\overset{\vee}{\leftarrow} \cdot {}^=_\alpha\!\leftarrow \cdot {}^*_\beta\overset{\vee}{\leftarrow}$$

Examples





Term Rewriting

$t \leftarrow s \rightarrow u$

Three possibilities (modulo symmetry)

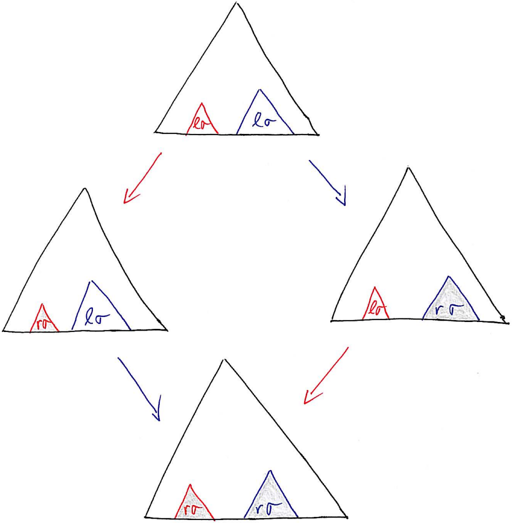
$t \;{}_{p,l\to r}\!\leftarrow s[l\sigma]_p = s = s[l\sigma]_q = l \rightarrow_{q,l\to r} u$

- $p \parallel q$ (parallel overlap)
- $p < q$ and $q \leqslant \mathcal{P}\mathrm{os}_{\mathcal{V}}(s[l]_p)$ (variable overlap)
- $p \leqslant q$ and $q \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(s[l]_p)$ (critical overlap)

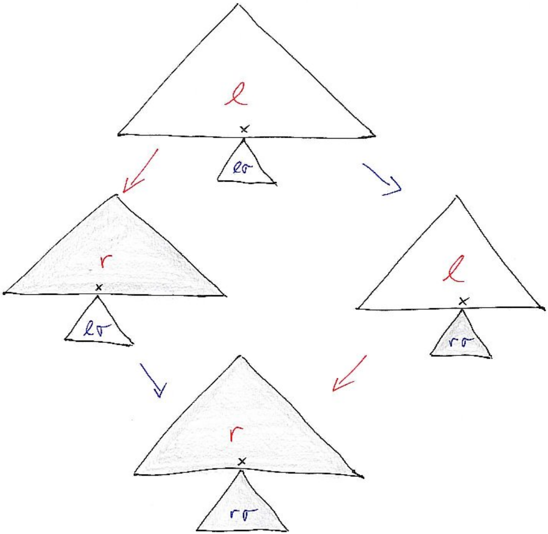
Remark

linear TRSs first

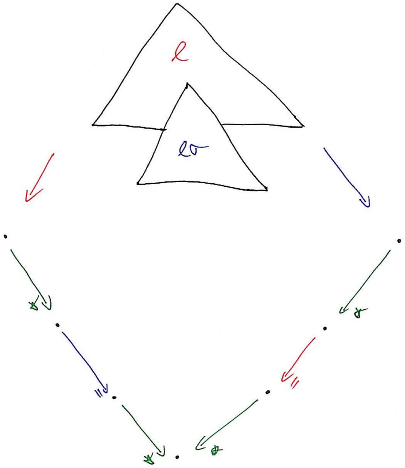
parallel



variable (linear)



critical



Rule Labeling ($\ell\colon \mathcal{R} \to \mathbb{N}$)

- parallel *OK*
- variable (linear) *OK*
- overlap check

Theorem (van Oostrom, 2008)

critical diagrams decreasing (rule labeling) $\longrightarrow$ linear TRS confluent

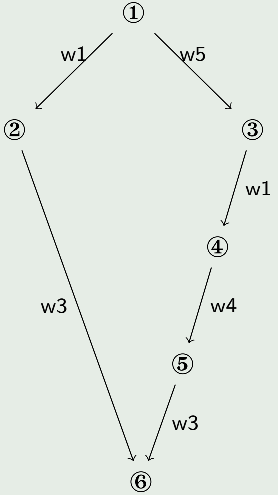
Implementation

- (Aoto, 2010)
- (Hirokawa & Middeldorp, 2010)

Example (van Oostrom, 2008)

$$
\begin{aligned}
1:&\quad \textcolor{red}{\mathsf{nats}} \to 0 : \mathsf{inc}(\mathsf{nats})\\
2:&\quad \mathsf{hd}(x : y) \to x\\
3:&\quad \mathsf{tl}(x : y) \to y\\
4:&\quad \mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y)\\
5:&\quad \mathsf{inc}(\mathsf{tl}(\textcolor{red}{\mathsf{nats}})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))
\end{aligned}
$$

① inc(tl(nats))
② inc(tl(0 : inc(nats)))
③ tl(inc(nats))
④ tl(inc(0 : inc(nats)))
⑤ tl(s(0) : inc(inc(nats)))
⑥ inc(inc(nats))

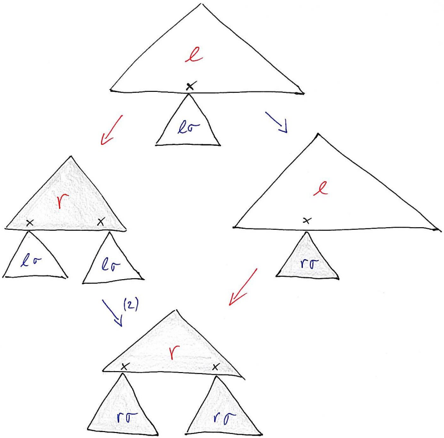


$w1 >_{\mathbb{N}} w3, w4$ shows confluence

Extensions (left-linear)

• parallel	*OK*
• variable (left-linear)	???
• overlap	*check*

variable (left-linear)



Idea

$\rightarrow$ strict decrease
$\rightarrow$ no increase

Idea (van Oostrom, 2008)

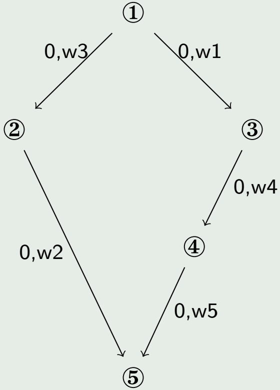
- $\#_f C_l[] > \#_f C_r[]$ $\quad x$ non-linear in $C_r[x]$
- $\#_f C_l[] \geqslant \#_f C_r[]$ $\quad x$ linear in $C_r[x]$
- critical diagrams decreasing $(\#_f C[] \times \text{rl})$

$\longrightarrow$ left-linear $\mathcal{R}$ confluent

Example (van Oostrom, 2008)

$$1 : \quad g(a) \to f(g(a))$$
$$2 : \quad g(b) \to c$$
$$3 : \quad a \to b$$
$$4 : \quad f(x) \to h(x, x)$$
$$5 : \quad h(x, y) \to c$$

① g(a)
② g(b)
③ f(g(a))
④ $h(g(a), g(a))$
⑤ c



$w1 > w2, w4, w5$ local decreasingness

Remark

(Aoto, 2010) applies

Problem

Example

$$1 : f(g(x, a)) \to g(f(x), f(x)) \qquad 2 : a \to b \qquad 3 : b \to a$$

Example (OO03)

$$
\begin{array}{llll}
1: & x + (y + z) \to (x + y) + z & 2: & (x + y) + z \to x + (y + z) \\
3: & s(x) + y \to x + s(y) & 4: & x + s(y) \to s(x) + y \\
5: & x \times s(y) \to x + (x \times y) & 6: & s(x) \times y \to (x \times y) + y \\
7: & x + y \to y + x & 8: & x \times y \to y \times x \\
9: & sq(x) \to x \times x & 10: & sq(s(x)) \to (x \times x) + s(x + x)
\end{array}
$$

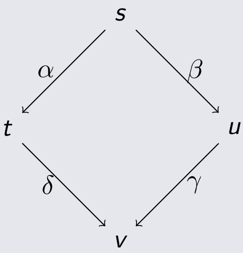
Definition (labeling)

$\Gamma = s \rightarrow_{p,l \rightarrow r} t \qquad \Delta = u \rightarrow_{q,l' \rightarrow r'} v$

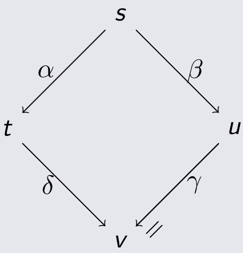
$(\ell, \geqslant, >)$ labeling if $\geqslant \cdot > \cdot \geqslant \subseteq >$

- $\ell(\Gamma) \geqslant \ell(\Delta) \longrightarrow \ell(C[\Gamma\sigma]) \geqslant \ell(C[\Delta\sigma])$
- $\ell(\Gamma) > \ell(\Delta) \longrightarrow \ell(C[\Gamma\sigma]) > \ell(C[\Delta\sigma])$

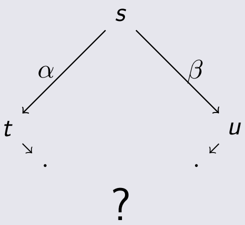
linear TRSs (3 kinds of peaks)



(a) parallel



(b) variable linear



(c) critical

L-labeling

Definition (L-labeling)

labeling ℓ is L-labeling if $\alpha \geqslant \gamma$, $\beta \geqslant \delta$ (parallel, variable linear)

Example

$(\ell_{\mathsf{rl}}, \geqslant_{\mathbb{N}}, >_{\mathbb{N}})$ is L-labeling ($\ell_{\mathsf{rl}} : \mathcal{R} \to \mathbb{N}$)
$(\ell_{\mathsf{sn}}, \to^*_{\mathcal{R}}, \to^+_{\mathcal{S}/\mathcal{R}})$ is L-labeling ($\ell_{\mathsf{sn}}(s \to t) = s$)
if $\to_{\mathcal{S}} \subseteq \to_{\mathcal{R}}$ and $\mathcal{S}/\mathcal{R}$ is terminating

Lemma

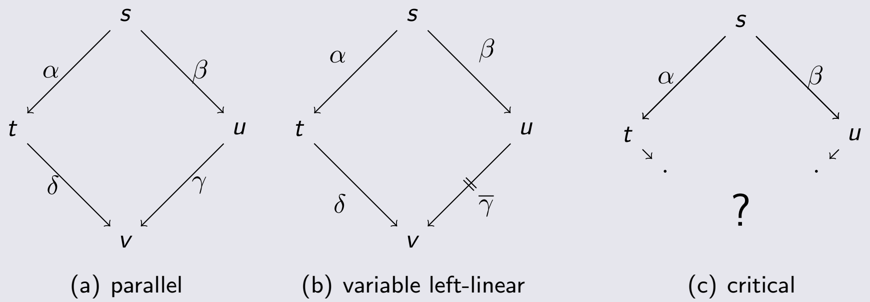
ℓ_1, ℓ_2 *L-labelings* $\longrightarrow$ $\ell_1 \times \ell_2$ *L-labeling*

Theorem

$\mathcal{R}$ *linear, critical peaks decreasing for L-labeling* $\longrightarrow$ $\mathcal{R}$ *confluent*

LL-labeling

left-linear TRSs (3 kinds of peaks)



Definition (LL-labeling)

L-labeling ℓ is LL-labeling if $\alpha > \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

Example

ℓ_{rl} is not LL-labeling
ℓ_{sn} is LL-labeling if $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ terminating

Definition (LL-labeling)

L-labeling ℓ is weak LL-labeling if $\alpha \geqslant \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

Example

ℓ_{rl} weak LL-labeling
every LL-labeling is weak LL-labeling

Lemma

LL-labeling ℓ_1, *weak LL-labeling* $\ell_2 \longrightarrow \ell_1 \times \ell_2$, $\ell_2 \times \ell_1$ *LL-labelings*

Theorem

$\mathcal{R}$ *left-linear, critical peaks decreasing for LL-labeling* $\longrightarrow \mathcal{R}$ *confluent*

First result

Corollary

$\mathcal{R}$ left-linear, critical peaks decreasing for weak LL-labeling, $\mathcal{R}_d/\mathcal{R}_{nd}$ terminating $\longrightarrow$ $\mathcal{R}$ confluent

Example (OO03 cont'd)

$$\begin{array}{rlrl}
1: & x + (y + z) \to (x + y) + z & 2: & (x + y) + z \to x + (y + z) \\
3: & \mathsf{s}(x) + y \to x + \mathsf{s}(y) & 4: & x + \mathsf{s}(y) \to \mathsf{s}(x) + y \\
5: & x \times \mathsf{s}(y) \to x + (x \times y) & 6: & \mathsf{s}(x) \times y \to (x \times y) + y \\
7: & x + y \to y + x & 8: & x \times y \to y \times x \\
9: & \mathsf{sq}(x) \to x \times x & 10: & \mathsf{sq}(\mathsf{s}(x)) \to (x \times x) + \mathsf{s}(x + x)
\end{array}$$

$\mathcal{R}_{\mathrm{d}}/\mathcal{R}_{\mathrm{nd}}$ terminating

$+_{\mathbb{N}}(x, y) = x + y$ $\quad \mathsf{s}_{\mathbb{N}}(x) = x + 1$ $\quad \times_{\mathbb{N}}(x, y) = x^2 + xy + y^2$ $\quad \mathsf{sq}_{\mathbb{N}}(x) = 3x^2 + 1$

weak *LL*-labeling $\ell_{\mathrm{rl}}(8) = \ell_{\mathrm{rl}}(9) = 2$, $\ell_{\mathrm{rl}}(6) = \ell_{\mathrm{rl}}(10) = 1$, $\ell_{\mathrm{rl}}(\cdot) = 0$

Towards a second result

Definition

$$\star(f(t_1,\ldots,t_n),p) = \begin{cases} f_i(\star(t_i,q)) & \text{if } p = iq \\ x & \text{if } p = \epsilon \end{cases}$$

$$\mathcal{R}^\star_\circ = \{\star(l,p) \to \star(r,q) \mid l \to r \in \mathcal{R},\ l|_p = r|_q \in \mathcal{V}, \text{ and } |r|_y \circ 1\}$$

$\circ \in \{>,=\}$ $\qquad \star(\mathcal{R}) := \mathcal{R}^\star_>/\mathcal{R}^\star_=$

Example

$1 : \mathsf{f}(\mathsf{g}(x,\mathsf{a})) \to \mathsf{g}(\mathsf{f}(x),\mathsf{f}(x)) \quad 2 : \mathsf{a} \to \mathsf{b} \quad 3 : \mathsf{b} \to \mathsf{a} \quad 4 : \mathsf{h}(x) \to \mathsf{h}(x)$

$\mathcal{R}^\star_> = \{\mathsf{f}_1(\mathsf{g}_1(x)) \to \mathsf{g}_1(\mathsf{f}_1(x)), \mathsf{f}_1(\mathsf{g}_1(x)) \to \mathsf{g}_2(\mathsf{f}_1(x))\}$
$\mathcal{R}^\star_= = \{\mathsf{h}_1(x) \to \mathsf{h}_1(x)\}$

Definition

$\ell_\star(s \to_{p,l\to r} t) = \star(s,p)$

Lemma

$(\geqslant, >)$ monotone reduction pair, $\mathcal{R}^\star_> \subseteq >$, $\mathcal{R}^\star_> \cup \mathcal{R}^\star_= \subseteq \geqslant \longrightarrow (\ell_\star, \geqslant, >)$ LL-labeling

Corollary

$\mathcal{R}$ left-linear, $\mathcal{R}^\star_>/\mathcal{R}^\star_=$ terminating, ℓ weak LL-labeling, critical peaks decreasing for $\ell_\star \times \ell \longrightarrow \mathcal{R}$ confluent

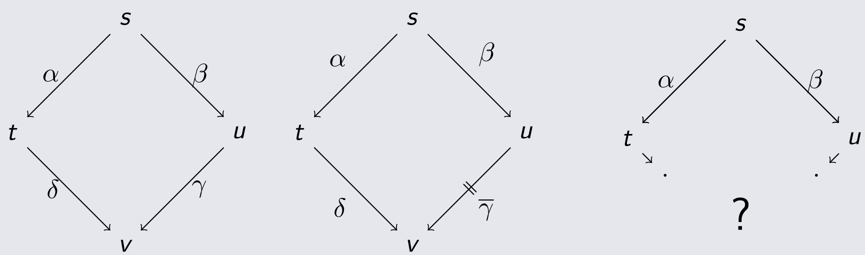
Example (OO03)

$$
\begin{array}{llll}
1: & x + (y + z) \to (x + y) + z & 2: & (x + y) + z \to x + (y + z) \\
3: & \mathsf{s}(x) + y \to x + \mathsf{s}(y) & 4: & x + \mathsf{s}(y) \to \mathsf{s}(x) + y \\
5: & x \times \mathsf{s}(y) \to x + (x \times y) & & \dots
\end{array}
$$

$\mathcal{R}^\star_>$ contains nonterminating $\times_1(x) \to +_2(\times_1(x))$

LL-labeling

left-linear TRSs (3 kinds of peaks)



(a) parallel (b) variable left-linear (c) critical

Definition (*LL*-labeling)

L-labeling ℓ is *LL*-labeling if $\alpha > \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

$\alpha \geqslant \gamma_1$, $\alpha > \gamma_i$ $(i > 1)$

LL-labeling improved

Definition

$\overset{\star}{\star}(\mathcal{R}) := \mathcal{R}_{>}^{\star\star} / \mathcal{R}_{=}^{\star\star}$

Lemma

$(\geqslant, >)$ monotone reduction pair, $\mathcal{R}_{>}^{\star\star} \subseteq >$, $\mathcal{R}_{>}^{\star\star} \cup \mathcal{R}_{=}^{\star\star} \subseteq \geqslant \longrightarrow$ $(\ell\star, \geqslant, >)$ LL-labeling

Corollary

ℓ weak LL-labeling, $\overset{\star}{\star}(\mathcal{R})$ terminating, critical peaks decreasing for $\ell_\star \times \ell$ $\longrightarrow \mathcal{R}$ confluent

Example (OO03)

$\mathcal{R}^\star_>$

$$\begin{aligned}
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_1(\times_1(x))\\
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_1(\times_2(x))\\
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_2(\mathsf{s}_1(+_1(x)))\\
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_2(\mathsf{s}_1(+_2(x)))\\
\mathsf{sq}_1(x) &\to \times_1(x)\\
\mathsf{sq}_1(x) &\to \times_2(x)\\
\times_2(y) &\to +_1(\times_2(y))\\
\times_2(y) &\to +_2(y)\\
\times_1(x) &\to +_1(x)\\
\times_1(x) &\to +_2(\times_1(x))
\end{aligned}$$

$\mathcal{R}^\star_=$

$$\begin{aligned}
\times_1(x) &\to \times_2(x) & \times_1(\mathsf{s}_1(x)) &\to +_1(\times_1(x))\\
\times_2(y) &\to \times_1(y) & \times_2(\mathsf{s}_1(y)) &\to +_2(\times_2(y))\\
+_1(x) &\to +_2(x) & +_1(\mathsf{s}_1(x)) &\to +_1(x)\\
+_2(y) &\to +_1(y) & +_1(+_1(x)) &\to +_1(x)\\
+_1(x) &\to +_1(\mathsf{s}_1(x)) & +_1(+_2(y)) &\to +_2(+_1(y))\\
+_1(x) &\to +_1(+_1(x)) & +_2(+_2(z)) &\to +_2(z)\\
+_2(z) &\to +_2(+_2(z)) & +_2(\mathsf{s}_1(y)) &\to +_2(y)\\
+_2(y) &\to +_2(\mathsf{s}_1(y)) & +_2(+_1(y)) &\to +_1(+_2(y))\\
\times_1(x) &\to +_2(\times_1(x)) & \times_2(y) &\to +_1(\times_2(y))
\end{aligned}$$

$\mathsf{sq}_{1\mathbb{N}}(x) = x + 2 \quad \times_{1\mathbb{N}}(x) = \times_{2\mathbb{N}}(x) = x + 1 \quad +_{1\mathbb{N}}(x) = +_{2\mathbb{N}}(x) = \mathsf{s}_1(x) = x$

Implementation

Label with $\mathbb{N}$ (for lexicographic combination)

ℓ_{rl} ✓ ℓ_{sn} ? ✓ $\ell_{\star}$? ? ✓

ℓ_{rl}

(Aoto, 2010), (Hirokawa & Middeldorp, 2010)

ℓ_{sn}

peak $t \leftarrow s \rightarrow u$

partial termination proof of $\mathsf{CDS}(\mathcal{R})/\mathcal{R} \longrightarrow \mathcal{S}'/\mathcal{R}'$

$\ell_{\mathsf{sn}}(t_i \rightarrow t_{i+1}) = 1$ if $s \rightarrow^*_{\mathcal{S}'} t_i$

$\ell_{\mathsf{sn}}(t_i \rightarrow t_{i+1}) = 0$ otherwise

$\ell_{\star}$

prove termination of $\overset{\star}{\star}(\mathcal{R})$ with matrix interpretations

Experiments

53 left-linear TRSs (Aoto, 2010)

method	pre	CR(ℓ_{rl})	CR(ℓ_{sn})	CR
rule labeling	40	35	–	35
SN($\mathcal{R}_d/\mathcal{R}_{nd}$)	45	40	37	42
SN($\star(\mathcal{R})$)	45	42	34	42
SN($\overset{\star}{\star}(\mathcal{R})$)	48	45	36	45
ACP	–	42	–	48
CSI	–	–	–	49

106 TRSs (Aoto, 2010)

tool	CR	not CR
ACP	64	18
CSI	61	20

Conclusion

Summary

- decreasing diagrams
- (linear) ℓ_{rl}, ℓ_{sn}, $\ell_{\star}$
- (left-linear) $\mathsf{SN}(\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}})$, $\mathsf{SN}(\star(\mathcal{R}))$, $\mathsf{SN}(\overset{\star}{\star}(\mathcal{R}))$

Future Work

- study parallel reduction