Constraints for Argument Filterings

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joint work with

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- Motivation
- Term Rewriting
- Knuth-Bendix Order
- SAT Encoding
- Implementation Issues
- Experimental Results
- Dependency Pairs
- Remarks

Why Encode Termination Problems as Satisfiability Problems?

- execution speed
- ease of implementation
- developments in SAT community are directly available

Why KBO?

- more challenging than LPO

Kurihara & Kondo	*1997 2004*	*LPO*
Codish, Lagoon & Stuckey	*2006*	*LPO, QLPO*

- existing implementations (in T$_\mathsf{T}$T and AProVE) based on polynomial time algorithm

  Korovin & Voronkov *2003*

  are slow

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signature

0 constant $\mathsf{s}, \mathsf{p}, \mathsf{fac}$ unary $+, \times$ binary

rewrite rules

$$\begin{aligned} \mathsf{fac}(0) &\to \mathsf{s}(0) & 0 + y &\to y \\ \mathsf{fac}(\mathsf{s}(x)) &\to \mathsf{s}(x) \times \mathsf{fac}(\mathsf{p}(\mathsf{s}(x))) & \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\ \mathsf{p}(\mathsf{s}(x)) &\to x & 0 \times y &\to 0 \\ & \qquad\qquad \text{TRS} & \mathsf{s}(x) \times y &\to x \times y + y \end{aligned}$$

rewriting

$$\begin{aligned} \mathsf{fac}(\mathsf{s}(0)) &\to \mathsf{s}(0) \times \mathsf{fac}(\mathsf{p}(\mathsf{s}(0))) \to \mathsf{s}(0) \times \mathsf{fac}(0) \to \mathsf{s}(0) \times \mathsf{s}(0) \\ &\to 0 \times \mathsf{s}(0) + \mathsf{s}(0) \to 0 + \mathsf{s}(0) \to \mathsf{s}(0) \qquad \text{normal form} \end{aligned}$$

Definition

TRS is terminating if there are no infinite rewrite sequences

Termination Methods

Knuth-Bendix order, polynomial interpretations, lexicographic path order, multiset order, multiset path order, recursive path order, semantic path order, recursive decomposition order, transformation order, elementary interpretations, well-founded monotone algebra, general path order, semantic labeling, type introduction, freezing, top-down labeling, dependency pair method, matchbounds, size-change principle, matrix interpretations, ...

Termination Tools

AProVE, Cariboo, Cime, JamBox, MatchBox, MultumNonMulta, MuTerm, Teparla, Torpa, TPA, T$_\mathsf{T}$T, T$_\mathsf{T}$Tbox

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Definition

- quasi-precedence $\gtrsim$ is quasi-order on signature $\mathcal{F}$
- weight function (w, w_0) consists of mapping $w \colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for all constants $c \in \mathcal{F}$
- weight of term t is

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- weight function (w, w_0) is admissible for quasi-precedence $\gtrsim$ if

$$f \gtrsim g \quad \forall\, g \in \mathcal{F} \setminus \{f\}$$

whenever f is unary function symbol in $\mathcal{F}$ with $w(f) = 0$

Definition

Knuth-Bendix order $>_{\mathsf{kbo}}$ on terms:
$s >_{\mathsf{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either

1. $w(s) > w(t)$
2. $w(s) = w(t)$ and either
   1. $\exists\, n > 0\ \exists\, x \in \mathcal{V}$ such that $s = f_n(\dots(f_1(x)\dots)$ and $t = x$
   2. $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f \sim g$ and $\exists\, i$
      $\forall\, j < i \quad s_j = t_j \qquad s_i >_{\mathsf{kbo}} t_i$
   3. $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f > g$

Theorem

TRS $\mathcal{R}$ is terminating if

$\exists$ quasi-precedence $\gtrsim$ $\quad\exists$ admissible weight function (w, w_0)

such that $l >_{\mathsf{kbo}} r$ for all $l \to r \in \mathcal{R}$

Example TRS/HM_t000

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
\vdots & \vdots & & \vdots \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x + (y : z) \to y : (x + z) & 0 : x \to x \\
(x : y) + z \to x : (y + z) & x : (y : z) \to (x + y) : z
\end{array}
$$

TTT

$$w(0) = w(1) = w(2) = w(3) = w(4) = w(+) = w_0 = 1$$

$$w(5) = w(6) = w(7) = w(8) = w(9) = 3$$

$$w(:) = 2$$

$$+ > : \qquad + > 5 \qquad + > 6 \qquad + > 7 \qquad + > 8$$

Example TRS/SK90_2.42

$$\begin{aligned}
\mathsf{flat}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{flat}(\mathsf{unit}(x)) &\to \mathsf{flat}(x) \\
\mathsf{flat}(x \mathbin{+\!\!+} y) &\to \mathsf{flat}(x) \mathbin{+\!\!+} \mathsf{flat}(y) \\
\mathsf{flat}(\mathsf{unit}(x) \mathbin{+\!\!+} y) &\to \mathsf{flat}(x) \mathbin{+\!\!+} \mathsf{flat}(y) \\
\mathsf{flat}(\mathsf{flat}(x)) &\to \mathsf{flat}(x) \\
x \mathbin{+\!\!+} \mathsf{nil} &\to x
\end{aligned}
\qquad
\begin{aligned}
\mathsf{rev}(\mathsf{nil}) &\to \mathsf{nil} \\
\mathsf{rev}(\mathsf{unit}(x)) &\to \mathsf{unit}(x) \\
\mathsf{rev}(x \mathbin{+\!\!+} y) &\to \mathsf{rev}(y) \mathbin{+\!\!+} \mathsf{rev}(x) \\
\mathsf{rev}(\mathsf{rev}(x)) &\to x \\
(x \mathbin{+\!\!+} y) \mathbin{+\!\!+} z &\to x \mathbin{+\!\!+} (y \mathbin{+\!\!+} z) \\
\mathsf{nil} \mathbin{+\!\!+} y &\to y
\end{aligned}$$

$$w(\mathsf{flat}) = w(\mathsf{rev}) = w(\mathbin{+\!\!+}) = 0$$

$$w(\mathsf{unit}) = w(\mathsf{nil}) = w_0 = 1$$

$$\mathsf{flat} \sim \mathsf{rev} > \mathsf{unit} > \mathbin{+\!\!+} > \mathsf{nil}$$

Example SRS/Zantema_z101

$\mathsf{aa} \to \mathsf{bbb}$ $\mathsf{bbbbb} \to \mathsf{aaa}$

$w(\mathsf{a}) = 13 \qquad w(\mathsf{b}) = 8$

$w(\mathsf{a}) = 3 \qquad w(\mathsf{b}) = 2 \qquad \mathsf{a} > \mathsf{b}$

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Aim

define propositional formula $\varphi(\mathcal{R})$ depending on TRS $\mathcal{R}$ such that

$$\begin{aligned} \vDash \varphi(\mathcal{R}) \quad &\Longrightarrow \quad \mathcal{R} \text{ is KBO terminating} \\ \nvDash \varphi(\mathcal{R}) \quad &\Longrightarrow \quad \mathcal{R} \text{ is not KBO terminating} \end{aligned}$$

Problem

weight function $w \colon \mathcal{F} \to \mathbb{N}$

Solution

restrict range of weight function to $\{\, 0, \ldots, 2^k - 1 \,\}$ $\quad$ (k bits)

Revised Aim

define propositional formula $\varphi_k(\mathcal{R})$ such that

$$\begin{aligned} \vDash \varphi_k(\mathcal{R}) \quad &\Longrightarrow \quad \mathcal{R} \text{ is KBO terminating} \\ \nvDash \varphi_k(\mathcal{R}) \quad &\Longrightarrow \quad \mathcal{R} \text{ is not KBO terminating in } k \text{ bits} \end{aligned}$$

- atom-based approach Kurihara & Kondo 1997 2004

$$\forall\, f \neq g \quad \begin{cases} X_{fg} = \text{true} & \iff \quad f > g \\ Y_{fg} = \text{true} & \iff \quad f \sim g \end{cases}$$

$$\mathrm{QO}(\mathcal{F}) \;=\; \bigwedge_{\substack{f,\,g,\,h \in \mathcal{F} \\ f \neq g \neq h \neq f}} \Big(X_{fg} \wedge X_{gh} \rightarrow X_{fh} \Big) \;\wedge\; \bigwedge_{\substack{f,\,g \in \mathcal{F} \\ f \neq g}} \Big(X_{fg} \rightarrow \neg(Y_{fg} \vee Y_{gf}) \Big) \;\wedge\; \cdots$$

- symbol-based approach Codish, Lagoon & Stuckey 2006

interpret function symbols in $\{\,0, \ldots, |\mathcal{F}| - 1\,\}$

Definition

- $\mathbf{a} = \langle a_k, \ldots, a_1 \rangle$
- $\ulcorner \mathbf{a} >_j \mathbf{b} \urcorner = \begin{cases} a_1 \wedge \neg b_1 & \text{if } j = 1 \\ a_j \wedge \neg b_j \vee (a_j \leftrightarrow b_j) \wedge \ulcorner \mathbf{a} >_{j-1} \mathbf{b} \urcorner & \text{if } j > 1 \end{cases}$
- $\ulcorner \mathbf{a} > \mathbf{b} \urcorner = \ulcorner \mathbf{a} >_k \mathbf{b} \urcorner$
- $\ulcorner \mathbf{a} = \mathbf{b} \urcorner = \bigwedge_{i=1}^{k} (a_i \leftrightarrow b_i)$
- $\ulcorner \mathbf{a} \geqslant \mathbf{b} \urcorner = \ulcorner \mathbf{a} > \mathbf{b} \urcorner \vee \ulcorner \mathbf{a} = \mathbf{b} \urcorner$

Definition

$$\ulcorner (\mathbf{a}, \varphi) + (\mathbf{b}, \psi) \urcorner \ = \ (\mathbf{s}, \varphi \wedge \psi \wedge \gamma \wedge \sigma)$$

with

$$\gamma \ = \ \neg c_k \wedge \neg c_0 \wedge \bigwedge_{i=1}^{k} \big(c_i \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_{i-1}) \vee (b_i \wedge c_{i-1}))\big)$$

and

$$\sigma \ = \ \bigwedge_{i=1}^{k} \big(s_i \leftrightarrow (a_i \oplus b_i \oplus c_{i-1})\big)$$

- fresh variables c_i $(0 \leqslant i \leqslant k)$ for carry and s_i $(1 \leqslant i \leqslant k)$ for sum
- $\oplus$ denotes exclusive or
- $\neg c_k$ prevents overflow

Definition

- weight of term t

$$\mathsf{w}(t) \;=\; \begin{cases} (\mathbf{w_0}, \top) & \text{if } t \in \mathcal{V} \\ \ulcorner (\mathbf{f}, \top) + \sum_{i=1}^{n} \mathsf{w}(t_i) \urcorner & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- comparing weights

$$\ulcorner (\mathbf{f}, \varphi) > (\mathbf{g}, \psi) \urcorner \;=\; \ulcorner \mathbf{f} > \mathbf{g} \urcorner \wedge \varphi \wedge \psi$$

- admissibility condition

$$\mathsf{ADM}(\mathcal{F}) \;=\; \ulcorner \mathbf{w_0} > \mathbf{0} \urcorner \;\wedge \bigwedge_{c \,\in\, \mathcal{F}^{(0)}} \ulcorner \mathbf{c} \geqslant \mathbf{w_0} \urcorner \;\wedge$$
$$\bigwedge_{f \,\in\, \mathcal{F}^{(1)}} \big(\ulcorner \mathbf{f} = \mathbf{0} \urcorner \rightarrow \bigwedge_{g \,\in\, \mathcal{F}} (\mathsf{X}_{fg} \vee \mathsf{Y}_{fg})\big)$$

Definition

$$\ulcorner s >_{\mathsf{kbo}} t \urcorner = \begin{cases} \bot & \text{if } s \in \mathcal{V} \text{ or } s = t \text{ or } \exists\, x\ |s|_x < |t|_x \\ \ulcorner \mathrm{w}(s) > \mathrm{w}(t) \urcorner \vee \ulcorner \mathrm{w}(s) = \mathrm{w}(t) \urcorner \wedge \ulcorner s >'_{\mathsf{kbo}} t \urcorner & \\ & \text{otherwise} \end{cases}$$

with

$$\ulcorner s >'_{\mathsf{kbo}} t \urcorner = \begin{cases} \top & \text{if } s = f_n(\ldots(f_1(t)\ldots) \text{ and } t \in \mathcal{V} \\ \ulcorner s_i >_{\mathsf{kbo}} t_i \urcorner & \text{if } s = f(s_1,\ldots,s_n) \text{ and } t = f(t_1,\ldots,t_n) \\ \mathrm{X}_{fg} \vee \mathrm{Y}_{fg} \wedge \ulcorner s_i >_{\mathsf{kbo}} t_i \urcorner & \\ & \text{if } s = f(s_1,\ldots,s_n) \text{ and } t = g(t_1,\ldots,t_m) \end{cases}$$

where i is least $1 \leqslant j \leqslant \min\{m, n\}$ with $s_j \neq t_j$

Theorem

TRS $\mathcal{R}$ is KBO terminating if

$$\varphi_k(\mathcal{R}) \;=\; \mathsf{ADM}(\mathcal{F}) \;\wedge\; \mathsf{QO}(\mathcal{F}) \;\wedge \bigwedge_{l \to r \,\in\, \mathcal{R}} \ulcorner l >_{\mathsf{kbo}} r \urcorner$$

is satisfiable

Remark

satisfying assignment encodes quasi-precedence and weight function

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- BDDs vs SAT MiniSat
- Tseitin's translation to CNF
- optimizations to reduce size of generated formulas
  - $w(\mathrm{f}(y, \mathrm{g}(x), \mathrm{h}(x))) > w(\mathrm{f}(\mathrm{g}(x), \mathrm{g}(x), y))$:

$$\ulcorner(\mathbf{f}, \top) + (\mathbf{w_0}, \top) + (\mathbf{g}, \top) + (\mathbf{w_0}, \top) + (\mathbf{h}, \top) + (\mathbf{w_0}, \top) \; > \; (\mathbf{f}, \top) + (\mathbf{g}, \top) + (\mathbf{w_0}, \top) + (\mathbf{g}, \top) + (\mathbf{w_0}, \top) + (\mathbf{w_0}, \top)\urcorner$$

$$\Downarrow$$

$$\ulcorner(\mathbf{h}, \top) > (\mathbf{g}, \top)\urcorner$$

  - $\ulcorner\mathbf{f} \; = \; \mathbf{0}\urcorner \quad \Rightarrow \quad \neg \bigvee_{i=1}^{k} f_i$

  - use cache for propositional addition

Strict-Precedence

865 TRSs in version 3.2 of TPDB

method(# bits)	total time	# successes	# timeouts (60*s*)
sat(2)	19.2	72 73	0
sat(3)	20.2	77 78	0
sat(4)	21.9	78 79	0
sat(10)	86.1	78 79	1
$\mathsf{T_TT}$	169.5	77 -	1

Example TRS/various_21

$$
\begin{aligned}
\mathsf{p1} + \mathsf{p1} &\to \mathsf{p2} & \mathsf{p1} + (\mathsf{p1} + x) &\to \mathsf{p2} + x \\
\mathsf{p2} + \mathsf{p1} &\to \mathsf{p1} + \mathsf{p2} & \mathsf{p2} + (\mathsf{p1} + x) &\to \mathsf{p1} + (\mathsf{p2} + x) \\
\mathsf{p5} + \mathsf{p1} &\to \mathsf{p1} + \mathsf{p5} & \mathsf{p5} + (\mathsf{p1} + x) &\to \mathsf{p1} + (\mathsf{p5} + x) \\
\mathsf{p5} + \mathsf{p2} &\to \mathsf{p2} + \mathsf{p5} & \mathsf{p5} + (\mathsf{p2} + x) &\to \mathsf{p2} + (\mathsf{p5} + x) \\
\mathsf{p5} + \mathsf{p5} &\to \mathsf{p10} & \mathsf{p5} + (\mathsf{p5} + x) &\to \mathsf{p10} + x \\
\mathsf{p10} + \mathsf{p1} &\to \mathsf{p1} + \mathsf{p10} & \mathsf{p10} + (\mathsf{p1} + x) &\to \mathsf{p1} + (\mathsf{p10} + x) \\
\mathsf{p10} + \mathsf{p2} &\to \mathsf{p2} + \mathsf{p10} & \mathsf{p10} + (\mathsf{p2} + x) &\to \mathsf{p2} + (\mathsf{p10} + x) \\
\mathsf{p10} + \mathsf{p5} &\to \mathsf{p5} + \mathsf{p10} & \mathsf{p10} + (\mathsf{p5} + x) &\to \mathsf{p5} + (\mathsf{p10} + x) \\
\mathsf{p1} + (\mathsf{p2} + \mathsf{p2}) &\to \mathsf{p5} & \mathsf{p1} + (\mathsf{p2} + (\mathsf{p2} + x)) &\to \mathsf{p5} + x \\
\mathsf{p2} + (\mathsf{p2} + \mathsf{p2}) &\to \mathsf{p1} + \mathsf{p5} & \mathsf{p2} + (\mathsf{p2} + (\mathsf{p2} + x)) &\to \mathsf{p1} + (\mathsf{p5} + x) \\
& & (x + y) + z &\to x + (y + z)
\end{aligned}
$$

$$w(\mathsf{p1}) = w(\mathsf{p2}) = 4 \qquad w(\mathsf{p5}) = 6 \qquad w(\mathsf{p10}) = 11 \qquad w(+) = 0$$

$$\mathsf{p2} > \mathsf{p1} \qquad \mathsf{p5} > \mathsf{p1} \qquad \mathsf{p10} > \mathsf{p1}$$

sat(4)	0.19
$\mathsf{T_TT}$	4016.23

Example TRS/HM_t000

$$\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
\vdots & \vdots & & \vdots \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 \\
x+(y:z) \to y:(x+z) & & & 0:x \to x \\
(x:y)+z \to x:(y+z) & & & x:(y:z) \to (x+y):z
\end{array}$$

$$w(0) = w(1) = w(2) = w(3) = w(4) = 1 \qquad w(+) = 7$$

$$w(5) = w(6) = w(7) = w(8) = w(9) = 2 \qquad w(:) = 8 \qquad + > :$$

sat(4)	1.77
T_TT	??

String Rewrite Systems

322 SRSs in version 3.2 of TPDB

strict-precedence

method(# bits)	total time	# successes	# timeouts (60s)
sat(2)	9.1	8	0
sat(3)	12.1	17	0
sat(4)	15.1	24	0
sat(7)	17.0	33	0
sat(10)	21.6	33	0
$\mathsf{T_TT}$	72.4	29	1

Example SRS/Zantema_z113

$11 \to 43$	$33 \to 56$	$55 \to 62$
$12 \to 21$	$34 \to 11$	$56 \to 12$
$22 \to 111$	$44 \to 3$	$66 \to 21$

$\mathsf{T_TT}$

$$w(1) = 32471712256 \qquad w(4) = 21696293888$$
$$w(2) = 48725750528 \qquad w(5) = 44731872512$$
$$w(3) = 43247130624 \qquad w(6) = 40598731520$$
$$3 > 1 > 2 \qquad 1 > 4$$

sat(7)

$$w(1) = 31 \qquad w(2) = 47 \qquad w(3) = 41$$
$$w(4) = 21 \qquad w(5) = 43 \qquad w(6) = 39$$
$$3 > 5 > 6 > 1 > 4 \qquad 1 > 2$$

Dependency Pairs

- correspond to "recursive function calls"

$$l > r \ \forall l \to r \in \mathcal{R}$$

replaced by

$$l \geqslant r \ \forall l \to r \in \mathcal{R} \text{ and } l > r \ \forall \mathrm{DP}(\mathcal{R})$$

- more sofisticated techniques using DP
  - dependency graph
  - recursive SCC algorithm
  - argument filterings

Theorem

TRS $\mathcal{R}$ is terminating if $\forall$ cycle $\mathcal{C}$ in dependency graph of $\mathcal{R}$

$$\mathsf{AF}(\mathcal{F}) \;\wedge\; \mathsf{ADM}^{\pi}(\mathcal{F}) \;\wedge\; \mathsf{PO}(\mathcal{F}) \;\wedge\; \Big(\bigwedge_{l \to r \,\in\, \mathcal{U}(\mathcal{C},\pi) \,\cup\, \mathcal{C}} \ulcorner\, l \geqslant^{\pi}_{\mathsf{kbo}} r \,\urcorner\Big) \;\wedge\; \Big(\bigvee_{l \to r \,\in\, \mathcal{C}} \ulcorner\, l >^{\pi}_{\mathsf{kbo}} r \,\urcorner\Big)$$

is satisfiable

Theorem

TRS $\mathcal{R}$ is terminating if $\forall$ cycle $\mathcal{C}$ in dependency graph of $\mathcal{R}$

$$\mathsf{AF}(\mathcal{F}) \;\wedge\; \mathsf{PO}(\mathcal{F}) \;\wedge\; \Big(\bigwedge_{l \to r \,\in\, \mathcal{U}(\mathcal{C},\pi) \,\cup\, \mathcal{C}} \ulcorner\, l \geqslant^{\pi}_{\mathsf{lpo}} r \,\urcorner\Big) \;\wedge\; \Big(\bigvee_{l \to r \,\in\, \mathcal{C}} \ulcorner\, l >^{\pi}_{\mathsf{lpo}} r \,\urcorner\Big)$$

is satisfiable

865 TRSs in 2006 edition of TPDB

KBO ‖ LPO	$\mathsf{T_TT}$ (L ; K)/(K ; L)	sat(2)		sat(3)		sat(4)	
solved	310/ 295	305	337	338	369	343	377
timeout	121/ 136	6	9	9	11	14	16
time	7025/9025	1664	1940	2076	2351	2623	2898

advanced usable rules

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- $\forall\, k\ \exists$ TRS $\mathcal{R}_k$ such that $\mathcal{R}_k$ is terminating in k bits but not in $k-1$ bits

$$\mathsf{f}(\mathsf{g}(x,y)) \to \mathsf{g}(\mathsf{f}(x),\mathsf{f}(y)) \qquad \mathsf{s}(x) \to \mathsf{f}(\mathsf{f}(x)) \qquad \mathsf{init}(x) \to \mathsf{s}^{2^k}(x)$$

- SAT encoding of other termination methods
  - LPO, MPO, RPO
  - polynomial interpretations
  - matrix interpretations
  - general version of KBO (weight function $\Rightarrow$ monotone algebra)
  - ...
- Pseudo Boolean Constraints