Constraints for Argument Filterings

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- Motivation
- Term Rewriting
- SAT Encoding
- Implementation Issues
- Experimental Results
- Remarks

Why Encode Termination Problems as Satisfiability Problems?

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- execution speed

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- execution speed
- ease of implementation
- developments in SAT community are directly available

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signature

0 constant $\mathsf{s}, \mathsf{p}, \mathsf{fac}$ unary $+, \times$ binary

rewrite rules

$$
\begin{aligned}
\mathsf{fac}(0) &\to \mathsf{s}(0) & 0 + y &\to y \\
\mathsf{fac}(\mathsf{s}(x)) &\to \mathsf{s}(x) \times \mathsf{fac}(\mathsf{p}(\mathsf{s}(x))) & \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\
\mathsf{p}(\mathsf{s}(x)) &\to x & 0 \times y &\to 0 \\
& & \mathsf{s}(x) \times y &\to x \times y + y
\end{aligned}
$$

TRS

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rewriting

$\mathsf{fac}(\mathsf{s}(0))$

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$$\mathsf{fac}(0) \to \mathsf{s}(0)$$
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normal form

Definition

TRS is terminating if there are no infinite rewrite sequences

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Theorem

TRS $\mathcal{R}$ is terminating if there is a reduction order $>$ with $\mathcal{R} \subseteq >$.

- $\rhd_{\mathsf{emb}}, >_{\mathsf{lpo}}, >_{\mathsf{kbo}}, \ldots$ *are reduction orders*

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Example (emb)

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$\mathsf{s}(x) \rightarrow x \qquad \mathsf{fac}(x) \rightarrow x \qquad \mathsf{p}(x) \rightarrow x$

$x + y \rightarrow x \qquad x + y \rightarrow y \qquad x \times y \rightarrow x \qquad x \times y \rightarrow y$

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relation

$$\mathsf{s}(\mathsf{p}(x)) \rhd_{\mathsf{emb}} x \qquad x + 0 \rhd_{\mathsf{emb}} x$$

$$\mathsf{s}(x) \times y \not\rhd_{\mathsf{emb}} (x \times y) + y \qquad \mathsf{s}(x) \times y \rhd_{\mathsf{emb}} x \times y$$

Theorem

TRS $\mathcal{R}$ is terminating if $\forall$ cycle $\mathcal{C}$ in dependency graph of $\mathcal{R}$
$\exists$ reduction pair $(\geqslant, >)$ such that

1. $\mathcal{R} \cup \mathcal{C} \subseteq \geqslant$
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TRS $\mathcal{R}$ is terminating if $\forall$ cycle $\mathcal{C}$ in dependency graph of $\mathcal{R}$ $\exists$ argument filtering π $\exists$ reduction pair $(\geqslant, >)$ such that

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- $$\pi(t) = \begin{cases} t & \text{if } t \text{ is variable} \\ \pi(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ and ①} \\ f(\pi(t_{i_1}), \ldots, \pi(t_{i_m})) & \text{if } t = f(t_1, \ldots, t_n) \text{ and ②} \end{cases}$$

Example (argument filtering)

signature

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filtering

$$\begin{aligned}
\pi(0) &= [] \\
\pi(\mathsf{s}) &= [] \\
\pi(\mathsf{p}) &= 1 \\
\pi(\mathsf{fac}) &= [1] \\
\pi(+) &= [2] \\
\pi(\times) &= [1,2]
\end{aligned}$$

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$$\pi(0) = 0$$

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Example (argument filtering)

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$$\pi(0) = [] \qquad \pi(0) = 0$$
$$\pi(s) = [] \qquad \pi(s(x)) = s$$
$$\pi(p) = 1 \qquad \pi(p(x)) = x$$
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$$\pi(\mathsf{fac}) = [1] \qquad \pi(\mathsf{fac}(x)) = \mathsf{fac}(x)$$
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- $n + 1$ propositional variables for n-ary function symbol f:
  - X_f select between ① and ②
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Definition

induced assignment α_π for argument filtering π:

$$\alpha_\pi(X_f) = \begin{cases} \text{true} & \text{if } \pi(f) = [i_1, \dots, i_m] \\ \text{false} & \text{if } \pi(f) = i \end{cases}$$

$$\alpha_\pi(X_f^i) = \begin{cases} \text{true} & \text{if } i \in \pi(f) \\ \text{false} & \text{if } i \notin \pi(f) \end{cases}$$

Definition

- assignment α is argument filtering consistent if for all f

$$\text{if } \alpha \not\models X_f \text{ then } \exists! \; i \text{ such that } \alpha \models X_f^i$$

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- propositional formula $\quad \mathrm{AF}(\mathcal{F}) = \bigwedge_{f \in \mathcal{F}} \left(X_f \vee \bigvee_{i=1}^{\mathrm{arity}(f)} \left(X_f^i \wedge \bigwedge_{j \neq i} \neg X_f^j \right) \right)$

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Lemma

assignment α for $\mathcal{X}_{\mathcal{F}}$ is argument filtering consistent iff $\alpha \vDash \mathsf{AF}(\mathcal{F})$

Definition

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Lemma

assignment α for $\mathcal{X}_\mathcal{F}$ is argument filtering consistent iff $\alpha \models \mathsf{AF}(\mathcal{F})$

Definition

induced argument filtering π_α for argument filtering consistent assignment α:

$$\pi_\alpha(f) = \begin{cases} [\, i \mid \alpha \models X_f^i \,] & \text{if } \alpha \models X_f \\ i & \text{if } \alpha \not\models X_f \text{ and } \alpha \models X_f^i \end{cases}$$

Aim

define propositional formulas $\ulcorner s \rhd^{\pi}_{\mathsf{emb}} t \urcorner$ and $\ulcorner s \unrhd^{\pi}_{\mathsf{emb}} t \urcorner$ such that

$$\pi_\alpha(s) \rhd_{\mathsf{emb}} \pi_\alpha(t) \quad \text{when} \quad \alpha \vDash \ulcorner s \rhd^{\pi}_{\mathsf{emb}} t \urcorner \wedge \mathsf{AF}(\mathcal{F})$$

and

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$$\Downarrow$$

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- SAT Encoding
- Implementation Issues
- **Experimental Results**
- Remarks

Embedding

865 TRSs in version 3.2 of TPDB

embedding	AProVE	$\mathsf{T_TT}$	sat
solved	194	194	194
timeout (60 seconds)	12	6	0
time (in seconds)	735	407	146

KBO and LPO

865 TRSs in 2006 edition of TPDB

KBO ‖ LPO	$\mathsf{T_TT}$ (L ; K)	$\mathsf{T_TT}$ (K ; L)	sat(2)	sat(3)	sat(4)
solved	310	295	305	338	343
timeout	121	136	6	9	14
time	7025	9025	1664	2076	2623

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865 TRSs in 2006 edition of TPDB

KBO ∥ LPO	T_TT (L ; K)	T_TT (K ; L)	sat(2)		sat(3)		sat(4)	
solved	310	295	305	337	338	369	343	377
timeout	121	136	6	9	9	11	14	16
time	7025	9025	1664	1940	2076	2351	2623	2898

advanced usable rules

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  - ...
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  - $(\sum_{1 \leqslant i \leqslant n} a_i * x_i) \circ m \qquad \circ \in \{\geqslant, =, \leqslant\}$
  - get $+$ for free (KBO)
  - get $+$ and $\times$ for free (polynomial-, matrix interpretations, ...)