Infinite Games in the Cantor Space over Admissible Set Theories

Naohi Eguchi

Institute of Computer Science
University of Innsbruck
Austria

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Introduction

- Axiom of determinacy for infinite games: Set-theoretic statement over second order language stemming from descriptive set theory.
- This work: A fine-grained analysis of Δ^0_2-definable games in the Cantor space over admissible set theories.
  - Why Δ^0_2-games? - The first class for which the different hierarchy makes sense.
  - Why in the Cantor space? - The logical strength of the axiom gets weaker than in the Baire space (Π^1_1-TR_0 to ATR_0).
  - Why admissible set theories? - A natural hierarchy reaching ATR_0 is known.

Axiom of determinacy

Two players game A: $(x_0, x_1, \ldots y_0, y_1, \cdots \in X\ )$

Player I	x_0		x_1		...	
Player II		y_0		y_1		...

- A strategy σ for Player I is a partial function $X^{<\mathbb{N}} \to X$ s.t. $\sigma(\langle x_0, y_0, \ldots, x_{j-1}, y_{j-1}\rangle) = x_j$.
- A strategy σ for Player II is a partial function $X^{<\mathbb{N}} \to X$ s.t. $\sigma(\langle x_0, y_0, \ldots, x_{j-1}, y_{j-1}, x_j\rangle) = y_j$.

Player I wins the game $A \iff \langle x_0, y_0, x_1, y_1, \ldots\rangle \in A$
for any strategy for Player II.

Player II wins the game $A \iff \langle x_0, y_0, x_1, y_1, \ldots\rangle \notin A$
for any strategy for Player I.

Axiom of determinacy in the Cantor space

Let Φ: class of sets.

Axiom of determinacy: Either Player I or II wins the game $A \in \Phi$.

1. Φ-Det: In case $X = \mathbb{N}$.
2. Φ-Det^*: In case $X = 2 = \{0, 1\}$.

Theorem (Nemoto-MedSalem-Tanaka '07)

1. $\mathrm{RCA}_0 \vdash \Sigma^0_1\text{-}\mathrm{Det}^* \leftrightarrow \mathrm{WKL}_0$.
2. $\mathrm{RCA}_0 \vdash \Delta^0_2\text{-}\mathrm{Det}^* \leftrightarrow \mathrm{ATR}_0$.

Shoenfield Limit lemma

Any Δ^0_2 set can be approximated by the symmetric difference of recursively enumerable sets.

Theorem (Shoenfield)

For any Δ^0_2-set, there exists a recursive function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that $\lim_s f(x, s) = A(x)$. ($A(x) \Leftrightarrow x \in A \Leftrightarrow \chi_A(x) = 1$)

This induces the Ershov hierarchy, the symmetric difference of a recursively enumerable sets for an element a of Klneene's ordinal notation system $\mathcal{O}$.

Kleene's $\mathcal{O}$

Definition (Kleene's $\mathcal{O}$)

The set $\mathcal{O} \subseteq \mathbb{N}$ of notations, a function $|\cdot|_{\mathcal{O}} : \mathcal{O} \to Ord$ and a strict partial order $<_{\mathcal{O}}$ on $\mathcal{O}$ are defined simultaneously.

1. $1 \in \mathcal{O}$ and $|1|_{\mathcal{O}} = 0$.
2. If $a \in \mathcal{O}$ and $|a|_{\mathcal{O}} = \alpha$, then $2^a \in \mathcal{O}$ and $|2^a|_{\mathcal{O}} = \alpha + 1$.
3. If e is a code of a total recursive function such that $|\{e\}(n)|_{\mathcal{O}} = \alpha_n$ and $\{e\}(n) <_{\mathcal{O}} \{e\}(n+1)$ hold for all $n \in \mathbb{N}$, then $3 \cdot 5^e \in \mathcal{O}$ and $|3 \cdot 5^e|_{\mathcal{O}} = \lim_n \alpha_n$.

Fact

1. *$<_{\mathcal{O}}$ and $\mathcal{O}$ are Π^1_1-definable sets.*
2. *$<_{\mathcal{O}}$ is a well-founded partial order on $\mathcal{O}$.*
3. *$<_{\mathcal{O}}\restriction a$ is a linear order for any $a \in \mathcal{O}$, .*

a-r.e. sets

Definition (a-r.e. sets)

Let $a \in \mathcal{O}$. $A \subseteq \mathbb{N}$ is a-r.e. if there exist recursive functions $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ and $o : \mathbb{N} \times \mathbb{N} \to \mathcal{O}$ s.t.

1. $f(x, 0) = 0$ and $o(x, 0) <_{\mathcal{O}} a$ for all x.
2. $o(x, s+1) \leqslant_{\mathcal{O}} o(x, s)$ for all x and s.
3. For all x and for all s, if $f(x, s+1) \neq f(x, s)$, then $o(x, s+1) <_{\mathcal{O}} o(x, s)$.
4. $\lim_s f(x, s) = A(x)$ for all x.

Theorem (Stephan-Yang-Yu '10)

For any Δ^0_2 set $A \subseteq \mathbb{N}$, there exists $a \in \mathcal{O}$ such that $|a|_{\mathcal{O}} = \omega^2$ and A is an a-r.e. set.

Remarks

- Original idea: to layer the Δ^0_2-Det^* by the Ershov hierarchy.
- Oversight of the speaker: The theorem fails for $A \subseteq 2^{\mathbb{N}}$ (addressed by T. Kihara).
  - Δ^0_2 subsets of $2^{\mathbb{N}}$ will not be exhausted at ω^2.
  - The Ershov hierarchy might not be appropriate for fine-grained analysis of determinacy of Δ^0_2-definable games.
- This talk presents very partial results.

$(\Sigma^0_1)_a$-formula

Definition

Let $a \in \mathcal{O}$. Assume the relation $<_{\mathcal{O}} \restriction a$ can be expressed in an underlying formal system.
Then we say a formula is $(\Sigma^0_1)_a$-formula if it is of the form $(\exists b <_{\mathcal{O}} a)\,[\varphi(b) \wedge (\forall c <_{\mathcal{O}} b)\neg\varphi(c)]$ for some Σ^0_1-formula φ.

Intuitively, a $(\Sigma^0_1)_a$-formula expresses:

$$(\exists b <_{\mathcal{O}} a)\,[\exists s\ f(s,b) = 0 \wedge (\forall c <_{\mathcal{O}} b)\forall s\ f(s,c) = 1]$$

Admissible set theory

A system KPu^0 of admissible set theory:
Weak subsystem of ZF without (Power) over $\mathcal{L}_{\mathrm{ZF}} \cup \{\mathsf{Ad}\}$ s.t.

1. Axiom of Separation is limited to Δ_0-formulas.
2. Axiom of Replacement is limited the axiom of Collection for Δ_0- formulas.
3. Axioms for Ad: $\mathsf{Ad}(z)$ means z is an admissible set, i.e., z satisfies (Δ_0-Sep) and (Δ_0-Col).

Note:

- $\mathrm{KPu}^0 \vdash \Delta^1_1\text{-}\mathrm{CA}_0$. Hence KPu^0 is strong enough for a base system.
- Unlike KPu (or KP), transfinite induction holds in KPu^0 only for Δ_0-formulas.

Admissible set theory with iterated admissible universes

$\mathrm{KPu}^0 + (\mathcal{U}_n)$ (over $\mathcal{L}_{\mathrm{ZF}} \cup \{\mathsf{Ad}\} \cup \{d_0, \dots, d_{n-1}\}$):

$$\mathsf{Ad}(d_0) \wedge \cdots \wedge \mathsf{Ad}(d_{n-1}) \wedge d_0 \in d_1 \wedge \cdots \wedge d_{n-2} \in d_{n-1} \qquad (\mathcal{U}_n)$$

The set d_0 could be interpreted as $L_{\omega_1^{\mathrm{CK}}}$.

Theorem (Jäger '84)

$|T|$: *maximal order type of recursive well ordering provable in* T.
$(\alpha, \beta) \mapsto \varphi(\alpha, \beta)$: *Veblen function.*

1. $|\mathrm{KPu}^0 + (\mathcal{U}_1)| = \varphi(\varepsilon_0, 0)$.
2. $|\mathrm{KPu}^0 + (\mathcal{U}_{n+2})| = \varphi(|\mathrm{KPu}^0 + (\mathcal{U}_{n+1})|, 0)$.

Therefore $|\bigcup_{n<\omega} \mathrm{KPu}^0 + (\mathcal{U}_n)| = |\mathrm{ATR}_0| = \Gamma_0$.

Fixed point axiom holds in $\bigcup_{n<\omega} \mathrm{KPu} + (\mathcal{U}_n)$

Admissible sets have a closure property: The fixed point axiom for arithmetically definably operators holds in $\bigcup_{n<\omega} \mathrm{KPu} + (\mathcal{U}_n)$.

Lemma (Jäger '84)

$\varphi(X, \vec{Y}, x)$: X-positive arithmetical formula.

$$\mathrm{KPu}{+}(\mathcal{U}_{n+1}) \vdash (\forall \vec{Y} \in d_{n-1})(\exists X \in d_n)(\forall x) \left(x \in X \leftrightarrow \varphi(X, \vec{Y}, x) \right)$$

(Hence at most n-fold iterated application of fixed point axiom is possible)

Note: due to absence of transfinite recursion, the leastness of the fixed point is not provable.

ATR_0 holds in $\bigcup_{n<\omega} \mathrm{KPu} + (\mathcal{U}_n)$

ATR_0 holds in $\bigcup_{n<\omega} \mathrm{KPu}^0 + (\mathcal{U}_n)$.

Lemma (Jäger '84)

φ: arithmetical formula.

$$\begin{aligned}
&(\forall <, \vec{Y} \in d_n)\\
&\mathrm{WO}(<) \to\\
&(\exists X \in d_n)(\forall \alpha \in \mathsf{field}(<))\forall x \left(x \in X_\alpha \leftrightarrow \varphi(X_{<\alpha}, \vec{Y}, x)\right)
\end{aligned}$$

holds in $\mathrm{KPu}^0 + (\mathcal{U}_{n+1})$.

Well-ordering of $<_{\mathcal{O}}$ up to $\omega \cdot n$

Lemma

Let $n < \omega$ and $a_n = 3 \cdot 5^{e_n} \in \mathcal{O}$ represent $\omega \cdot (n+1)$.

1. *$<_{\mathcal{O}} \restriction a_n$ of $<_{\mathcal{O}}$ is definable in $\mathrm{KPu} + (\mathcal{U}_{n+1})$.*
2. *$\mathrm{KPu} + (\mathcal{U}_{n+1}) \vdash \mathrm{WO}(<_{\mathcal{O}} \restriction a_n)$.*

Proof.

By n-fold application of FP axiom, define a relation $<_n \in d_n$:

$$b <_0 a \quad \leftrightarrow \quad (b = 1 \land a = 2^1) \lor \exists c(b \leqslant_0 c \land a = 2^c)$$

$$b <_{n+1} a \quad \leftrightarrow \quad \left\{ \begin{array}{l} b <_n a \lor \exists c(b \leqslant_{n+1} c \land a = 2^c) \lor \\ [a = a_n \land \forall m(\{e_n\}(m) <_n \{e_n\}(m+1)) \land \\ \exists m(b <_n \{e_n\}(m))] \end{array} \right.$$

See $<_n = <_{\mathcal{O}} \restriction a_n$. Show $\mathrm{KPu} + (\mathcal{U}_{n+1}) \vdash \mathrm{WO}(<_n)$ by ind on n. □

Theorem

Theorem

Let $1 \leq n$. Suppose that $a \in \mathcal{O}$ is a notation for $\omega \cdot n$.
Then $\mathrm{KPu}^0 + (\mathcal{U}_n) \vdash (\Sigma^0_1)_a\text{-}\mathrm{Det}^*$.

Outline of Proof.

Given a $(\Sigma^0_1)_a$ formula $\varphi(f)$, define a set $W_b \in d_{n-1}$ $(b <_{\mathcal{O}} \restriction a)$ of winning positions $s \in 2^{<\mathbb{N}}$ by (ATR):

$$s \in W_b \leftrightarrow \psi(s, W_{<_{\mathcal{O}} b}),$$

where $\psi \in \Pi^1_0$ is defined from φ. Define a new Σ^0_1 game $\varphi'(f) :\equiv \exists m (\exists b <_{\mathcal{O}} a) \langle f(0), \ldots, f(2m-1) \rangle \in W_b$.

1. If Player I wins $\varphi'(f)$, then I wins $\varphi(f)$.
2. If Player II wins $\varphi'(f)$, then II wins $\varphi(f)$.

Note: $\Sigma^0_1\text{-}\mathrm{Det}^*$ holds in $\mathrm{KPu}^0 + (\mathcal{U}_n)$. $\square$

Conclusion

Summary

- Aiming fine-grained analysis of determinacy of Δ^0_2-definable games in the Cantor space.
- Layering based on the Ershov hierarchy, which turns out to be questionable.
- Obtained partial results strongly rely on the definability and provability of the well ordering of $<_{\mathcal{O}} \upharpoonright a$.
- This observation is consistent with the results about $(\Sigma^0_1)_\alpha\text{-}\mathrm{Det}^*$ $(\alpha < \Gamma_0)$ by Nemoto-Sato.

Thank you for your listening!

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