Equational Reasoning for Termination of Rewriting

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Overview

- Dependency Pairs
- Dependency Graph
- Waldmeister

Definition (Dependency Pairs [AG00])

$\mathrm{DP}(\mathcal{R}) = \{l^\sharp \to u^\sharp \mid l \to r \in \mathcal{R}, r \trianglerighteq u, root(u) \text{ defined}\}$

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Example

$$\mathsf{0} + y \to y \qquad \mathsf{0} \times y \to \mathsf{0}$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

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with DPs

$$\mathsf{s}(x) +^\sharp y \to x +^\sharp y \quad \mathsf{s}(x) \times^\sharp y \to (x \times y) +^\sharp y \quad \mathsf{s}(x) \times^\sharp y \to x \times^\sharp y$$

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Theorem (Termination)

$\mathcal{R}$ terminating iff no sequence $s_0 \to_{\mathrm{DP}(\mathcal{R})} t_0 \to_{\mathcal{R}}^* s_1 \to_{\mathrm{DP}(\mathcal{R})} t_1 \to_{\mathcal{R}}^* \cdots$

Definition (Dependency Graph)

graph DG $:= (N, E)$ with

- $N = \mathrm{DP}(\mathcal{R})$
- $E = \{(s \to t,\ u \to v) \in N \times N \mid \exists \sigma, \tau : t\sigma \to_{\mathcal{R}}^{*} u\tau\}$

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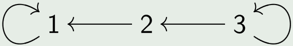
\begin{aligned}
0 + y &\to y & \qquad 0 \times y &\to 0 \\
\mathsf{s}(x) + y &\to \mathsf{s}(x + y) & \qquad \mathsf{s}(x) \times y &\to (x \times y) + y
\end{aligned}
$$

$1 : \mathsf{s}(x) +^{\sharp} y \to x +^{\sharp} y \quad 2 : \mathsf{s}(x) \times^{\sharp} y \to (x \times y) +^{\sharp} y \quad 3 : \mathsf{s}(x) \times^{\sharp} y \to x \times^{\sharp} y$



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Example

$$0 + y \;\geq\; y \qquad\qquad 0 \times y \;\geq\; 0$$

$$\mathsf{s}(x) + y \;\geq\; \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \;\geq\; (x \times y) + y$$

$$1 : \mathsf{s}(x) +^{\sharp} y \;>\; x +^{\sharp} y \quad 2 : \mathsf{s}(x) \times^{\sharp} y \to (x \times y) +^{\sharp} y \quad 3 : \mathsf{s}(x) \times^{\sharp} y \;\to\; x \times^{\sharp} y$$

$$\circlearrowleft 1 \longleftarrow 2 \longleftarrow 3 \circlearrowright$$

Theorem

$\forall$ *SCC* $\mathcal{S}$ $\exists$ $(\geq, >)$ *s.t.* $\mathcal{S} \cup \mathcal{R} \subseteq \geq$ *and* $\mathcal{S} \cap > \neq \varnothing$ *then* $\mathcal{R}$ *terminating*

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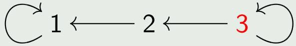
Example

$$0 + y \geq y \qquad 0 \times y \geq 0$$

$$\mathsf{s}(x) + y \geq \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \geq (x \times y) + y$$

$$1 : \mathsf{s}(x) +^\sharp y \to x +^\sharp y \quad 2 : \mathsf{s}(x) \times^\sharp y \to (x \times y) +^\sharp y \quad 3 : \mathsf{s}(x) \times^\sharp y > x \times^\sharp y$$



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$\forall$ SCC $\mathcal{S}$ $\exists$ $(\geq, >)$ s.t. $\mathcal{S} \cup \mathcal{R} \subseteq \geq$ and $\mathcal{S} \cap > \neq \varnothing$ then $\mathcal{R}$ terminating

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$$\mathcal{R} : \qquad \mathsf{f}(\mathsf{0}, \mathsf{1}, x) \to \mathsf{f}(\mathsf{h}(x), \mathsf{h}(x), x)$$

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ren,cap [AG00]

- ren: rename variables (fresh variables)
- cap: replace term with defined root by fresh variable
- idea: $\exists \sigma, \tau : t\sigma \to_{\mathcal{R}}^{*} u\tau$ implies $\mathsf{ren}(\mathsf{cap}(t)) \overset{\mathsf{u}}{\approx} u$

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$$\mathsf{F}(\mathsf{h}(y), \mathsf{h}(z), w) \quad \overset{\mathsf{u}}{\not\approx} \mathsf{F}(0, 1, x)$$

Example

$$\mathcal{R}: \quad \mathsf{f}(0,1,x) \to \mathsf{f}(\mathsf{h}(x),\mathsf{h}(x),x) \quad \mathsf{h}(x) \to \mathsf{g}(x)$$

Example

$$\mathcal{R}: \qquad f(0,1,x) \to f(h(x),h(x),x) \quad h(x) \to g(x)$$

$$\mathsf{ren}(\mathsf{cap}(F(h(x),h(x),x))) \overset{u}{\approx} F(0,1,x)$$

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$$\mathcal{R}: \qquad f(0,1,x) \to f(h(x),h(x),x) \quad h(x) \to g(x)$$

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ren, qɐɔ [Mi02,HM05]

- qɐɔ: replace term with defined root (wrt $\rotatebox[origin=c]{180}{$\mathcal{R}$}$) by fresh variable
- idea: use system $\mathcal{R}$ to test $\mathsf{ren}(\mathsf{cap}(t)) \overset{\mathsf{u}}{\approx} u$ $(\exists \sigma, \tau : t\sigma \to^*_{\mathcal{R}} u\tau)$
  idea: use system $\rotatebox[origin=c]{180}{$\mathcal{R}$}$ to test $t \overset{\mathsf{u}}{\approx} \mathsf{ren}(\text{qɐɔ}(u))$ $(\exists \sigma, \tau : t\sigma \;{}_{\rotatebox[origin=c]{180}{$\mathcal{R}$}}^{*}\!\!\leftarrow u\tau)$
- non-collapsing TRSs only

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$$\mathcal{R}: \qquad f(0,1,x) \to f(h(x),h(x),x) \quad h(x) \to g(x)$$

$$F(h(x),h(x),x) \stackrel{u}{\not\approx} \qquad\qquad F(0,1,y)$$

Example

$\mathcal{R}:\quad \mathsf{f}(0,1,x) \to \mathsf{f}(\mathsf{h}(\mathsf{a}(x)),\mathsf{h}(\mathsf{a}(x)),x) \quad \mathsf{h}(\mathsf{b}(x)) \to 0 \quad \mathsf{h}(\mathsf{b}(x)) \to 1$

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$$F(h(a(x)),h(a(x)),x) \overset{u}{\approx} F(y,z,w)$$

tcap [GTS 05]

- $\text{tcap}(x) = y$ (fresh variable)
- $\text{tcap}(f(t_1,\ldots,t_n)) = f(\text{tcap}(t_1),\ldots,\text{tcap}(t_n))$
  if $f(\text{tcap}(t_1),\ldots,\text{tcap}(t_n))$ does not unify with lhs from $\mathcal{R}$
- $\text{tcap}(f(t_1,\ldots,t_n)) = y$ (fresh variable) otherwise

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$$\mathcal{R}: \quad f(0,1,x) \to f(h(a(x)), h(a(x)), x) \quad h(b(x)) \to 0 \quad h(b(x)) \to 1$$

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Example

$$\mathcal{R}: \quad f(0,1,x) \to f(h(x),h(x),x) \quad h(x) \to 2 \quad 3 \to 0 \quad 3 \to 1$$

Example

$\mathcal{R}: \quad \mathsf{f}(0,1,x) \to \mathsf{f}(\mathsf{h}(x),\mathsf{h}(x),x) \quad \mathsf{h}(x) \to 2 \quad 3 \to 0 \quad 3 \to 1$

$\mathsf{tcap}(\mathsf{F}(\mathsf{h}(x),\mathsf{h}(x),x)) \stackrel{\mathsf{u}}{\approx} \mathsf{F}(0,1,x)$

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Remark

also $\widetilde{\mathrm{cap}}$ does not help

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Equational Reasoning

- $t\sigma \to^*_{\mathcal{R}} u\tau$ implies $t\sigma \leftrightarrow^*_{\mathcal{R}} u\tau$ means $\mathcal{E}_{\mathcal{R}} \models t \approx u$
- Waldmeister can refute $\mathcal{E}_{\mathcal{R}} \models t \approx u$

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Equational Reasoning

- $t\sigma \to_{\mathcal{R}}^{*} u\tau$ implies $t\sigma \leftrightarrow_{\mathcal{R}}^{*} u\tau$ means $\mathcal{E}_{\mathcal{R}} \models t \approx u$
- Waldmeister can refute $\mathcal{E}_{\mathcal{R}} \models t \approx u$ (sometimes)

Basic Idea

$\mathcal{E}_{\mathcal{R}} \quad \longrightarrow$

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$\Longrightarrow$ yes, no, timeout

Basic Idea

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$t \approx u \longrightarrow$

$\Longrightarrow$ yes, no, timeout

Example (various/11)

$$f(0,1,x) \to f(h(x),h(x),x) \qquad h(0) \to 0 \qquad h(g(x,y)) \to y$$

with DPs

$$F(0,1,x) \to F(h(x),h(x),x) \qquad F(0,1,x) \to H(x)$$

```
NAME        various11
MODE        PROOF
SORTS       ANY
SIGNATURE   g: ANY ANY -> ANY          0: -> ANY
            h: ANY -> ANY              1: -> ANY
            F: ANY ANY ANY -> ANY      f: ANY ANY ANY -> ANY
ORDERING    LPO g > 1 > 0 > h > F > f
VARIABLES   z, w, y, x : ANY
EQUATIONS   f(0, 1, x) = f(h(x), h(x), x)
            h(0) = 0
            h(g(x, y)) = y
            F(x, x, x) = F(x, x, x)
CONCLUSION  F(h(z), h(z), z) = F(0, 1, w)
```

```
Initial eqs: (    1)  x1 = h(g(x2,x1))
             (    2)  h(0) = 0
             (    3)  f(h(x1),h(x1),x1) = f(0,1,x1)
             (    4)  F(x1,x1,x1) = F(x1,x1,x1)
             (    5)  eq(x1,x1) = true
             (    6)  eq(F(h(x1),h(x1),x1),F(0,1,x2)) = false
Goals:       (    1)  true ?= false
  using narrowing to prove F(h(x1),h(x1),x1) ?= F(0,1,x2)
******************** COMPLETION - PROOF ********************
new rule:         1  F(h(x1),h(x1),x1) ? F(0,1,x2)
new rule:         2  h(g(x1,x2)) -> x2
new rule:         3  F(x1,x1,g(x2,x1)) ? F(0,1,x3)
new rule:         4  h(0) -> 0
new rule:         5  F(0,0,0) ? F(0,1,x1)
new rule:         6  f(h(x1),h(x1),x1) -> f(0,1,x1)
new rule:         7  eq(x1,x1) -> true
new rule:         8  f(0,0,0) -> f(0,1,0)
new equation:     1  f(0,1,g(x1,x2)) = f(x2,x2,g(x1,x2))
Refuted Goals: No.  1: true ?= false, current true ?= false
  using narrowing to prove F(h(x1),h(x1),x1) ?= F(0,1,x2)
1 goal was specified, which was refuted.
Waldmeister states: System completed.
```

Experimental Results ($\mathsf{T_TT_2}$)

	sccs	competition
edg***		
wdg		

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	sccs	competition
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- 1391 TRSs from TPDB 5.0
- 5 sec timeout
- 0.1 sec timeout (per edge)

Experimental Results ($\mathsf{T_TT_2}$)

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- 0.1 sec timeout (per edge)

Experimental Results (T_TT_2)

	sccs	competition
edg***	60	... not essential for competition
wdg	71	

- 1391 TRSs from TPDB 5.0
- 5 sec timeout
- 0.1 sec timeout (per edge)

Experimental Results ($\mathsf{T_TT_2}$)

	sccs	competition
edg***	60	but still . . .
wdg	71	

- 1391 TRSs from TPDB 5.0
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Experimental Results ($\mathsf{T_TT_2}$)

	sccs	competition
edg***	60	779
wdg	71	778

- 1391 TRSs from TPDB 5.0
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- 0.1 sec timeout (per edge)

Experimental Results ($\mathsf{T_TT_2}$)

	sccs	competition
edg***	60	779
wdg	71	778

- 1391 TRSs from TPDB 5.0
- 5 sec timeout
- 0.1 sec timeout (per edge)
- competition:
  gain various/11, secret06/tpa01
  lose two TRCSR/*, various/14 (5 sec limit)
- AProVE, Jambox, TPA could (partly) solve these systems

Conclusion

Contribution

DP graph estimation based on equational reasoning

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DP graph estimation based on equational reasoning

Related Work

- apart from [AG00, Mi02, HM05, GTS05]
- [KM09] → tree automata techniques
- [ZM09] → (increasing) interpretations

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DP graph estimation based on equational reasoning

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DP graph estimation based on equational reasoning

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Future Work

- innermost termination (usable rules)
- other completion tools suitable?