Automating Ordinal Interpretations

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Overview

- Algebras
- Ordinals
- Ordinal Interpretations
- Implementation and Evaluation
- Conclusion

Definition

well-founded monotone algebra $(\mathcal{A}, >)$ for signature $\mathcal{F}$ has

- carrier A
- interpretation $f_{\mathcal{A}}$ for each $f \in \mathcal{F}$
- $a > b$ implies $f_{\mathcal{A}}(\dots, a, \dots) > f_{\mathcal{A}}(\dots, b, \dots)$ for each $f \in \mathcal{F}$
- $>$ well-founded

Definition

$s >_{\mathcal{A}} t$ if $\alpha_{\mathcal{A}}[s] > \alpha_{\mathcal{A}}[t]$ for all assignments α

Example

polynomial interpretations, matrix interpretations, arctic interpretations, (match-bounds)

Theorem

TRS terminating if compatible with well-founded monotone algebra

Definition

TRS $\mathcal{R}$ compatible with algebra $\mathcal{A}$ if $\ell >_{\mathcal{A}} r$ for every $\ell \to r \in \mathcal{R}$

(Polynomial) Interpretations

		$\mathsf{f}(x) \to \mathsf{g}(x,x)$	
$\mathsf{f}_{\mathbb{N}}(x) = 2x + 1$	$\mathsf{g}_{\mathbb{N}}(x,y) = x + y$	$2x + 1 > 2x$	✓
$\mathsf{f}_{\mathbb{N}}(x) = x + 1$	$\mathsf{g}_{\mathbb{N}}(x,y) = x - y$	$x + 1 > 0$	×
$\mathsf{f}_{\mathbb{N}}(x) = x + 1$	$\mathsf{g}_{\mathbb{N}}(x,y) = \max\{x,y\}$	$x + 1 > x$	×
$\mathsf{f}_{\mathbb{N}}(x) = x + 1$	$\mathsf{g}_{\mathbb{N}}(x,y) = \min\{x,y\}$	$x + 1 > x$	×

Theorem

TRS terminating if compatible with well-founded weakly-monotone simple algebra

Definition

simple $f_{\mathcal{A}}(\dots, a, \dots) \geqslant a$

weakly-monotone $a > b$ implies $f(\dots, a, \dots) \geqslant f(\dots, b, \dots)$

(Polynomial) Interpretations

		$\mathsf{f}(x) \to \mathsf{g}(x, x)$	
$\mathsf{f}_{\mathbb{N}}(x) = 2x + 1$	$\mathsf{g}_{\mathbb{N}}(x, y) = x + y$	$2x + 1 > 2x$	✓
$\mathsf{f}_{\mathbb{N}}(x) = x + 1$	$\mathsf{g}_{\mathbb{N}}(x, y) = x - y$	$x + 1 > 0$	×
$\mathsf{f}_{\mathbb{N}}(x) = x + 1$	$\mathsf{g}_{\mathbb{N}}(x, y) = \max\{x, y\}$	$x + 1 > x$	✓
$\mathsf{f}_{\mathbb{N}}(x) = x + 1$	$\mathsf{g}_{\mathbb{N}}(x, y) = \min\{x, y\}$	$x + 1 > x$	×

Remark

Arctic / Quasi-periodic / Ordinal Interpretations are weakly-monotone

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Definition

ordinal is set α such that

1. α is totally ordered with respect to membership
2. every element of α is subset of α

Example

$\varnothing$	☺	0
$\{\varnothing\}$	☺	1
$\{\{\varnothing\}\}$	☹	
$\{\varnothing, \{\varnothing\}\}$	☺	2
$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$	☺	3
$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \dots\}$	☺	ω
$\omega \cup \{\omega\}$	☺	$\omega + 1$

Lemma

ordinal α *is either*

- $\beta \cup \{\beta\} = \beta + 1$ *successor ordinal*
- $\bigcup \alpha$ *limit ordinal*

Definition (Addition)

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
3. $\alpha + \beta = \bigcup_{\xi < \beta} (\alpha + \xi)$ for all limit ordinals $\beta > 0$

Example

$$1 + \omega = \bigcup_{\xi < \omega} 1 + \xi = \omega \neq \omega + 1$$

Definition (Multiplication)

1. $\alpha \cdot 0 = 0$
2. $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$
3. $\alpha \cdot \beta = \bigcup_{\xi < \beta} \alpha \cdot \xi$ for all limit ordinals $\beta > 0$

Example

$$2 \cdot \omega = \bigcup_{\xi < \omega} 2 \cdot \xi = \omega \neq \omega + \omega = \omega \cdot 2$$

Definition (Exponentation)

1. $\alpha^0 = \alpha$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$
3. $\alpha^\beta = \bigcup_{\xi<\beta} \alpha^\xi$ for all limit ordinals $\beta > 0$

Definition

$\epsilon_0 = \bigcup_{n<\omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$

Remark

ϵ_0 is least ordinal ϵ such that $\omega^\epsilon = \epsilon$

$$\epsilon_0 = \omega^{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}$$

Remark

addition is weakly monotone but not monotone:

$$1 > 0 \qquad \textit{and} \qquad 1 + \omega \not> 0 + \omega$$

Theorem (Cantor Normal Form)

every ordinal $\alpha < \epsilon_0$ *can be uniquely written as*

$$\alpha = \omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n$$

such that $\alpha > \alpha_1 \geqslant \cdots \geqslant \alpha_n$ *and* $k_1, \ldots, k_n$ *are natural numbers* $\neq 0$

Notation

O is set of ordinals smaller than ϵ_0 $\qquad\qquad \mathsf{O} = \epsilon_0$

Overview

Definition

algebra $\mathcal{O}$

- carrier O
- every $f_{\mathcal{O}}$ is *polynomial*
- standard order $>$

Example

- $4 > 3$ ✓
- $\omega > 3$ ✓
- $\omega + 1 > \omega$ ✓
- $1 + \omega \not> \omega$ ×
- $\omega + 1 > 1 + \omega$ ✓
- $\omega \cdot 2 > 2 \cdot \omega$ ✓

Example ($\mathcal{R}_1$)

$$\mathsf{a}\,\mathsf{b} \to \mathsf{b}\,\mathsf{a}\,\mathsf{a} \qquad \mathsf{b} \to \epsilon$$

$$a_{\mathcal{O}}(x) = x + 1 \qquad b_{\mathcal{O}}(x) = x + \omega$$

$$x + \omega + 1 > x + \omega = x + 1 + 1 + \omega \qquad x + \omega > x$$

Note: $\mathsf{a}\,\mathsf{b}^n \to^* \mathsf{a}^{2^n}$

Example ($\mathcal{R}_2$)

$$\mathsf{a}\,\mathsf{b} \to \mathsf{b}\,\mathsf{a}\,\mathsf{a} \qquad \mathsf{b} \to \epsilon \qquad \mathsf{b}\,\mathsf{c} \to \mathsf{c}\,\mathsf{b}\,\mathsf{b} \qquad \mathsf{c} \to \epsilon$$

$$a_{\mathcal{O}}(x) = x + 1 \qquad b_{\mathcal{O}}(x) = x + \omega \qquad c_{\mathcal{O}}(x) = x + \omega^2$$

$$x + \omega + 1 > x + \omega \quad x + \omega > x \quad x + \omega^2 + \omega > x + \omega^2 \quad x + \omega^2 > x$$

Note: $\mathsf{a}\,\mathsf{b}\,\mathsf{c}^n \to^* \mathsf{a}^{2^{2^n}}$

Definition ($\mathcal{R}_m$)

$$\mathsf{a}_i\,\mathsf{a}_{i+1} \to \mathsf{a}_{i+1}\,\mathsf{a}_i\,\mathsf{a}_i \qquad \mathsf{a}_{i+1} \to \epsilon \qquad 1 \leqslant i \leqslant m$$

Lemma

$$\mathsf{a}_0\,\mathsf{a}_1 \cdots \mathsf{a}_{m-1}\,\mathsf{a}_m^n \to^*_{\mathcal{R}_m} \mathsf{a}_0^{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}} \qquad \textit{(tower of 2's has height m)}$$

Definition (linear ordinal interpretation of degree d)

$$a_{\mathcal{O}}(x) = xa' + \omega^d a_d + \omega^{d-1} a_{d-1} + \cdots + \omega a_1 + a_0$$

with $a', a_d, \ldots, a_0 \in \mathbb{N}$

Lemma

For every $\mathcal{R}_m$ with $m \in \mathbb{N}$ there exists a compatible linear ordinal interpretation of degree m but not of degree $m - 1$

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polynomials using ordinals below ω^ω (for SRSs)

canonical form ($f', f_d, \ldots, f_0 \in \mathbb{N}$)

$$f_{\mathcal{O}}(x) = xf' + \omega^d f_d + \cdots + \omega^1 f_1 + f_0$$

Example

- encoding $\mathrm{a}(\mathrm{b}(x))$ with $d = 1$

$$\begin{aligned} &(xb' + \omega b_1 + b_0)a' + \omega a_1 + a_0 \\ &= xb'a' + \omega b_1 a' + b_0 a' + \omega a_1 + a_0 \\ &= xb'a' + \omega(b_1 a' + a_1) + (a_1 > 0 \mathbin{?} 0 : b_0 a') + a_0 \end{aligned}$$

- compatibility $\quad xl' + \omega l_1 + l_0 > xr' + \omega r_1 + r_0$

$$l' \geqslant r' \wedge (l_1 > r_1 \vee (l_1 = r_1 \wedge l_0 > r_0))$$

- simplicity/weak-monotonicity

$$f' \geqslant 1$$

720 SRSs from TPDB 7.0.2

2 bits for $f', f_d, \ldots, f_0$, 5 bits for intermediate results

method	yes	time (avg)	timeout (60s)
linear interpretations ($\mathbb{N}$)	6	0.5	0
linear ordinal interpretations (O, degree 1)	27	0.7	0
linear ordinal interpretations (O, degree 2)	29	1.1	0
linear ordinal interpretations (O, degree 3)	29	1.4	0

4 bits for $f', f_d, \ldots, f_0$, 8 bits for intermediate results

method	yes	time (avg)	timeout (60s)
linear interpretations ($\mathbb{N}$)	19	0.9	1
linear ordinal interpretations (O, degree 1)	40	2.5	1
linear ordinal interpretations (O, degree 2)	40	3.8	6
linear ordinal interpretations (O, degree 3)	38	2.1	21

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Summary

- linear ordinal interpretations
- multiple exponential derivational complexity
- implementation for SRSs

Future Work

- implementation for TRSs
- problem: $f(x, y) \to g(y, x)$
  compatibility $xf_1 + yf_2 + f_0 \geqslant yg_1 + xg_2 + g_0$?
  $f_1 \geqslant g_2, f_2 \geqslant g_1, f_0 \geqslant g_0$ not sufficient (e.g. $f_2, f_1, f_0, g_2, g_1, g_0, x = 1,\ y = \omega$)
  $$1 + \omega + 1 \not\geqslant \omega + 1 + 1$$
- solution: *natural* addition/multiplication of ordinals

Ultimate Goal

prove battle of *Hercules and Hydra* automatically