Predicative Lexicographic Path Orders

Towards a Maximal Model for Primitive Recursive Functions

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Introduction

- Computable functions are classified depending on their computational complexity (polynomial time, elementary recursive, primitive recursive, multiple recursive etc.).
- Péter discussed about distinction between primitive recursive and multiple recursive functions.

Recursive Functions. R. Péter, 1967.

Theorem (Péter '67)

The class of primitive recursive functions is closed under primitive recursion with parameter substitution (PRP)*, unnested multiple recursion* (UMR) *and simple nested recursion* (SNR)*.*

$$\begin{array}{lrcl}
(\mathrm{PRP}) & f(x+1, y) & = & h(x, y, f(x, p(x, y))) \\
(\mathrm{UMR}) & f(x+1, y+1) & = & h(x, y, f(x, p(x, y)), f(x+1, y)) \\
(\mathrm{SNR}) & f(x+1, y) & = & h(x, y, f(x, p(x, y, f(x, y))))
\end{array}$$

Outline (1/2)

- To assess complexity of a given function, it is natural to look at the maximal length of rewriting sequences in the corresponding rewrite system - derivational complexity.
  - Derivational complexity can be exponentially related to complexity of the given function. (Cichon & Weiermann '97)
  - Derivational complexity can be polynomially related to complexity of the given function. (Avanzini & Moser '10)
- A rewriting proof of Péter's result. (Cichon & Weiermann '97)
- (PRP), (UMR) and (SNR) are compatible with termination orders known as lexicographic path orders (LPOs).

$$
\begin{aligned}
f(s(x), y) &>_{\mathsf{lpo}} h(x, y, f(x, p(x, y))) \\
f(s(x), s(y)) &>_{\mathsf{lpo}} h(x, y, f(x, p(x, y)), f(s(x), y)) \\
f(s(x), y) &>_{\mathsf{lpo}} h(x, y, f(x, p(x, y, f(x, y))))
\end{aligned}
$$

- However LPOs in general imply mutiply recursive derivational complexity. (Weiermann '95)

Outline (2/2)

Simmons discussed about restrictive (higher-order) primitive recursion, known as predicative recursion.

The Realm of Primitive Recursion. H. Simmons, 1988.

- to cover all known reductions of non-trivial primitive recursive equations to primitive recursion.
- to explain why those reductions work.

This talk: Predicative Lexicographic Path Orders (PLPOs)

1. A syntactic restriction of LPO compatible with (PRP), (UMR) and (SNR). (Based on predicative recursion)
2. Only induces primitive recursive derivational complexity.

- An alternative proof of Péter's result.
- An attempt to find a maximal model for primitive recursive functions in terms of termination orders.

Lexicographic path orders

Definition (Lexicographic path orders)

Assume a well-founded partial order $>$ on the signature.
Then $s = f(s_1, \ldots, s_k) >_{\mathsf{lpo}} t$ if one of the following holds.

1. $s_i \geqslant_{\mathsf{lpo}} t$ for some $i \in \{1, \ldots, k\}$.
2. $t = g(t_1, \ldots, t_l)$, $f > g$ and $s >_{\mathsf{lpo}} t_j$ for all $j \in \{1, \ldots, l\}$.
3. $t = f(t_1, \ldots, t_k)$ and there exists $i \in \{1, \ldots, k\}$ such that
   - $s_j = t_j$ for all $j < i$,
   - $s_i >_{\mathsf{lpo}} t_i$, and
   - $s >_{\mathsf{lpo}} t_j$ for all $j > i$.

Theorem (Weiermann '95)

Derivational complexity of every rewrite system compatible with an LPO is bounded by a multiply recursive function in the size of a starting term.

Examples

Every multiple recursive function can be represented by a rewrite system compatible with an LPO.

Example

$$
\begin{array}{lrl}
(\mathrm{PRP}) & f(s(x), y) & >_{\mathsf{lpo}} h(x, y, f(x, p(x, y))) \\
(\mathrm{UMR}) & f(s(x), s(y)) & >_{\mathsf{lpo}} h(x, y, f(x, p(x, y)), f(s(x), y)) \\
(\mathrm{SNR}) & f(s(x), y) & >_{\mathsf{lpo}} h(x, y, f(x, p(x, y, f(x, y)))) \\
(\mathrm{Ack}) & Ack(s(x), s(y)) & >_{\mathsf{lpo}} Ack(x, Ack(s(x), y))
\end{array}
$$

Theorem (shown by Cichon & Weiermann '97)

Rewrite systems corresponding to (PRP), (UMR) *and* (SNR) *only have primitive recursive derivational complexity.*

Proof.

By primitive recursive number-theoretic interpretations. $\square$

Simple nested recursion

Distinction between simple & non-simple nested recursion.

Example ($\mathcal{T}_{exp}$)

$$\begin{aligned} exp(0, y) &\to s(y) \\ exp(s(x), y) &\to exp(x, exp(x, y)) \end{aligned}$$

$\mathcal{T}_{exp}$ computes an exponential $2^x + y$.
In any (SNR) a specific argument separation is possible.

$$exp(0; y) \to s(y) \qquad exp(s(x); y) \to exp(x; exp(x; y))$$

- Observed by Simmons (1988).
- Called predicative recursion. (Bellantoni & Cook '92)
- Intuition: f(recursion performed; recursion term substituted)

Predicative recursion

Restrictive primitive recursion. (Bellantoni & Cook '92)

(Predicative recursion) $\begin{cases} f(0, \vec{y}; \vec{z}) & = & g(\vec{y}; \vec{z}) \\ f(x+1, \vec{y}; \vec{z}) & = & h(x, \vec{y}; \vec{z}, f(x, \vec{y}; \vec{z})) \end{cases}$

(Predicative composition) $f(\vec{x}; \vec{y}) = h(\vec{p}(\vec{x};); \vec{g}(\vec{x}; \vec{y}))$

Aim: to weaken the power of primitive recursion.
A variant: implicitly considered by Simmons (1988).

$$f(x+1; y) = h(x; y, f(x; p(x; y, f(x; y))))$$

The argument separation is taken into account only for multiple recursion.
Aim: to preserve the power of primitive recursion but to weaken the power of multiple recursion.

Predicative lexicographic path orders (1/2)

Introducing Predicative Lexicographic Path Order (PLPO) $>_{\mathsf{plpo}}$.
An auxiliary relation $\sqsupset_{\mathsf{plpo}} \subseteq >_{\mathsf{plpo}}$.

Definition

$f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) \sqsupset_{\mathsf{plpo}} t$ if either 1 or 2 holds.

1. $s_i \sqsupseteq_{\mathsf{plpo}} t$ for some $i \in \{1, \ldots, k\}$.
2. $t = g(t_1, \ldots, t_m)$, $f > g$, and $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{1, \ldots, m\}$.

Expresses the relation $f(\vec{x}; \vec{y}) \sqsupset_{\mathsf{plpo}} p(\vec{x};)$.

Definition

1. $(s_1, \ldots, s_k) \geqslant_{\mathsf{plpo}} (t_1, \ldots, t_k)$
   if $s_j \geqslant_{\mathsf{plpo}} t_j$ for all $j \in \{1, \ldots, k\}$.
2. $(s_1, \ldots, s_k) >_{\mathsf{plpo}} (t_1, \ldots, t_k)$
   if $(s_1, \ldots, s_k) \geqslant_{\mathsf{plpo}} (t_1, \ldots, t_k)$ and $s_i >_{\mathsf{plpo}} t_i$ for some i.

Predicative lexicographic path orders (2/2)

Definition (Predicative lexicographic path orders)

$s = f(s_1, \dots, s_k; s_{k+1}, \dots, s_{k+l}) >_{\mathsf{plpo}} t$ if one of 1–4 holds.

1. $s_i \geqslant_{\mathsf{plpo}} t$ for some $i \in \{1, \dots, k+l\}$.
2. $t = g(t_1, \dots, t_m; t_{m+1}, \dots, t_{m+n})$, $f > g$,
    - $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{1, \dots, m\}$, and
    - $s >_{\mathsf{plpo}} t_j$ for all $j \in \{m+1, \dots, m+n\}$.
3. $t = f(t_1, \dots, t_k; t_{k+1}, \dots, t_{k+l})$, $f \notin \mathcal{D}_{\mathsf{lex}}$,
    - $(s_1, \dots, s_k) \geqslant_{\mathsf{plpo}} (t_1, \dots, t_k)$, and
    - $(s_{k+1}, \dots, s_{k+l}) >_{\mathsf{plpo}} (t_{k+1}, \dots, t_{k+l})$.
4. $t = f(t_1, \dots, t_k; t_{k+1}, \dots, t_{k+l})$, $f \in \mathcal{D}_{\mathsf{lex}}$ and $\exists i \leq k$ s.t.
    - $s_j = t_j$ for all $j < i$,
    - $s_i >_{\mathsf{plpo}} t_i$,
    - $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{i+1, \dots, k\}$, and
    - $s >_{\mathsf{plpo}} t_j$ for all $j \in \{k+1, \dots, k+l\}$.

Examples (1/3)

In the definition of PLPOs, a specific subset $\mathcal{D}_{\mathsf{lex}}$ of (defined) function symbols is assumed.

Example (Primitive recursion)

$f(; s(; x), y) >_{\mathsf{plpo}} h(; x, y, f(; x, y))$

Let $f > h$ and $\mathcal{D}_{\mathsf{lex}} = \emptyset$.

1. $(s(x), y) >_{\mathsf{plpo}} (x, y)$.
2. $f(; s(x), y) >_{\mathsf{plpo}} f(; x, y)$. (by 1 & $f \notin \mathcal{D}_{\mathsf{lex}}$)
3. $f(; s(x), y) >_{\mathsf{plpo}} x, y$.
4. $f(; s(x), y) >_{\mathsf{plpo}} h(; x, y, f(; x, y))$. (by 2, 3 & $f > h$)

Examples (2/3)

Example (Primitive recursion with parameter substitution)

$f(s(;x);y) >_{\mathsf{plpo}} h(x;y,f(x;p(x;y)))$

Let $f > h, p$ and $\mathcal{D}_{\mathsf{lex}} = \{f\}$.

1. $f(s(x);y) >_{\mathsf{plpo}} p(x;y)$. (since $f(s(x);y) \sqsupset_{\mathsf{plpo}} x$ & $f > p$)
2. $s(x) >_{\mathsf{plpo}} x$.
3. $f(s(x);y) >_{\mathsf{plpo}} f(x;p(x;y))$. (by 1, 2 & $f \in \mathcal{D}_{\mathsf{lex}}$)
4. $f(s(x);y) \sqsupset_{\mathsf{plpo}} x$ and $f(s(x);y) >_{\mathsf{plpo}} y$.
5. $f(s(x);y) >_{\mathsf{plpo}} h(x;y,f(x;p(x;y)))$. (by 3, 4 & $f > h$)

Example (Simple nested recursion)

$f(s(;x);y) >_{\mathsf{plpo}} h(x;y,f(x;p(x;y,f(x;y))))$

Let $f > h, p$ and $\mathcal{D}_{\mathsf{lex}} = \{f\}$.

Examples (3/3)

Example (Unnested multiple recursion)

$f(s(;x), s(;y);) >_{\mathsf{plpo}} h(x, y; f(x, p(x, y;);), f(s(;x), y;))$

Let $f > h, p$ and $\mathcal{D}_{\mathsf{lex}} = \{f\}$.

1. $f(s(x), s(y);) \sqsupset_{\mathsf{plpo}} x, y$.
2. $f(s(x), s(y);) \sqsupset_{\mathsf{plpo}} p(x, y;)$. (by 1 & $f > p$)
3. $s(x) >_{\mathsf{plpo}} x$.
4. $f(s(x), s(y);) >_{\mathsf{plpo}} f(x, p(x, y;);)$. (by 2, 3 & $f \in \mathcal{D}_{\mathsf{lex}}$)
5. $f(s(x), s(y);) >_{\mathsf{plpo}} f(s(x), y;)$. (since $f \in \mathcal{D}_{\mathsf{lex}}$)
6. $f(s(x), s(y);) >_{\mathsf{plpo}} h(x, y; f(x, p(x, y;);), f(s(x), y;))$.
   (by 1, 4, 5 & $f > h$)

Example (Ackermann function)

$Ack(s(x), s(y);) \not>_{\mathsf{plpo}} Ack(x, Ack(s(x), y;);)$

Complexity result

Theorem

Derivational complexity of any rewrite system compatible with a PLPO is bounded by a primitive recursive function in the size of a starting term.

Proof.

By primitive recursive interpretation stemming from Cichon '92. $\square$

Corollary

The class of primitive recursive functions is closed under (PRP), (UMR) *and* (SNR).

Contrast to related orders

- Every PLPO is a suborder of an LPO.
- Exponential path orders EPOs. (Avanzini-E.-Moser '11)
  Every EPO is a suborder of a PLPO.
- PLPO is incomparable with polynomial path orders POPs (Avanzini-Moser '08) or light multiset path orders LMPO (Marion '03).
- Ramified lexicographic path orders RLPOs. (Cichon '92)
  For any rewrite system $\mathcal{R}$ compatible with an RLPO there exists an extension $\mathcal{R}'$ of $\mathcal{R}$ defining the same function and compatible with a PLPO.
- The same holds for light lexicographic path orders LLPO (Cichon-Marion).

Summary

Predicative lexicographic path orders (PLPO), a syntactic restriction of LPOs based on predicative recursion, introduced.

- As well as LPOs, equations of (PRP), (UMR) and (SNR) can be oriented with PLPOs.
- In contrast to LPOs, PLPOs only induce primitive recursive derivational complexity for compatible rewrite systems.
- A rewriting application to non-trivial closure conditions: The class of primitive recursive functions is closed under (PRP), (UMR) and (SNR).

Possible extension (1/2)

Definition (Predicative lexicographic path orders)

$s = f(s_1, \ldots, s_k; s_{k+1}, \ldots, s_{k+l}) >_{\mathsf{plpo}} t$ if one of 1–4 holds.

1. $s_i \geqslant_{\mathsf{plpo}} t$ for some $i \in \{1, \ldots, k + l\}$.
2. $t = g(t_m, \ldots, t_m; t_{m+1}, \ldots, t_{m+n})$, $f > g$,
    - $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{1, \ldots, m\}$, and
    - $s >_{\mathsf{plpo}} t_j$ for all $j \in \{m + 1, \ldots, m + n\}$.
3. $t = f(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l})$, $f \notin \mathcal{D}_{\mathsf{lex}}$,
    - $(s_1, \ldots, s_k) \geqslant_{\mathsf{plpo}} (t_1, \ldots, t_k)$, and
    - $(s_{k+1}, \ldots, s_{k+l}) >_{\mathsf{plpo}} (t_{k+1}, \ldots, t_{k+l})$.
4. $t = f(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l})$, $f \in \mathcal{D}_{\mathsf{lex}}$ and $\exists i \leq k$ s.t.
    - $s_j = t_j$ for all $j < i$,
    - $s_i >_{\mathsf{plpo}} t_i$,
    - $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{i + 1, \ldots, k\}$, and
    - $s >_{\mathsf{plpo}} t_j$ for all $j \in \{k + 1, \ldots, k + l\}$.

Possible extension (2/2)

- Permutation π on $\{k+1, \ldots, k+l\}$ is allowed.
  $t = f(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l})$, $f \notin \mathcal{D}_{\mathrm{lex}}$,
  - $(s_1, \ldots, s_k) \geqslant_{\mathrm{plpo}} (t_1, \ldots, t_k)$, and
  - $(s_{k+1}, \ldots, s_{k+l}) >_{\mathrm{plpo}} (t_{\pi(k+1)}, \ldots, t_{\pi(k+l)})$.
- Question: Permutation π on $\{1, \ldots, k\}$ allowed?
  $t = f(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l})$, $f \notin \mathcal{D}_{\mathrm{lex}}$,
  - $(s_1, \ldots, s_k) \geqslant_{\mathrm{plpo}} (t_{\pi(1)}, \ldots, t_{\pi(k)})$, and
  - $(s_{k+1}, \ldots, s_{k+l}) >_{\mathrm{plpo}} (t_{k+1}, \ldots, t_{k+l})$.
- Question: Multiset comparison allowed?
  $t = f(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+l})$, $f \in \mathcal{D}_{\mathrm{mul}}$,
  - $(s_1, \ldots, s_k)(\geqslant_{\mathrm{plpo}})_{\mathrm{mul}}(t_1, \ldots, t_k)$, and
  - $(s_{k+1}, \ldots, s_{k+l})(>_{\mathrm{plpo}})_{\mathrm{mul}}(t_{k+1}, \ldots, t_{k+l})$.
- Can be reformulated in higher order recursive path orders?
  Recall higher order predicative recursion by Simmons.

References

Predicative Lexicographic Path Orders:
Towards a Maximal Model for Primitive Recursive Functions
Naohi Eguchi
Technical report, arXiv: 1308.0247 [math.LO].

Thank you for your attention!

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