An Iterative Path Ordering

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Abstract

In a recursive path ordering with status, terms starting with the same function symbol are compared by recursively comparing their arguments, either as sequences, ordered lexicographically, or as multisets, ordered by the multiset extension of the recursive path ordering. Klop, van Oostrom, and de Vrijer present an iterative approach to the lexicographic path ordering which uses only the first way of comparing arguments. We extend their approach to include also a comparison of multisets of arguments. This approach is proved equivalent to the recursive path ordering with status.

1 Introduction

An important method for proving normalisation of a term rewriting system is the *recursive path ordering* (RPO), defined originally by Dershowitz [2]. RPO is a recursively defined relation $\succ$ on the set of terms, based on a well-founded ordering on the function symbols. It was demonstrated that RPO is a reduction ordering, which means that termination of a rewriting relation is guaranteed if $l \succ r$ for each of its rewrite rules $l \to r$.

Many variations and adaptations of RPO have been defined. One of them is the *lexicographic path ordering* (LPO) by Kamin and Lévy [3]. The difference with RPO is in the way two terms with the same root symbol are compared. For RPO, $f(s_1, \ldots, s_n) \succ f(t_1, \ldots, t_n)$ if the multiset $\{\{s_1, \ldots, s_n\}\}$ is greater than the multiset $\{\{t_1, \ldots, t_n\}\}$ in the multiset extension of (the recursive calls of) RPO. For LPO, $f(s_1, \ldots, s_n) \succ f(t_1, \ldots, t_n)$ if the sequence $[s_1, \ldots, s_n]$ is greater than the sequence $[t_1, \ldots, t_n]$ in the lexicographic extension of (the recursive calls of) LPO.

An alternative approach to RPO is presented by Bergstra and Klop [1]. Starting from a well-founded ordering (or more generally a relation) on the set of function symbols, an auxiliary term rewriting system is defined. This auxiliary system depends only on the ordering on function symbols, and not on the term rewriting system that we want to prove terminating (just like an instance of RPO only depends on the ordering on function symbols). Now the transitive closure $\rightarrow^+$ of the auxiliary term rewriting system $\rightarrow$ plays the role of RPO. That is, if $l \rightarrow^+ r$ in the auxiliary term rewriting system, then the term rewriting system under consideration is terminating.

The exact relationship between the recursive and iterative approach is studied by Klop, van Oostrom, and de Vrijer [4]. On the recursive side, they consider LPO. On the iterative side, an auxiliary term rewriting system *Lex* is defined. Its transitive closure is called the *iterative lexicographic path ordering* (ILPO). It is shown that ILPO is well-founded, and that LPO and ILPO coincide, if we start from a transitive relation on function symbols (which is the case if we start with an ordering).

The starting point of this note is [4]. We explore the robustness of the iterative approach in the light of the recursive path ordering with status, where terms starting with the same function symbol are compared by comparing their arguments either as multisets or as (lexicographically ordered) vectors, depending on the status of the function symbol. Building on the rules of ILPO, we present an iterative approach for this version of RPO, and go on to prove that this new system IPO coincides with its recursive counterpart. Termination is not proved separately here, but results as a consequence of the wellfoundedness of RPO.

2 Preliminaries

We assume familiarity with first-order term rewriting with the usual notations. In particular, terms are written as $s, t, u, v, \ldots$. We do recall the definitions of the lexicographic and multiset extensions of a relation, which play an important role in the definition of RPO. For sequences we use the notation $[\ldots]$, and for multisets the notation $\{\!\{\ldots\}\!\}$.

Definition 1 (lexicographical extension)**.** Given a relation $>$ on terms, its lexicographical extension $>_{\text{lex}}$ on sequences of terms is defined as: $[s_1, \ldots, s_n] >_{\text{lex}} [t_1, \ldots, t_m]$ iff $n < m$ or both $n = m$ and, for some $i \leq n$, $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$ and $s_i > t_i$.

In the remainder, we will only compare sequences of equal length.

Definition 2 (multiset extension)**.** Given a relation $>$ on terms, its multiset extension $>_{\text{mul}}$ on multisets of terms is defined as: $X >_{\text{mul}} Y$ iff there are multisets A, B, C such that $X = A \uplus B$, $Y = A \uplus C$, $B \neq \emptyset$ and $\forall c \in C \exists b \in B [b > c]$.

It has been proved that if $>$ is a well-founded relation, then so are its lexicographical and multiset extensions.

In the literature many variations of the definition of RPO are given. In all definitions, a relation $\rhd$ on the set of function symbols is "lifted" to a relation on the set of terms. The requirements on $\rhd$ and the inductive lifting rules may vary. Here we assume a set of function symbols Σ and a well-founded ordering $\rhd$ on Σ. We consider *RPO with status*, which means that comparing two terms with the same root function symbol is done by comparing the arguments with either the lexicographic or the multiset extension of (the recursive calls of) RPO, depending on the "status" of the function symbol. The status is defined by taking Σ the disjoint union of Σ_{LEX} and Σ_{MUL}.

Definition 3 (RPO)**.** The *recursive path ordering* (RPO) induced by $\rhd$ is the relation $\succ_{\text{rpo}}$ generated by the following rules:
$s = f(s_1, \ldots, s_n) \succ_{\text{rpo}} t$ iff:

(RPO1) $s_i \succeq_{\text{rpo}} t$ for some $i \in \{1, \ldots, n\}$, OR

(RPO2) $t = g(t_1, \ldots, t_m)$ AND:

- **(RPO2a)** $f \rhd g$ and $s \succ_{\text{rpo}} t_1, \ldots, s \succ_{\text{rpo}} t_m$, OR
- **(RPO2b)** $f = g \in \Sigma_{\mathsf{LEX}}$ (hence $n = m$), and $[s_1, \ldots, s_n](\succ_{\text{rpo}})_{\text{lex}}[t_1, \ldots, t_n]$ and $s \succ_{\text{rpo}} t_1, \ldots, s \succ_{\text{rpo}} t_n$, OR
- **(RPO2c)** $f = g \in \Sigma_{\mathsf{MUL}}$ (hence $n = m$), and $\{\!\{s_1, \ldots, s_n\}\!\}(\succ_{\text{rpo}})_{\text{mul}}\{\!\{t_1, \ldots, t_n\}\!\}$.

Since $\succ_{\text{rpo}}$ is wellfounded, a rewrite relation $\to$ can be proved terminating by finding a well-founded ordering $\rhd$ on the set of function symbols that that $l \succ_{\text{rpo}} r$ for every rewrite rule $l \to r$.

LPO (the lexicographical path ordering) is the relation obtained without (RPO2c), taking $\Sigma = \Sigma_{\mathsf{LEX}}$. Some variations start with a quasi-order on function symbols rather than an order, and allow $f \sim g$ in rules (RPO2b) and (RPO2c). Instead of using this seemingly more powerful alternative, one can usually adapt the term rewriting system such that this is not needed.

Now we move on to iterative definitions. We denote by Σ^* a copy of Σ in which all function symbols are marked by a $*$. First recall the iterative definition of LPO as defined in [4]. It makes use of the following term rewriting system:

Definition 4 (ILPO)**.** Given a finite signature Σ and well-founded ordering $\rhd$ on Σ, the relation $\to_{\text{ilpo}}$ on terms over $\Sigma \cup \Sigma^*$ is the rewrite relation induced by the following rules:

$$\begin{array}{rll}
f(\vec{x}) & \to_{\text{put}} & f^*(\vec{x}) \\
f^*(\vec{x}) & \to_{\text{select}} & x_i \ (1 \le i \le n) \\
f^*(\vec{x}) & \to_{\text{copy}} & g(f^*(\vec{x}), \ldots, f^*(\vec{x})) \ (f \rhd g) \\
f^*(\vec{x}, g(\vec{y}), \vec{z}) & \to_{\text{lex}} & f(\vec{x}, g^*(\vec{y}), l, \ldots, l) \quad (l = f^*(\vec{x}, g(\vec{y}), \vec{z}))
\end{array}$$

It is proved in [4] that the transitive closure $\rightarrow^+_{\text{ilpo}}$ is a well-founded ordering on the set of unmarked terms. Because it is a rewrite relation, closure under contexts and substitutions is immediate. Hence $\rightarrow^+_{\text{ilpo}}$ is a reduction ordering, and termination of a rewrite relation follows if we can prove $l \rightarrow^+_{\text{ilpo}} r$ for each of its rewrite rules (for some well-founded relation $\rhd$ on the function symbols). This is called ILPO-termination. They prove, moreover, that ILPO-termination coincides with LPO-termination.

3 An iterative path ordering

We want to extend the auxiliary term rewriting system from [4] to include also the multiset comparison of arguments, so as to provide an alternative to the definition of RPO with status. To add a multiset extension to the system, there are different possibilities. We could add a direct rule scheme

$$f^*(s_1, \dots, s_n) \rightarrow_{\text{bigmul}} f(t_1, \dots, t_n)$$

if there are $A, A' \subset \{1, \dots, n\}$ such that $\{\!\{s_i | i \in A\}\!\} = \{\!\{t_i | i \in A'\}\!\}$, and $\forall i \notin A' \exists j \notin A [t_i = s_j^*]$.
Alternatively, we could add two rule schemes for smaller steps:

$$f^*(x_1, \dots, x_n) \rightarrow_{\text{ord}} f^*(x_{\pi_1}, \dots, x_{\pi_n})$$

for any permutation π, and

$$f^*(\vec{x}, g(\vec{y}), \vec{z}) \rightarrow_{\text{smallmul}} f([x_1|g^*(\vec{y})], \dots, [x_{|\vec{x}|}|g^*(\vec{y})], g^*(\vec{y}), [z_1|g^*(\vec{y})], \dots, [z_{|\vec{z}|}|g^*(\vec{y})])$$

where $[a|b]$ means either one or the other can be chosen.

We choose to introduce the latter two rules, as these are both conceptually simpler, and more in the spirit of an iterative approach with small steps.

Definition 5 (IPO). Given a finite signature $\Sigma = \Sigma_{\mathsf{MUL}} \uplus \Sigma_{\mathsf{LEX}}$ and a wellfounded order $\rhd$ on Σ. The *iterative path order* induced by $\rhd$ is the term rewrite relation $\rightarrow_{\text{ipo}}$ over $\Sigma \cup \Sigma^*$ defined by the following rules:

$$\begin{array}{rll}
f(\vec{x}) & \rightarrow_{\text{put}} & f^*(\vec{x}) \\
f^*(\vec{x}) & \rightarrow_{\text{select}} & x_i \ (1 \le i \le n) \\
f^*(\vec{x}) & \rightarrow_{\text{copy}} & g(f^*(\vec{x}), \dots, f^*(\vec{x})) \ (f \rhd g) \\
f^*(\vec{x}, g(\vec{y}), \vec{z}) & \rightarrow_{\text{lex}} & f(\vec{x}, g^*(\vec{y}), l, \dots, l) \\
 & & \quad (l = f^*(\vec{x}, g(\vec{y}), \vec{z}),\ f \in \Sigma_{\mathsf{LEX}}) \\
f^*(\vec{x}, g(\vec{y}), \vec{z}) & \rightarrow_{\text{mul}} & f([x_1|g^*(\vec{y})], \dots, g^*(\vec{y}), \dots, [z_n|g^*(\vec{y})]) \\
 & & \quad (f \in \Sigma_{\mathsf{MUL}}) \\
f^*(\vec{x}) & \rightarrow_{\text{ord}} & f^*(x_{\pi(1)}, \dots, x_{\pi(n)}) \\
 & & \quad (\pi \text{ some permutation},\ f \in \Sigma_{\mathsf{MUL}})
\end{array}$$

IPO-termination is defined similarly to ILPO-termination: a system is IPO-terminating if for some well-founded ordering $\rhd$ on the function symbols we can prove $l \rightarrow^{+}_{\text{ipo}} r$ for every rewrite rule $l \rightarrow r$. We still need to show that this method is sound but will not do so here; this will be a consequence of the equivalence of IPO-termination and RPO-termination.

Example 1. *Let $\Sigma = \{f : 3, g : 1, a : 0\}$ (the number following each term denotes its arity), $R = \{f(x, y, g(z)) \rightarrow f(a, z, x)\}$. The relation $\rightarrow_{\text{R}}$ is terminating, for choosing $g \rhd a$ we have: $f(x, y, g(z)) \rightarrow_{\text{put}} f^*(x, y, g(z)) \rightarrow_{\text{ord}} f^*(g(z), y, x) \rightarrow_{\text{mul}} f(g^*(z), g^*(z), x) \rightarrow_{\text{copy}} f(a, g^*(z), x) \rightarrow_{\text{select}} f(a, z, x)$.*

We use the notation s^* for s with its root symbol marked, so $(f(s_1, \ldots, s_n))^* = f^*(s_1, \ldots, s_n)$. This notation is used only if s is not a variable, and is not a term with a marked root symbol.

4 Multisets

As a first step towards equivalence of RPO-termination and IPO-termination we focus on some issues concerning multisets of equal size. At first sight, one may wonder whether the rewrite rules `mul` and `ord` really have the power to obtain a "multiset extension". Is the combination of these rules as strong as the alternative `bigmul` rule, which implements the idea of a multiset order more directly? In this section we will see that yes, `ord` and `mul` together are at least as strong as `bigmul`. In the next section we prove equivalence of RPO-termination and IPO-termination.

In addition to multiset extensions, we define the *minimul* extension of a relation $>$ as follows: $X >_{\text{minimul}} Y$ iff there is a multiset $A \subseteq X \cap Y$ and some $x \in X - A$ such that $\forall y \in Y - A[x > y]$, and $|X| = |Y|$.

Lemma 1. *For any transitive relation $>$, $>_{\text{mul}}$ is the transitive closure of $>_{\text{minimul}}$.*

Proof. It is evident that $>_{\text{minimul}}$ is contained in $>_{\text{mul}}$ and thus, since the latter is transitive (this is a wellknown result), the transitive closure of $>_{\text{minimul}}$ must also be contained in it. For the other direction, we must see that whenever $X >_{\text{mul}} Y$ there is a sequence $X = Z_1 >_{\text{minimul}} \ldots >_{\text{minimul}} Z_n = Y$. Let $X = A \uplus B, Y = A \uplus C$ (with $\forall c \in C \exists b \in B[b > c]$). X and Y have equal size, and therefore B and C are both non-empty. Find a function $g : C \rightarrow B$ such that always $g(c) > c$; define $B_1 = g(C),\ B_2 = B \setminus B_1$. Let $B_1 = \{\!\{b_1, \ldots, b_n\}\!\}$ ($n > 0$ since C is nonempty). Now define $Z'_1 := A \uplus \{\!\{b_1, \ldots, b_n\}\!\}$, and for each $i < n$: $Z'_{i+1} := (Z'_i - \{\!\{b_i\}\!\}) \uplus \{\!\{c \in C | g(c) = b_i\}\!\}$. Each Z'_{i+1} is at least as large as Z'_i, and $Z_n = Y$. If we define $E_1 := B_2$ and always $E_{i+1} \subseteq E_i$, in such a way that $|E_i| = |X| - |Z'_i|$, then we can define $Z_i := Z'_i \uplus E_i$ and have $Z_i >_{\text{minimul}} Z_{i+1}$ for all i. $\square$

Lemma 2. *Defining $>$ as the relation "$s > t$ if $t = s^*$", $\{\!\{s_1, \dots, s_n\}\!\} >_{\text{minimul}} \{\!\{t_1, \dots, t_n\}\!\}$ iff $f^*(\vec{s}) \to_{\text{ord}} \cdot \to_{\text{mul}} f(\vec{t})$.*

Proof. If $f^*(\vec{s}) \to_{\text{ord}} f^*(\vec{r}) \to_{\text{mul}} f(\vec{t})$, then $\{\!\{\vec{s}\}\!\} = \{\!\{\vec{r}\}\!\} >_{\text{minimul}} \{\!\{\vec{t}\}\!\}$ follows directly from the definition. For the other direction, suppose $\{\!\{\vec{s}\}\!\} >_{\text{minimul}} \{\!\{\vec{t}\}\!\}$. There are $A, B \subset \{1, \dots, n\}$ such that $\{\!\{s_i | i \in A\}\!\} = \{\!\{t_i | i \in B\}\!\}$, and some $k \notin A$ such that $\forall j \in \{1, \dots, n\} - B[t_j = s_k^*]$. Find a permutation π of $\{1, \dots, n\}$ that maps B to A in such a way that $s_{\pi(i)} = t_i$ for all $i \in B$ (it automatically holds that $t_i = s_{\pi(\pi^{-1}(k))}$ for $i \notin B$). It is clear that $f^*(\vec{s}) \to_{\text{ord}} f^*(s_{\pi(1)}, \dots, s_{\pi(n)}) \to_{\text{mul}} f(\vec{t})$. □

Lemma 3. *Defining $>$ as the relation "$s > t$ iff $t = s^*$", $\{\!\{s_1, \dots, s_n\}\!\} >_{\text{mul}} \{\!\{t_1, \dots, t_n\}\!\}$ iff $f^*(\vec{s}) \to_{\text{bigmul}} f(\vec{t})$.*

Proof. Evident. □

Combining Lemmas 1, 2 and 3, we see that `bigmul` is a derived rule of IPO.

Theorem 1. *$s^* \to_{\text{bigmul}} t$ if $s(\to_{\text{put}} \cdot \to_{\text{ord}} \cdot \to_{\text{mul}})^+ t$.*

□

5 Equivalence of RPO and IPO

In this section we will prove the equivalence on the set of terms of the recursive path ordering $\succ_{\text{rpo}}$ and the transitive closure of $\to_{\text{ipo}}$. As in [4], we will need the condition that $\rhd$ is transitive: if this is not required, an easy counterexample would be the rewrite system $A \to C$ with $A \rhd B \rhd C$. It is not provable that $A \succ_{\text{rpo}} C$, but $A \to_{\text{put}} A^* \to_{\text{copy}} B \to_{\text{put}} B^* \to_{\text{copy}} C$.

We first prove the easy direction of the equivalence: the recursive path ordering $\succ_{\text{rpo}}$ is contained in $\to_{\text{ipo}}^+$.

Theorem 2. *Let s and t be unlabelled terms. If $s \succ_{\text{rpo}} t$ then s is markable and $s^* \to_{\text{ipo}}^* t$.*

Proof. By induction over the definition of $\succ_{\text{rpo}}$. Checking the rules of rpo we may always write $s = f(s_1, \dots, s_n)$ if $s \succ_{\text{rpo}} t$, so s is markable. To prove the second clause, commit case distinction over the rule used to derive $s \succ_{\text{rpo}} t$.

(RPO1) $f^*(s_1, \dots, s_n) \to_{\text{select}} s_i$. If $s_i = t$ we are done, otherwise $s_i \to_{\text{put}} s_i^* \to_{\text{ipo}}^* t$ by induction hypothesis.

(RPO2a) $f^*(s_1, \dots, s_n) \to_{\text{copy}} g(f^*(\vec{s}), \dots, f^*(\vec{s}))$. By the induction hypothesis $f^*(\vec{s}) \to_{\text{ipo}}^* t_i$ for all $1 \le i \le m$, so $g(f^*(\vec{s}), \dots, f^*(\vec{s})) \to_{\text{ipo}}^* g(t_1, \dots, t_m)$.

(RPO2b) By the definition of lexicographical extension, there is $i \leq n$ such that $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}, s_i \succ_{\text{rpo}} t_i$. By the induction hypothesis s_i is markable and $s_i^* \to_{\text{ipo}}^* t_i$. Also by induction hypothesis $s^* \to_{\text{ipo}}^* t_{i+1}, \ldots, s^* \to_{\text{ipo}}^* t_n$. Hence $f^*(\vec{s}) \to_{\text{lex}} f(s_1, \ldots, s_{i-1}, s_i^*, f^*(\vec{s}), \ldots, f^*(\vec{s})) \to_{\text{ipo}}^* f(\vec{t})$.

(RPO2c) Following the definition of multisets, write $\{\{s_1, \ldots, s_n\}\} = A \uplus B$, $\{\{t_1, \ldots, t_n\}\} = A \uplus C$. Let $\vec{r}$ be a vector of terms such that $r_i = t_i$ if $t_i \in A$, $r_i = s_j^*$ if $t_i \in C$ and $s_j \in B$ is a term with $s_j \succ_{\text{rpo}} t_i$. Again writing $>$ for the marking relation, we have $\{\{\vec{s}\}\} >_{\text{mul}} \{\{\vec{r}\}\}$, so by Lemma 3, we have $f^*(\vec{s}) \to_{\text{bigmul}} f(\vec{r})$, which $\to_{\text{ipo}}^*$-reduces to $f(\vec{t})$ by the induction hypothesis. Since `bigmul` is a derived rule of IPO, the induction step follows. $\square$

The other direction takes a bit more effort; we will use some auxiliary lemmas. The fundamental idea is that a reduction can always be done "top-down": we start by doing a reduction on the top of the term, and then steadily work down on its arguments. To this end, it is crucial to see that marking can always be done at the top:

Lemma 4. *If $s \to_{\text{ipo}}^* t$ and s is unmarked, then either $s = t$ or $s^* \to_{\text{ipo}}^* t$.*

Proof. If $s \neq t$ there is some r such that $s \to_{\text{ipo}} r \to_{\text{ipo}}^* t$. Since s is unmarked, this first step is `put`, either on the top of the term or in one of its arguments. We proceed by induction on the depth of the put-step. If the put-step takes place at the top, we have $r = s^* \to_{\text{ipo}}^* t$, and we are done. If the put-step is not at the top, we have $s = f(s_1, \ldots, s_i, \ldots, s_n)$ and $r = f(s_1, \ldots, s_i', \ldots, s_n)$ with $s_i \to_{\text{put}} s_i'$. By the induction hypothesis $s_i^* \to_{\text{ipo}}^* s_i'$. If $f \in \Sigma_{\mathsf{MUL}}$ then $s^* \to_{\text{mul}} f(s_1, \ldots, s_i^*, \ldots, s_n) \to_{\text{ipo}}^* r \to_{\text{ipo}}^* t$. If $f \in \Sigma_{\mathsf{LEX}}$ then similarly we have $s^* \to_{\text{lex}} f(s_1, \ldots, s_i^*, f^*(\vec{s}), \ldots, f^*(\vec{s})) \to_{\text{select}}^* f(s_1, \ldots, s_i^*, \ldots, s_n) \to_{\text{ipo}}^* t$. $\square$

Next, we will transform an IPO-reduction into a so-called normal one. Write $R_1 \Rightarrow R_2$ for the statement "if we have a reduction sR_1t then we also have the reduction sR_2t. Below, we locally use the notation $\to_{\text{rule}}$ for the *topmost* application of `rule`. An internal step is denoted by $\to_{\text{in}}$.

Lemma 5. *Assuming transitivity of $\rhd$, reductions can be manipulated by the following rules:*

$$\begin{array}{lcll}
\to_{\text{in}} \cdot \to_{\text{put}} & \Rightarrow & \to_{\text{put}} \cdot \to_{\text{in}} & (1) \\
\to_{\text{in}} \cdot \to_{\text{select}} & \Rightarrow & \to_{\text{select}} \cdot \to_{\text{ipo}}^{=} & (2) \\
\to_{\text{copy}} \cdot \to_{\text{put}} \cdot \to_{\text{select}} & \Rightarrow & & (3) \\
\to_{\text{lex}} \cdot \to_{\text{put}} \cdot \to_{\text{select}} & \Rightarrow & (\to_{\text{select}} \cdot \to_{\text{put}}^{=})^{=} & (4) \\
\to_{\text{ord}} \cdot \to_{\text{select}} & \Rightarrow & \to_{\text{select}} & (5) \\
\to_{\text{mul}} \cdot \to_{\text{put}} \cdot \to_{\text{select}} & \Rightarrow & \to_{\text{select}} \cdot \to_{\text{put}}^{=} & (6)
\end{array}$$

$$
\begin{aligned}
\to_{\text{in}} \cdot \to_{\text{copy}} &\Rightarrow \to_{\text{copy}} \cdot \to^*_{\text{in}} && (7)\\
\to_{\text{copy}} \cdot \to_{\text{put}} \cdot \to_{\text{copy}} &\Rightarrow \to_{\text{copy}} \cdot \to^*_{\text{in}} && (8)\\
\to_{\text{lex}} \cdot \to_{\text{put}} \cdot \to_{\text{copy}} &\Rightarrow \to_{\text{copy}} \cdot \to^*_{\text{in}} && (9)\\
\to_{\text{ord}} \cdot \to_{\text{copy}} &\Rightarrow \to_{\text{copy}} \cdot \to^*_{\text{in}} && (10)\\
\to_{\text{mul}} \cdot \to_{\text{put}} \cdot \to_{\text{copy}} &\Rightarrow \to_{\text{copy}} \cdot \to^*_{\text{in}} && (11)\\
\to_{\text{copy}} \cdot \to_{\text{put}} \cdot \to^*_{\text{in}} \cdot \to_{\text{lex}} &\Rightarrow \to_{\text{copy}} \cdot \to^*_{\text{in}} && (12)\\
\to_{\text{copy}} \cdot \to_{\text{put}} \cdot \to_{\text{ord}} &\Rightarrow \to_{\text{copy}} \cdot \to_{\text{put}} && (13)\\
\to_{\text{copy}} \cdot \to_{\text{put}} \cdot \to^*_{\text{in}} \cdot \to_{\text{mul}} &\Rightarrow \to_{\text{copy}} \cdot \to^*_{\text{in}} && (14)\\
\to_{\text{in}} \cdot \to_{\text{ord}} &\Rightarrow \to_{\text{ord}} \cdot \to_{\text{in}} && (15)\\
\to_{\text{ord}} \cdot \to_{\text{ord}} &\Rightarrow \to_{\text{ord}} && (16)\\
\to_{\text{ord}} \cdot \to_{\text{mul}} \cdot \to_{\text{put}} \cdot \to_{\text{ord}} &\Rightarrow \to_{\text{ord}} \cdot \to_{\text{mul}} \cdot \to_{\text{put}} && (17)
\end{aligned}
$$

For reductions $s^* \to^*_{\text{ipo}} t$ with s unmarked, we can also rewrite at the start of this reduction:

$$
\begin{aligned}
\to^*_{\text{in}} \cdot \to_{\text{lex}} &\Rightarrow \to_{\text{lex}} \cdot \to^*_{\text{in}} && (18)\\
\to_{\text{lex}} \cdot \to_{\text{put}} \cdot \to^*_{\text{in}} \cdot \to_{\text{lex}} &\Rightarrow \to_{\text{lex}} \cdot \to^*_{\text{in}} && (19)\\
\to^*_{\text{in}} \cdot \to_{\text{mul}} &\Rightarrow \to_{\text{mul}} \cdot \to^*_{\text{in}} && (20)\\
\to^n_{\text{mul}} \cdot \to_{\text{put}} \cdot \to^*_{\text{in}} \cdot \to_{\text{mul}} &\Rightarrow \to^n_{\text{mul}} \cdot \to^=_{\text{mul}} \cdot \to^*_{\text{in}} && (21)
\end{aligned}
$$

Proof. Each of these manipulations follows by simply examining the rules involved. For example, in $\to_{\text{lex}} \cdot \to_{\text{put}} \to_{\text{select}} \Rightarrow (\to_{\text{select}} \cdot \to^=_{\text{put}})^=$, suppose $s = f^*(\vec{s}) \to_{\text{lex}} f(s_1, \dots, s^*_j, f^*(\vec{s}), \dots) = f(\vec{r}) \to_{\text{put}} \cdot \to_{\text{select}} r_i$. Then either $r_i = s_i$ ($s \to_{\text{select}} r_i$), $r_i = s^*_i$ ($s \to_{\text{select}} \cdot \to_{\text{put}} r_i$) or $r_i = s$. Each of these clauses matches $s(\to_{\text{select}} \cdot \to^=_{\text{put}})^= r_i$ (for a relation R, $R^=$ is its reflexive closure).

The only case that uses transitivity of $\rhd$ is the manipulation of `copy`$\cdot \to_{\text{put}} \cdot \to_{\text{copy}}$. This case is quite simple for the rest: $s = f^*(\vec{s}) \to_{\text{copy}} g(\vec{t}) \to_{\text{put}} g^*(\vec{t}) \to_{\text{copy}} h(\vec{r})$. Evidently $s \to^*_{\text{ipo}} r_i$ for each i (as $r_i = g^*(\vec{t})$), and by transitivity of $\rhd$, $f \rhd h$. Validity of the manipulation follows.

The hardest cases are the ones where a mark is added to a subterm, for example by `lex`; here we need Lemma 4. Take case 19. This manipulation is only allowed at the start of a reduction, so let s be an unmarked term such that $s^* = f^*(\vec{s}) \to_{\text{lex}} f(s_1, \dots, s^*_i, f^*(\vec{s}), \dots, f^*(\vec{s})) \to_{\text{put}} \cdot \to^*_{\text{in}} f^*(\vec{t}) \to_{\text{lex}} f(t_1, \dots, t^*_j, f^*(\vec{t}), \dots, f^*(\vec{t})) =: f(\vec{r})$. Either $i < j$ or $i \geq j$, and regardless of that $s^* \to^+_{\text{ipo}} f^*(\vec{t})$. Suppose $i < j$. Then $f(s_1, \dots, s^*_i, f^*(\vec{s}), \dots, f^*(\vec{s})) \to^*_{\text{in}} f(t_1, \dots, t_i, f^*(\vec{t}), \dots, f^*(\vec{t})) \to^*_{\text{in-select}} f(t_1, \dots, t_j, f^*(\vec{t}), \dots, f^*(\vec{t})) \to_{\text{in-put}} f(\vec{r})$. If $i \geq j$: $s \to_{\text{lex}} f(s_1, \dots, s^*_j, f^*(\vec{s}), \dots, f^*(\vec{s})) \to^*_{\text{in}} f(t_1, \dots, t_{j-1}, s^*_j, f^*(\vec{t}), \dots, f^*(\vec{t}))$, which $\to^*_{\text{in}}$-reduces to r because $s^*_j \to^*_{\text{ipo}} t^*_j$: either $s_j = t_j$, in which case this is direct, or $s_j \to^+_{\text{ipo}} t_j$, in which case $s^*_j \to^*_{\text{ipo}} t_j \to_{\text{put}} t^*_j$. □

Definition 6 (normal ipo reduction)**.** A *normal* ipo reduction $s^* \to^+_{\text{ipo}} t$ with s and t unmarked terms has one of the following four forms:

$s^* \to_{\text{select}} \cdot (\to_{\text{put}} \cdot \to_{\text{select}})^* t$
$s^* (\to_{\text{select}} \cdot \to_{\text{put}})^* \cdot \to_{\text{copy}} \cdot \to_{\text{in}}^* t$
$s^* (\to_{\text{select}} \cdot \to_{\text{put}})^* \cdot \to_{\text{lex}} \cdot \to_{\text{in}}^* t$
$s^* (\to_{\text{select}} \cdot \to_{\text{put}})^* \cdot \to_{\text{ord}} \cdot (\to_{\text{mul}} \cdot \to_{\text{put}})^* \cdot \to_{\text{mul}} \cdot \to_{\text{in}}^* t$

Lemma 6. *Let s and t be unmarked terms. If $s \to_{\text{ipo}}^+ t$, then there is a normal reduction $s \to_{\text{ipo}}^+ t$.*

Proof. By Lemma 4 we can find a reduction $s^* \to_{\text{ipo}}^* t$. First we push each topmost `select` step to the start of the reduction (possibly together with the `put` step that precedes it), using the first six rules in lemma 5; by induction over the size of s this is a finite undertaking. Eventually, we have a reduction $s^* \to_{\text{ipo}}^* r \to_{\text{ipo}}^* t$, where $s^* \to_{\text{ipo}}^* r$ by purely `select` and `put` steps, and $r \to_{\text{ipo}}^* t$ doesn't use any topmost `select` steps. If $r = t$, the last step in the former reduction can not be `put`, so the reduction has the first form. Otherwise, again by Lemma 4 assume r marked; the reduction $s^* \to_{\text{ipo}}^* r$ has the form $(\to_{\text{select}} \cdot \to_{\text{put}})^*$.

Now consider the reduction $r \to_{\text{ipo}}^* t$; it contains at least one top step since r is marked. Using Lemma 5 we can ensure that the reduction starts with a top step and that $\to_{\text{in}}$ never precedes $\to_{\text{put}}$ or $\to_{\text{ord}}$.

If `copy` occurs (topmost) anywhere in the reduction, we can "merge" it with any topmost step that precedes or follows it using rules 8-14 of lemma 5, possibly combined with rule 7; each merge reduces the number of top steps in the reduction, and eventually we have a reduction with just one `copy` step, at the start: $r \to_{\text{copy}} \cdot \to_{\text{in}}^* t$.

If `copy` doesn't occur, then the root symbol of r stays the same throughout the reduction (discounting marks). If the first step is `lex`, then so is any other topstep, and again the steps can be merged, this time using rule 19, to end up with a reduction $r \to_{\text{lex}} \cdot \to_{\text{in}}^* t$. If the first step is `ord`, there has to be a `mul` step in the reduction (to lose the root mark), and observing that `ord` leaves the term unmarked save for the root symbol, use rules 16, 17 (combined with 15) and 21 to remove further `ord` steps and pull `mul` steps forward as well; we eventually have $r \to_{\text{ord}} \cdot (\to_{\text{mul}} \cdot \to_{\text{put}})^* \cdot \to_{\text{mul}} \cdot \to_{\text{in}}^* t$. If the first step is `mul`, we can precede it with an (empty) `ord` step to be in the previous case. □

Theorem 3. *For unmarked terms s and t: if $s \to_{\text{ipo}}^+ t$, then $s \succ_{\text{rpo}} t$ if $\rhd$ is transitive.*

Proof. By induction over the size of s first, t second. By Lemmas 4 and 6 we know that $s^* \to_{\text{ipo}}^* t$ with a normal reduction. If the first step in this normal reduction is `select`, then we are immediately done if it's the only step, and by induction hypothesis if it is followed by `put` (applying (RPO1) in both cases). If not, the reduction takes the form $\to_{\text{copy}} \cdot \to_{\text{in}}^*$, $\to_{\text{copy}} \cdot \to_{\text{in}}^*$ or $\to_{\text{ord}} \cdot (\to_{\text{mul}} \cdot \to_{\text{put}}) \cdot \to_{\text{mul}} \cdot \to_{\text{in}}^*$ which, by introducing empty `ord` steps and combining lemmas 1, 2 and 3, is equivalent to $\to_{\text{bigmul}}$. In all three cases, we can simply apply the induction hypothesis and the corresponding subrule of (RPO2).[1] For example, the `copy` case: $f(\vec{s}) \succ_{\text{rpo}} g(\vec{t})$ because $f \rhd g$ and by induction $f(\vec{s}) \succ_{\text{rpo}} t_i$ for all i. □

6 Conclusion

We have extended the iterative lexicographic path ordering with two extra rules and a "status" on the function symbols. The extended system turns out to be equivalent to rpo with status. As an added bonus, the proof can be restricted to have an alternative proof of equivalence of ilpo and lpo, reminiscent of the "wave form" strategy presented in [4].

We have not proved that $\to_{\text{ipo}}^+$ is wellfounded on unmarked terms, but this follows as a consequence of the equivalence with $\to_{\text{rpo}}$. Alternatively, it could be proved directly with a simple extension of the proof method in [4], as is done for example in the higher order variant in [5]. Using the equivalence the other way around, the proof given here provides a way to derive transitivity of $\succ_{\text{rpo}}$.

References

[1] Bergstra and Klop. Algebra of communicating processes. *Theoretical computer science*, 37:77–121, 1985.

[2] Nachum Dershowitz. Orderings for term-rewriting systems. In *Foundations of Computer Science, 1979., 20th Annual Symposium on*, pages 123–131, 1979.

[3] S. Kamin and J-J. Lévy. Two generalizations of the recursive path ordering. University of Illinois, 1980.

[4] Jan Willem Klop, Vincent van Oostrom, and Roel de Vrijer. Iterative lexicographic path orders. *Lecture Notes in Computer Science*, pages 541–554, 2006.

[5] Cynthia Kop and Femke van Raamsdonk. A higher-order iterative path ordering. In *Logic for Programming, Artificial Intelligence, and Reasoning*, pages 697–711, 2008.

[1] Although in the case of `lex`, it requires the observation that $s_1 \succeq_{\text{rpo}} t_1, \ldots, s_{i-1} \succeq_{\text{rpo}} t_{i-1}, s_i \succ_{\text{rpo}} t_i$ implies $[s_1, \ldots, s_n](\succ_{\text{rpo}})_{\text{lex}}[t_1, \ldots, t_n]$.