Automating Elementary Interpretations

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Example (Addition)

$$x + 0 \to x$$
$$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$

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Theorem

A TRS is terminating if it is compatible with a well-founded monotone algebra.

Example (Addition)

$$x + 0 \to x \qquad\qquad x + 2 > x$$

$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad\qquad x + 2y + 2 > x + 2y + 1$$

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Example (Multiplication)

$$\begin{aligned} x + 0 &\to x \\ x + \mathsf{s}(y) &\to \mathsf{s}(x + y) \\ 0 \cdot x &\to 0 \\ \mathsf{s}(x) \cdot y &\to x \cdot y + y \end{aligned}$$

$$\begin{aligned} x + 2 &> x \\ x + 2y + 2 &> x + 2y + 1 \end{aligned}$$

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\mathsf{s}(x) \cdot y &\to x \cdot y + y & 2xy + 2x + 3y + 2 &> 2xy + 2x + 3y \\
\mathsf{fact}(0) &\to \mathsf{s}(0) \\
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0_{\mathbb{N}} = ? \quad \mathsf{s}_{\mathbb{N}}(x) = ? \qquad x +_{\mathbb{N}} y = ? \qquad x \cdot_{\mathbb{N}} y = ?
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$$
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Overview

- History
- Encoding
- Implementation
- Conclusion

Why elementary interpretations?

Some history

method	theory	implementation
polynomials	Lankford'79	

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Which functions

$+, \cdot, 2^x$ ✓ x^2, xy, y^x ? $\sin(x), \log(x), \sqrt[n]{m}$ ✗

Which shape?

Linear elementary interpretation (LEI)

- $f(\vec{x}) = p(\vec{x}) + b(\vec{x})^{q(\vec{x})}$ proposed by Lucas'09

  $p(\vec{x}), b(\vec{x}), q(\vec{x})$ linear functions

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Fixed base elementary interpretations (FBI)

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FBIs - General idea

$$f(\vec{x}) = \sum_{1 \leqslant i \leqslant n} x_i f_i + f_0 + b^{f'(\vec{x})}(\sum_{1 \leqslant i \leqslant n} x_i \dot{f}_i + \dot{f}_0)$$

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with $f_0, \ldots, f_n, \dot{f}_0, \ldots, \dot{f}_n \in \mathbb{N}$ and $f'(x)$ an FBI

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with $f_0, \ldots, f_n, \dot{f}_0, \ldots, \dot{f}_n \in \mathbb{N}$ and $f'(x)$ an FBI

Problem (FBIs not closed under typical operations)

- addition: $2^x + 2^y$ ✗
- multiplication: $2^x(x + 2^y) = 2^x x + 2^{x+y}$ ✗
- composition: $(x + 2^x x) \circ 2^x = 2^x + 2^{2^x} 2^x = 2^x + 2^{2^x + x}$ ✗
- scalar multiplication: $(x + 2^x y)4 = 4x + 2^x 4y$ ✓

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Solution

- use approximations
- if $[\alpha]^{\mu}_{\mathbb{N}}(\ell) > [\alpha]^{\nu}_{\mathbb{N}}(r)$ for all $\ell \to r \in \mathcal{R}$ then $\mathcal{R}$ is terminating

Desired properties

$$f(\vec{x}) = \sum_{1 \leqslant i \leqslant n} x_i f_i + f_0 + b^{f'(\vec{x})}\left(\sum_{1 \leqslant i \leqslant n} x_i \dot{f}_i + \dot{f}_0\right) = \overline{f}(\vec{x}) + b^{f'(\vec{x})} \dot{f}(\vec{x})$$

Monotonicity

$mon(f) := \bigwedge_{1 \leqslant i \leqslant n} \big(f_i > 0 \vee \dot{f}_i > 0 \vee (\dot{f}(\vec{x}) > 0 \wedge mon(f'(\vec{x}))\big)$

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Well-definedness ($\mathbb{N}_{\geqslant 1}$)

$wd(f) := [f(\vec{x}) \geqslant 1]$

Addition

$$f(\vec{x}) = \sum_{1 \leqslant i \leqslant n} x_i f_i + f_0 + b^{f'(\vec{x})}(\sum_{1 \leqslant i \leqslant n} x_i \dot{f}_i + \dot{f}_0) \qquad g(\vec{x}) = \sum_{1 \leqslant i \leqslant n} x_i g_i + g_0 + b^{g'(\vec{x})}(\sum_{1 \leqslant i \leqslant n} x_i \dot{g}_i + \dot{g}_0)$$

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$$b^{g'(\vec{x})} \cdot_\nu f(\vec{x}) = (\dot{f}(\vec{x}) = 0) \; ? \; b^{g'(\vec{x})}\left(\sum_{1 \leqslant i \leqslant n} x_i f_i + f_0 \right)$$
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- $2^x \cdot_\mu y = 2^x y$
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Composition

Let $f(\vec{g})(\vec{x}) := f(g_1(\vec{x}), \ldots, g_n(\vec{x}))$ and $\sum_{1 \leqslant i \leqslant n}^{\mu} h_i := h_1 +_\mu \cdots +_\mu h_n$.

$$f(\vec{g})_\mu(\vec{x}) = \sum_{1 \leqslant i \leqslant n}^{\mu} g_i(\vec{x}) f_i +_\mu f_0 +_\mu b^{f'(\vec{g})_\mu(\vec{x})} \cdot_\mu \Big(\sum_{1 \leqslant i \leqslant n}^{\mu} g_i(\vec{x}) \dot{f}_i +_\mu \dot{f}_0 \Big)$$

$$f(\vec{g})_\nu(\vec{x}) = \sum_{1 \leqslant i \leqslant n}^{\nu} g_i(\vec{x}) f_i +_\nu f_0 +_\nu b^{f'(\vec{g})_\nu(\vec{x})} \cdot_\nu \Big(\sum_{1 \leqslant i \leqslant n}^{\nu} g_i(\vec{x}) \dot{f}_i +_\nu \dot{f}_0 \Big)$$

Comparison

$$f(\vec{x}) = \overline{f}(\vec{x}) + b^{f'(\vec{x})}\dot{f}(\vec{x}) \qquad g(\vec{x}) = \overline{g}(\vec{x}) + b^{g'(\vec{x})}\dot{g}(\vec{x})$$

$[f(\vec{x}) \geqslant g(\vec{x})]$

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Heuristics

d limit degree of FBIs (based on call-graph)

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Example (Factorial)

$$x + 0 \to x \qquad 0 \cdot x \to 0 \qquad \mathsf{fact}(0) \to \mathsf{s}(0)$$

$$x + \mathsf{s}(y) \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \cdot y \to x \cdot y + y \qquad \mathsf{fact}(\mathsf{s}(x)) \to \mathsf{s}(x) \cdot \mathsf{fact}(x)$$

function symbol	0	s	$+$	$\cdot$	fact
degree					

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function symbol	0	s	+	·	fact
degree	0	0	0	1	

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function symbol	0	s	$+$	$\cdot$	fact
degree	0	0	0	1	2

Heuristics

- d limit degree of FBIs (based on call-graph)
- + limit shape/coefficients of FBIs

Example (Factorial)

$$x + 0 \to x \qquad 0 \cdot x \to 0 \qquad \mathsf{fact}(0) \to \mathsf{s}(0)$$
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function symbol	0	s	+	·	fact
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Experimental results

$\mathsf{T_TT_2}$

method	TPDB 8.0.6[a] YES	avg. time	Example (Factorial) time
poly	125	0.4	-

[a] TRS Standard

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$\mathsf{T_TT_2}$

method	TPDB 8.0.6[a] YES	avg. time	Example (Factorial) time
poly	125	0.4	-
fbi	32	31.8	*1243.7*
fbi(d)	159	5.8	20.2
fbi(d+)	162	5.7	9.1

[a] TRS Standard

Conclusion

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- implementation of (some) elementary functions
- harder than ordinal arithmetic

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Future work

- suitable for AC
- suitable for ordered completion