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ABSTRACT COMPLETION, FORMALIZED

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Abstract. Completion is one of the most studied techniques in term rewriting and fundamental to automated reasoning with equalities. In this paper we present new correctness proofs of abstract completion, both for finite and infinite runs. For the special case of ground completion we present a new proof based on random descent. We moreover extend the results to ordered completion, an important extension of completion that aims to produce ground-complete presentations of the initial equations. We present new proofs concerning the completeness of ordered completion for two settings. Moreover, we revisit and extend results of Métivier concerning canonicity of rewrite systems. All proofs presented in the paper have been formalized in Isabelle/HOL.

1. Introduction

Reasoning with equalities is pervasive in computer science and mathematics, and has consequently been one of the main research areas of automated deduction. Indeed completion as introduced by Knuth and Bendix [KB70] has evolved into a fundamental technique whose ideas appear throughout automated reasoning whenever equalities are present. Many variants of the original calculus have since been proposed.

Bachmair, Dershowitz, and Hsiang [BDH86] recast completion procedures as inference systems. This style of presentation, *abstract completion*, has become the standard to describe completion procedures and *proof orders* the accompanying tool to establish correctness [BDH86, BDP89, Bac91], that is, that under certain conditions, exhaustive application of the inference rules results in a terminating and confluent rewrite system whose equational theory is equivalent to the initial set of equations.

In this paper we present new, modular correctness proofs, not relying on proof orders, for five abstract completion systems presented in the literature. Here, we use *modular* in the

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following sense: Proof orders have to be powerful (and thus complex) enough to cover all intermediate results (that is, proof orders are a *global* method), while for our new proofs, we *locally* apply well-founded induction with an order that is just strong enough for the current intermediate result. All proofs are fully formalized in Isabelle/HOL. First, we consider finite ($\mathsf{KB_f}$) and infinite ($\mathsf{KB_i}$) runs of classical Knuth-Bendix completion [KB70]. These two settings demand different proofs since in the latter case the inference system exhibits a stronger side condition. While our correctness proof for $\mathsf{KB_f}$ relies on a new notion we dub *peak decreasingness*, for the case of $\mathsf{KB_i}$ we employ a simpler version of this criterion called *source decreasingness*. To enhance applicability by covering efficient implementations, our proofs support the critical pair criterion known as primality [KMN88].

The relevance of infinite runs is illustrated by the following example.

Example 1.1. Consider the set of equations $\mathcal{E} = \{\mathsf{aba} \approx \mathsf{bab}\}$ of the three-strand positive braid monoid. Kapur and Narendran [KN85] proved that $\mathcal{E}$ admits no *finite* complete presentation. However, taking the Knuth-Bendix order [KB70] with a and b of weight 1 and $\mathsf{a} > \mathsf{b}$ in the precedence, completion produces in the limit the following infinite complete presentation of $\mathcal{E}$

$$\{\mathsf{aba} \to \mathsf{bab}\} \cup \{\mathsf{ab}^n\mathsf{ab} \to \mathsf{babba}^{n-1} \mid n \geqslant 2\}$$

which can be used to decide the validity problem for $\mathcal{E}$.[1]

Completion procedures, when successful, produce a complete system. Natural questions include whether such systems are unique and whether all complete systems for a given set of equations can be obtained by completion. For canonical systems, which are complete systems that satisfy an additional normalization requirement, Métivier [Mét83] obtained interesting results. In this paper we revisit and extend his work.

A special case of $\mathsf{KB_f}$ that is known to be decidable is the completion of ground systems [Sny93]. We present new correctness and completeness proofs for the corresponding inference system $\mathsf{KB_g}$, based on the recent notion of random descent [vOT16].

On a given set of input equalities, Knuth-Bendix completion can behave in three different ways: it may (1) succeed to compute a complete system in finitely many steps, (2) fail due to unorientable equalities, or (3) continuously compute approximations of a complete system without ever terminating. As a remedy to problem (2), ordered completion was developed by Bachmair, Dershowitz, and Plaisted [BDP89]. Ordered completion never fails and can produce a ground-complete system in the limit. Although the price to be paid is that the resulting system is in general only complete on ground terms, this is actually sufficient for many applications in theorem proving. Refutational theorem proving [BDP89] owes its semi-decidability to the unfailing nature of ordered completion. Again employing peak decreasingness, we obtain a new correctness proof of ordered completion ($\mathsf{KB_o}$). Next, we turn to completeness results for ordered completion, that is, to sufficient criteria for an ordered completion procedure to produce a complete system. We first reprove the case of a total reduction order, which assumes a slightly stronger notion of simplifiedness than the original result [BDP89] though. Then we consider the completeness result for linear completion ($\mathsf{KB_l}$) due to Devie [Dev91].

For easy reference, Table 1 provides pointers to the main definitions and results we present in this paper.

[1]Burckel [Bur01] constructed a complete rewrite system consisting of four rules with an additional symbol, which is no longer a complete presentation of $\mathcal{E}$ but can be also used to decide the validity problem for $\mathcal{E}$.

Table 1: Roadmap.

	$\mathsf{KB_f}$	$\mathsf{KB_g}$	$\mathsf{KB_i}$	$\mathsf{KB_o}$	$\mathsf{KB_l}$
inference system	3.1	5.1	6.2	7.2	8.13
fairness	3.5	–	6.3	7.14	8.16
correctness	3.8	5.5	6.12	7.17 7.16	8.17
completeness	–	5.12	–	8.10	8.23

The remainder of this paper is organized as follows. We present required preliminaries in Section 2, followed by the abstract confluence criteria of peak and source decreasingness, as well as a fairly detailed analysis of critical pairs. In Section 3 we recall the inference rules for (abstract) Knuth-Bendix completion and present our formalized correctness proof for finite runs. In Section 4 we present our results on canonical systems and normalization equivalence. We discuss ground completion in Section 5. Infinite runs are the subject of Section 6 and in Section 7 we extend our correctness results to ordered completion. Completeness of ordered completion is the topic of Section 8. We conclude in Section 9 with a few suggestions for future research.

Our formalizations are part of the *Isabelle Formalization of Rewriting* IsaFoR [TS09][2] version 2.37. Below we list the relevant Isabelle theory files grouped by their subdirectories inside IsaFoR:

```
thys/Abstract_Completion/
  Abstract_Completion.thy
  Completion_Fairness.thy
  CP.thy
  Ground_Completion.thy
  Peak_Decreasingness.thy
  Prime_Critical_Pairs.thy

thys/Confluence_and_Completion/
  Ordered_Completion.thy

thys/Normalization_Equivalence/
  Encompassment.thy
  Normalization_Equivalence.thy
```

In the remainder we provide hyperlinks (marked by ☑) to an HTML rendering of our formalization. Moreover, whenever we say that a proof is "formalized," what we mean is that it is"formalized in Isabelle/HOL." And when we "present a formalized proof," we give a textual representation of a formalized proof.

This paper and the accompanying formalization are substantially extended and revised versions of some of our previous work we published in the ITP [HMS14] and FSCD [HMSW17] conferences. The former presented a new correctness proof for finite runs of Knuth-Bendix completion. Its modular design separates concerns rather than relying on a single proof order, thus rendering it more formalization friendly. In revised form, these results are included in Section 3. The FSCD contribution extended this novel proof approach to both infinite runs and ordered completion (see Sections 6 and 7). It moreover incorporated canonicity results (Section 4). In addition to these results we present new and formalized proofs of correctness and completeness of ground completion (Section 5), as well as completeness of ordered completion for two different cases (Section 8). At the end of each section, we remark on the novelty of the respective results and their proofs.

[2]http://cl-informatik.uibk.ac.at/isafor

2. Preliminaries

We assume familiarity with the basic notions of abstract rewrite systems, term rewrite systems, and completion [Bac91, BN98], but nevertheless shortly recapitulate terminology and notation that we use in the remainder.

2.1. **Rewrite Systems.** For an arbitrary binary relation $\to_\alpha$, we write ${}_\alpha\!\leftarrow$, $\underset{\alpha}{\leftrightarrow}$, $\to_\alpha^=$, $\to_\alpha^+$, and $\to_\alpha^*$ to denote its *inverse*, its *symmetric closure*, its *reflexive closure*, its *transitive closure*, and its *reflexive transitive closure*, respectively. The reflexive, transitive, and symmetric closure $\overset{*}{\underset{\alpha}{\leftrightarrow}}$ of $\to_\alpha$ is called *conversion*, and a sequence of the form $c_0 \underset{\alpha}{\leftrightarrow} c_1 \underset{\alpha}{\leftrightarrow} \cdots \underset{\alpha}{\leftrightarrow} c_n$ is referred to as a conversion between c_0 and c_n (of length n). For a binary relation R without arrow notation, we also write R^{-1} for its *inverse* and $R^\pm$ for its *symmetric closure* $R \cup R^{-1}$. We further use $\downarrow_\alpha$ as abbreviation for the *joinability relation* $\to_\alpha^* \cdot {}_\alpha^*\!\leftarrow$, where from here on $\cdot$ denotes relation composition. If $a \to_\alpha b$ for no b then we say that a is a *($\to_\alpha$-)normal form*. The set of all normal forms of a given relation $\to_\alpha$ is denoted by $\mathsf{NF}(\to_\alpha)$. By $a \to_\alpha^! b$ we abbreviate $a \to_\alpha^* b \wedge b \in \mathsf{NF}(\to_\alpha)$ and we call b a *normal form of* a. Given two binary relations $\to_\alpha$ and $\to_\beta$, we use $\to_\alpha / \to_\beta$ as shorthand for the *relative rewrite relation* $\to_\beta^* \cdot \to_\alpha \cdot \to_\beta^*$. An *abstract rewrite system* (ARS for short) $\mathcal{A}$ is a set A, the carrier, equipped with a binary relation $\to$. Sometimes we partition the binary relation into parts according to a set I of indices (or labels). Then we write $\mathcal{A} = \langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ where we denote the part of the relation with label α by $\to_\alpha$, that is, $\to = \bigcup \{\to_\alpha \mid \alpha \in I\}$.

We assume a given signature $\mathcal{F}$ and a set of variables $\mathcal{V}$. The set of terms built up from $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$, while $\mathcal{T}(\mathcal{F})$ denotes the set of ground terms. *Positions* are strings of positive integers which are used to address subterms. The set of positions in a term t is denoted by $\mathcal{P}\mathsf{os}(t)$. The subset consisting of the positions addressing function symbols in t is denoted by $\mathcal{P}\mathsf{os}_\mathcal{F}(t)$ whereas $\mathcal{P}\mathsf{os}_\mathcal{V}(t) = \mathcal{P}\mathsf{os}(t) - \mathcal{P}\mathsf{os}_\mathcal{F}(t)$ is the set of variable positions in t. We write $p \leqslant q$ if p is a prefix of q and $p \parallel q$ if neither $p \leqslant q$ nor $q \leqslant p$. If $p \leqslant q$ then the unique position r such that $pr = q$ is denoted by $q \backslash p$. A *substitution* is a mapping σ from variables to terms such that its domain $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. Applying a substitution σ to a term t is written $t\sigma$. A variable substitution is a substitution from $\mathcal{V}$ to $\mathcal{V}$ and a *renaming* is a bijective variable substitution. A term s is a *variant* of a term t if $s = t\sigma$ for some renaming σ. A pair of terms (s,t) is sometimes considered an *equation*, then we write $s \approx t$, and sometimes a *(rewrite) rule*, then we write $s \to t$. In the latter case we assume the *variable condition*, that is, that the left-hand side s is not a variable and that variables of the right-hand side t are all contained in t. A set $\mathcal{E}$ of equations is called an *equational system* (ES for short) and a set $\mathcal{R}$ of rules a *term rewrite system* (TRS for short). Sets of pairs of terms $\mathcal{E}$ induce a *rewrite relation* $\to_\mathcal{E}$ by closing their components under contexts and substitutions. A rewrite step $s \to_\mathcal{E} t$ at a position $p \in \mathcal{P}\mathsf{os}(s)$ is called *innermost* and denoted by $s \xrightarrow{\mathsf{i}}_\mathcal{E} t$ if no proper subterm of $s|_p$ is reducible in $\mathcal{E}$. The *equational theory* induced by $\mathcal{E}$ consists of all pairs of terms s and t such that $s \leftrightarrow_\mathcal{E}^* t$. If $\ell \to r$ is a rewrite rule and σ is a renaming then the rewrite rule $\ell\sigma \to r\sigma$ is a *variant* of $\ell \to r$. A TRS is said to be *variant-free* if it does not contain rewrite rules that are variants of each other.

Two terms s and t are called *literally similar*, written $s \doteq t$, if $s\sigma = t$ and $s = t\tau$ for some substitutions σ and τ. Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ are called *literally similar*, denoted by $\mathcal{R}_1 \doteq \mathcal{R}_2$, if every rewrite rule in $\mathcal{R}_1$ has a variant in $\mathcal{R}_2$ and vice versa. The following result is folklore; we formalized the non-trivial proof.

Lemma 2.1. *Two terms s and t are variants of each other if and only if $s \doteq t$.* ☑

We say that s *encompasses* t, written $s \trianglerighteq t$, whenever $s = C[t\sigma]$ for some context C and substitution σ. *Proper encompassment* is defined by $\vartriangleright = \trianglerighteq \setminus \trianglelefteq$ and known to be well-founded. The identity $\trianglerighteq = \vartriangleright \cup \doteq$ is well-known. For a well-founded order $>$, we write $>_{\mathsf{mul}}$ to denote its *multiset extension* and $>_{\mathsf{lex}}$ to denote its *lexicographic extension* as defined by Baader and Nipkow [BN98].

A TRS $\mathcal{R}$ is *terminating* if $\to_{\mathcal{R}}$ is well-founded, and *weakly normalizing* if every term has a normal form. It is *(ground-)confluent* if $s \;{}_{\mathcal{R}}^{*}\!\!\leftarrow \cdot \to_{\mathcal{R}}^{*} t$ implies $s \to_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{*}\!\!\leftarrow t$ for all (ground) terms s and t. It is *(ground-)complete* if it is terminating and (ground) confluent. We say that $\mathcal{R}$ is a *complete presentation* of an ES $\mathcal{E}$ if $\mathcal{R}$ is complete and $\leftrightarrow_{\mathcal{R}}^{*} = \leftrightarrow_{\mathcal{E}}^{*}$. A TRS $\mathcal{R}$ is *left-reduced* if $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$, and *right-reduced* if $r \in \mathsf{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$. A *reduced* TRS is left- and right-reduced. A reduced complete TRS is called *canonical*.

We make use of the following result due to Bachmair and Dershowitz [BD86], where *quasi-commutation* of R over S means that the inclusion $S \cdot R \subseteq R \cdot (R \cup S)^*$ holds.

Lemma 2.2. *Let R and S be binary relations.*

(1) If R quasi-commutes over S then well-foundedness of $R \,/\, S$ and R coincide. ☑
(2) If $R \,/\, S$ and S are well-founded then $R \cup S$ is well-founded. ☑

Lemma 2.3. *If R is a well-founded rewrite relation then $(R \cup \vartriangleright) \,/\, \trianglerighteq$ is well-founded.* ☑

Proof. First we show the inclusion $\trianglerighteq \cdot R \subseteq R \cdot \trianglerighteq$. Suppose $s \trianglerighteq t \; R \; u$. So $s = C[t\sigma]$ for some context C and substitution σ. Because R is closed under contexts and substitutions, $s \; R \; C[u\sigma]$. Moreover, $C[u\sigma] \trianglerighteq u$. This establishes the inclusion, and we conclude that R (quasi-)commutes over $\trianglerighteq$. Because R is well-founded, it follows from Lemma 2.2(1) that the relation $R \,/\, \trianglerighteq$ is well-founded too. Then $R \,/\, \vartriangleright$ is well-founded since it is contained in $R \,/\, \trianglerighteq$. As $\vartriangleright$ is well-founded, it follows from Lemma 2.2(2) that $R \cup \vartriangleright$ is well-founded. We have $\trianglerighteq \cdot \vartriangleright \subseteq \vartriangleright$ and thus $R \cup \vartriangleright$ quasi-commutes over $\trianglerighteq$. Another application of Lemma 2.2(1) yields the well-foundedness of $(R \cup \vartriangleright) \,/\, \trianglerighteq$. □

2.2. **Abstract Confluence Criteria.**

We use the following simple confluence criterion for ARSs to replace Newman's Lemma in the correctness proof of abstract completion. In the sequel, we will refer to a conversion of the form ${}_{\mathcal{A}}\!\!\leftarrow \cdot \to_{\mathcal{A}}$ as a *peak*.

Definition 2.4 (Peak Decreasingness ☑)**.** An ARS $\mathcal{A} = \langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ is *peak decreasing* if there exists a well-founded order $>$ on I such that for all $\alpha, \beta \in I$ the inclusion

$$ {}_{\alpha}\!\!\leftarrow \cdot \to_{\beta} \;\subseteq\; \xleftrightarrow[\vee\alpha\beta]{*} $$

holds. Here $\vee\alpha\beta$ denotes the set $\{\gamma \in I \mid \alpha > \gamma \text{ or } \beta > \gamma\}$ and if $J \subseteq I$ then $\xleftrightarrow[J]{*}$ denotes a conversion consisting of $\to_J = \bigcup \{\to_\gamma \mid \gamma \in J\}$ steps.

Peak decreasingness is a special case of decreasing diagrams [vO94], which is known as a very powerful confluence criterion. For the sake of completeness, we present an easy direct (and formalized) proof of the sufficiency of peak decreasingness for confluence. We denote by $\mathcal{M}(J)$ the set of all multisets over a set J.

Lemma 2.5. *Every peak decreasing ARS is confluent.* ☑

Proof. Let $>$ be a well-founded order on I which shows that the ARS $\mathcal{A} = \langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ is peak decreasing. With every conversion C in $\mathcal{A}$ we associate the multiset M_C consisting of the labels of its steps. These multisets are compared by the multiset extension $>_{\mathsf{mul}}$ of $>$, which is a well-founded order on $\mathcal{M}(I)$. We prove $\leftrightarrow^* \subseteq \downarrow$ by well-founded induction on $>_{\mathsf{mul}}$. Consider a conversion C between a and b. We either have $a \downarrow b$ or $a \leftrightarrow^* \cdot \leftarrow \cdot \to \cdot \leftrightarrow^* b$. In the former case we are done. In the latter case there exist labels $\alpha, \beta \in I$ and multisets $\Gamma_1, \Gamma_2 \in \mathcal{M}(A)$ such that $M_C = \Gamma_1 \uplus \{\alpha, \beta\} \uplus \Gamma_2$. By the peak decreasingness assumption there exists a conversion C' between a and b such that $M_{C'} = \Gamma_1 \uplus \Gamma \uplus \Gamma_2$ with $\Gamma \in \mathcal{M}(\vee\alpha\beta)$. We obviously have $\{\alpha, \beta\} >_{\mathsf{mul}} \Gamma$ and hence $M_C >_{\mathsf{mul}} M_{C'}$. Finally, we obtain $a \downarrow b$ from the induction hypothesis. ☐

A similar criterion to show the Church-Rosser modulo property will be used in Section 8. Here an ARS $\mathcal{A}$ is called *Church-Rosser modulo* an ARS $\mathcal{B}$ if the inclusion

$$\xleftrightarrow[\mathcal{A}\cup\mathcal{B}]{*} \subseteq \xrightarrow[\mathcal{A}]{*} \cdot \xleftrightarrow[\mathcal{B}]{*} \cdot \xleftarrow[\mathcal{A}]{*}$$

holds.

Definition 2.6 (Peak Decreasingness Modulo ☑)**.** Consider two ARSs $\mathcal{A} = \langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ and $\mathcal{B} = \langle B, \{\to_\beta\}_{\beta \in J}\rangle$. Then $\mathcal{A}$ is *peak decreasing modulo* $\mathcal{B}$ if there exists a well-founded order $>$ on $I \cup J$ such that for all $\alpha \in I$ and $\gamma \in I \cup J$ the inclusion

$${}_\alpha\!\leftarrow \cdot \to_\gamma \;\subseteq\; \xleftrightarrow[\vee\alpha\gamma]{*}$$

holds. Here $\vee\alpha\gamma$ denotes the set $\{\delta \in I \cup J \mid \alpha > \delta \text{ or } \gamma > \delta\}$.

Lemma 2.7. *If $\mathcal{A}$ is peak decreasing modulo $\mathcal{B}$ then $\mathcal{A}$ is Church-Rosser modulo $\mathcal{B}$.* ☑

Proof. Let $x_1 \leftrightarrow_{\alpha_1} \cdots \leftrightarrow_{\alpha_n} x_{n+1}$ and $M = \{\alpha_1, \ldots, \alpha_n\}$. We use induction on M with respect to $>_{\mathsf{mul}}$ to show $x_1 \to_{\mathcal{A}}^* \cdot \leftrightarrow_{\mathcal{B}}^* \cdot {}_{\mathcal{A}}^{*}\!\leftarrow x_{n+1}$. If the given conversion is not of the desired shape, there is an index $1 \leqslant i < n$ such that $x_i \;{}_\alpha\!\leftarrow x_{i+1} \to_\gamma x_{i+2}$ or $x_i \;{}_\gamma\!\leftarrow x_{i+1} \to_\alpha x_{i+2}$ for some $\alpha \in I$ and $\gamma \in I \cup J$. As the reasoning is similar, we only consider the former case. By peak decreasingness there are labels $\beta_1, \ldots, \beta_m$ with $x_i \leftrightarrow_{\beta_1} \cdots \leftrightarrow_{\beta_m} x_{i+2}$ such that $\beta_j \in \vee\alpha\gamma$ for all $1 \leqslant j \leqslant m$. Writing N for the multiset $\{\beta_1, \ldots, \beta_m\}$, we obtain $M >_{\mathsf{mul}} (M - \{\alpha, \gamma\}) \uplus N$ from $\alpha, \gamma \in M$ and $\{\alpha, \gamma\} >_{\mathsf{mul}} N$. Therefore, the claim follows from the induction hypothesis. ☐

For the correctness proof in Section 6 we use a simpler notion than peak decreasingness.

Definition 2.8 (Source Decreasingness ☑)**.** Let $\mathcal{A} = \langle A, \to\rangle$ be an ARS equipped with a well-founded relation $>$ on A, and we write $b \xrightarrow{a} c$ if $b \to c$ and $a = b$. We say that $\mathcal{A}$ is *source decreasing* if the inclusion

$$\leftarrow a \to \;\subseteq\; \xleftrightarrow{\vee a}{}^*$$

holds for all $a \in A$. Here $\leftarrow a \to$ denotes the binary relation consisting of all pairs (b, c) such that $a \to b$ and $a \to c$. Moreover, $\xleftrightarrow{\vee a}{}^*$ denotes the binary relation consisting of all pairs of elements that are connected by a conversion in which all steps are labeled with an element smaller than a.

Source decreasingness is the specialization of peak decreasingness to source labeling [vO08, Example 6]. It is closely related to the *connectedness-below* criterion of Winkler and Buchberger [WB86]. Unlike the latter, source decreasingness does not entail termination. For instance, for $\mathsf{a} > \mathsf{b}$ and $\mathsf{a} > \mathsf{c}$ the non-terminating ARS

$$\begin{tikzcd} \mathsf{b} \arrow[r, bend right] & \mathsf{a} \arrow[l, bend right] \arrow[r, bend left] & \mathsf{c} \arrow[l, bend left] \end{tikzcd}$$

is source decreasing but the connectedness-below criterion does not apply.

Lemma 2.9. *Every source decreasing ARS is peak decreasing.*

Peak decreasingness as a special case of decreasing diagrams was first considered in our ITP publication [HMS14] (the modulo version in Definition 2.6 is new). Source decreasingness originates from our later FSCD contribution [HMSW17].

2.3. **Critical Peaks.** Completion is based on critical pair analysis. In this subsection we present a version of the critical pair lemma that incorporates primality (cf. Definition 2.13 below).

Definition 2.10 (Overlaps). An *overlap* of a TRS $\mathcal{R}$ is a triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$, consisting of two rewrite rules and a position, satisfying the following properties:

- there are renamings π_1 and π_2 such that $\pi_1(\ell_1 \to r_1), \pi_2(\ell_2 \to r_2) \in \mathcal{R}$ (that is, the rules are variants of rules in $\mathcal{R}$),
- $\mathcal{V}\mathsf{ar}(\ell_1 \to r_1) \cap \mathcal{V}\mathsf{ar}(\ell_2 \to r_2) = \varnothing$ (that is, the rules have no common variables),
- $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$,
- ℓ_1 and $\ell_2|_p$ are unifiable,
- if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants of each other.

In general this definition may lead to an infinite set of overlaps, since there are infinitely many possibilities of taking variable disjoint variants of rules. Fortunately it can be shown that overlaps that originate from the same two rules are variants of each other. Overlaps give rise to critical peaks and pairs.

Definition 2.11 (Critical Peaks and Pairs). Suppose $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ is an overlap of a TRS $\mathcal{R}$. Let σ be a most general unifier of ℓ_1 and $\ell_2|_p$. The term $\ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma$ can be reduced in two different ways:

$$\begin{tikzcd} & \ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma \arrow[dl, "\ell_1 \to r_1"', "p"] \arrow[dr, "\ell_2 \to r_2", "\epsilon"'] & \\ \ell_2\sigma[r_1\sigma]_p & & r_2\sigma \end{tikzcd}$$

We call the quadruple $(\ell_2\sigma[r_1\sigma]_p, p, \ell_2\sigma, r_2\sigma)$ a *critical peak* and the equation $\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ a *critical pair* of $\mathcal{R}$, obtained from the overlap. The set of all critical pairs of $\mathcal{R}$ is denoted by $\mathsf{CP}(\mathcal{R})$.

In our formalization of the above definition, instead of an arbitrary most general unifier, we use *the* most general unifier computed by the formalized unification algorithm that is part of IsaFoR (thereby removing one degree of freedom and making it easier to show that only finitely many critical pairs have to be considered for finite TRSs).

A critical peak (t, p, s, u) is usually denoted by $t \xleftarrow{p} s \xrightarrow{\epsilon} u$. It can be shown that different critical peaks and pairs obtained from two variants of the same overlap are variants of each other. Since rewriting is equivariant under permutations, it is enough to consult finitely many critical pairs or peaks for finite TRSs (one for each pair of rules and each appropriate position) in order to conclude rewriting related properties (like joinability or fairness, see below) for all of them.

We present a variation of the well-known critical pair lemma for critical peaks and its formalized proof. The slightly cumbersome statement is essential to avoid gaps in the proof of Lemma 2.15 below.

Lemma 2.12. *Let $\mathcal{R}$ be a TRS. If $t \;{}_{\mathcal{R}}\!\xleftarrow{p_1} s \xrightarrow{p_2}_{\mathcal{R}} u$ then one of the following holds:* ☑
(1) $t \downarrow_{\mathcal{R}} u$,
(2) $p_2 \leqslant p_1$ and $t|_{p_2} \xleftarrow{p_1 \backslash p_2} s|_{p_2} \xrightarrow{\epsilon} u|_{p_2}$ is an instance of a critical peak, or
(3) $p_1 \leqslant p_2$ and $u|_{p_1} \xleftarrow{p_2 \backslash p_1} s|_{p_1} \xrightarrow{\epsilon} t|_{p_1}$ is an instance of a critical peak.

Proof. Consider an arbitrary peak $t \;{}_{p_1,\ell_1' \to r_1',\sigma_1'}\!\leftarrow s \to_{p_2,\ell_2 \to r_2,\sigma_2} u$. If $p_1 \parallel p_2$ then

$$t \to_{p_2,\ell_2 \to r_2,\sigma_2} t[r_2\sigma_2]_{p_2} = u[r_1'\sigma_1']_{p_1} \;{}_{p_1,\ell_1' \to r_1',\sigma_1'}\!\leftarrow u$$

If the positions of the contracted redexes are not parallel then one of them is above the other. Without loss of generality we assume that $p_1 \geqslant p_2$. Let $p = p_1 \backslash p_2$. Moreover, let π be a permutation such that $\ell_1 \to r_1 = \pi(\ell_1' \to r_1')$ and $\ell_2 \to r_2$ have no variables in common. Such a permutation exists since we only have to avoid the finitely many variables of $\ell_2 \to r_2$ and assume an infinite set of variables. Furthermore, let $\sigma_1 = \pi^{-1} \cdot \sigma_1'$. We have $t = s[r_1\sigma_1]_{p_1} = s[\ell_2\sigma_2[r_1\sigma_1]_p]_{p_2}$ and $u = s[r_2\sigma_2]_{p_2}$. We consider two cases depending on whether $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$ in conjunction with the fact that whenever $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants, is true or not.

- Suppose $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$ and $p = \epsilon$ implies that $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants. Let $\sigma'(x) = \sigma_1(x)$ for $x \in \mathcal{V}\mathsf{ar}(\ell_1 \to r_1)$ and $\sigma'(x) = \sigma_2(x)$, otherwise. The substitution σ' is a unifier of $\ell_2|_p$ and ℓ_1: $(\ell_2|_p)\sigma' = (\ell_2\sigma_2)|_p = \ell_1\sigma_1 = \ell_1\sigma'$. Then $\langle \ell_1 \to r_1, p, \ell_2 \to r_2\rangle$ is an overlap. Let σ be a most general unifier of $\ell_2|_p$ and ℓ_1. Hence $\ell_2\sigma[r_1\sigma]_p \xleftarrow{p} \ell_2\sigma \xrightarrow{\epsilon} r_2\sigma$ is a critical peak and there exists a substitution τ such that $\sigma' = \sigma\tau$. Therefore
$$\ell_2\sigma_2[r_1\sigma_1]_p = (\ell_2\sigma[r_1\sigma]_p)\tau \xleftarrow{p} (\ell_2\sigma)\tau \xrightarrow{\epsilon} (r_2\sigma)\tau = r_2\sigma_2$$
and thus (2) is obtained.
- Otherwise, either $p = \epsilon$ and $\ell_1 \to r_1$, $\ell_2 \to r_2$ are variants, or $p \notin \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$. In the former case it is easy to show that $r_1\sigma_1 = r_2\sigma_2$ and hence $t = u$. In the latter case, there exist positions q_1, q_2 such that $p = q_1q_2$ and $q_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\ell_2)$. Let $\ell_2|_{q_1}$ be the variable x. We have $\sigma_2(x)|_{q_2} = \ell_1\sigma_1$. Define the substitution σ_2' as follows:
$$\sigma_2'(y) = \begin{cases} \sigma_2(y)[r_1\sigma_1]_{q_2} & \text{if } y = x \\ \sigma_2(y) & \text{if } y \neq x \end{cases}$$
Clearly $\sigma_2(x) \to_{\mathcal{R}} \sigma_2'(x)$, and thus $r_2\sigma_2 \to^* r_2\sigma_2'$. We also have
$$\ell_2\sigma_2[r_1\sigma_1]_p = \ell_2\sigma_2[\sigma_2'(x)]_{q_1} \to^* \ell_2\sigma_2' \to r_2\sigma_2'$$
Consequently, $t \to^* s[r_2\sigma_2']_{p_2} \;{}^*\!\leftarrow u$. Hence, (1) is concluded. □

An easy consequence of the above lemma is that for every peak $t \;{}_{\mathcal{R}}\!\leftarrow s \to_{\mathcal{R}} u$ we have $t \downarrow_{\mathcal{R}} u$ or $t \leftrightarrow_{\mathsf{CP}(\mathcal{R})} u$. It might be interesting to note that in our formalization of the above proof we do actually not need the fact that left-hand sides of rules are not variables.

Definition 2.13 (Prime Critical Peaks and Pairs ☑)**.** A critical peak $t \xleftarrow{p} s \xrightarrow{\epsilon} u$ is *prime* if all proper subterms of $s|_p$ are normal forms. A critical pair is called prime if it is derived from a prime critical peak. We write $\mathsf{PCP}(\mathcal{R})$ to denote the set of all prime critical pairs of a TRS $\mathcal{R}$.

Definition 2.14. Given a TRS $\mathcal{R}$ and terms s, t, and u, we write $t \mathrel{\triangledown_s} u$ if $s \to_{\mathcal{R}}^{+} t$, $s \to_{\mathcal{R}}^{+} u$, and $t \downarrow_{\mathcal{R}} u$ or $t \leftrightarrow_{\mathsf{PCP}(\mathcal{R})} u$.

Lemma 2.15. *Let $\mathcal{R}$ be a TRS. If $t \xleftarrow{p} s \xrightarrow{\epsilon} u$ is a critical peak then $t \mathrel{\triangledown_s^2} u$.*

Proof. First suppose that all proper subterms of $s|_p$ are normal forms. Then $t \approx u \in \mathsf{PCP}(\mathcal{R})$ and thus $t \mathrel{\triangledown_s} u$. Since also $u \mathrel{\triangledown_s} u$, we obtain the desired $t \mathrel{\triangledown_s^2} u$. This leaves us with the case that there is a proper subterm of $s|_p$ that is not a normal form. By considering an innermost redex in $s|_p$ we obtain a position $q > p$ and a term v such that $s \xrightarrow{q} v$ and all proper subterms of $s|_q$ are normal forms. Now, if $v \xleftarrow{q} s \xrightarrow{\epsilon} u$ is an instance of a critical peak then $v \to_{\mathsf{PCP}(\mathcal{R})} u$. Otherwise, $v \downarrow_{\mathcal{R}} u$ by Lemma 2.12, since $q \nleqslant \epsilon$. In both cases we obtain $v \mathrel{\triangledown_s} u$. Finally, we analyze the peak $t \xleftarrow{p} s \xrightarrow{q} v$ by another application of Lemma 2.12.

(1) If $t \downarrow_{\mathcal{R}} v$, we obtain $t \mathrel{\triangledown_s} v$ and thus $t \mathrel{\triangledown_s^2} u$, since also $v \mathrel{\triangledown_s} u$.
(2) Since $p < q$, only the case that $v|_p \xleftarrow{q \backslash p} s|_p \xrightarrow{\epsilon} t|_p$ is an instance of a critical peak remains. Moreover, all proper subterms of $s|_q$ are normal forms and thus we have an instance of a prime critical peak. Hence $t \leftrightarrow_{\mathsf{PCP}(\mathcal{R})} v$ and together with $v \mathrel{\triangledown_s} u$ we conclude $t \mathrel{\triangledown_s^2} u$. □

Lemma 2.16. *Let $\mathcal{R}$ be a TRS. If $t \mathrel{{}_{\mathcal{R}}{\leftarrow}} s \to_{\mathcal{R}} u$ then $t \mathrel{\triangledown_s^2} u$.*

Proof. From Lemma 2.12, either $t \downarrow_{\mathcal{R}} u$ and we are done, or $t \mathrel{{}_{\mathcal{R}}{\leftarrow}} s \to_{\mathcal{R}} u$ contains a (possibly reversed) instance of a critical peak. By Lemma 2.15 we conclude the proof, since rewriting is closed under substitutions and contexts. □

The following result is due to Kapur et al. [KMN88, Corollary 4].

Corollary 2.17. *A terminating TRS is confluent if and only if all its prime critical pairs are joinable.*

Proof. Let $\mathcal{R}$ be a terminating TRS such that $\mathsf{PCP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$. We claim that $\mathcal{R}$ is source decreasing. As well-founded order we take $> = \to_{\mathcal{R}}^{+}$. Consider an arbitrary peak $t \mathrel{{}_{\mathcal{R}}{\leftarrow}} s \to_{\mathcal{R}} u$. Lemma 2.16 yields a term v such that $t \mathrel{\triangledown_s} v \mathrel{\triangledown_s} u$. From the assumption $\mathsf{PCP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$ we obtain $t \downarrow_{\mathcal{R}} v \downarrow_{\mathcal{R}} u$. Since $s \to_{\mathcal{R}}^{+} v$, all steps in the conversion $t \downarrow_{\mathcal{R}} v \downarrow_{\mathcal{R}} u$ are labeled with a term that is smaller than s. Since the two steps in the peak receive the same label s, source decreasingness is established and hence we obtain the confluence of $\mathcal{R}$ from Lemma 2.5. The reverse direction is trivial. □

Note that unlike for ordinary critical pairs, joinability of prime critical pairs does not imply local confluence.

Example 2.18. Consider the TRS $\mathcal{R}$ given by the three rules:

$$\mathsf{f}(\mathsf{a}) \to \mathsf{b} \qquad\qquad \mathsf{f}(\mathsf{a}) \to \mathsf{c} \qquad\qquad \mathsf{a} \to \mathsf{a}$$

The set $\mathsf{PCP}(\mathcal{R})$ consists of the two pairs $\mathsf{f}(\mathsf{a}) \approx \mathsf{b}$ and $\mathsf{f}(\mathsf{a}) \approx \mathsf{c}$, which are trivially joinable. But $\mathcal{R}$ is not locally confluent because the peak $\mathsf{b} \mathrel{{}_{\mathcal{R}}{\leftarrow}} \mathsf{f}(\mathsf{a}) \to_{\mathcal{R}} \mathsf{c}$ is not joinable.

The critical pair lemma (Lemma 2.12) in this section is due to Knuth and Bendix [KB70] and Huet [Hue80]. The primality critical pair criterion was first presented by Kapur, Musser, and Narendran [KMN88]. Our presentation is based on the simpler correctness arguments from our earlier work [HMS14, HMSW17].

3. Correctness for Finite Runs

The original completion procedure by Knuth and Bendix [KB70] was presented as a concrete algorithm. Later on, Bachmair, Dershowitz, and Hsiang [BDH86] presented an inference system for completion and showed that all *fair* implementations thereof (in particular the original procedure) are correct. Abstracting from a concrete strategy, their approach thus has the advantage to cover a variety of implementations. Below, we recall the inference system, which constitutes the basis of the results presented in this section.

Definition 3.1 (Knuth-Bendix Completion ☑)**.** The inference system $\mathsf{KB_f}$ of abstract (Knuth-Bendix) completion operates on pairs $(\mathcal{E}, \mathcal{R})$ of sets of equations $\mathcal{E}$ and rules $\mathcal{R}$ over a common signature $\mathcal{F}$. It consists of the following inference rules, where we write $\mathcal{E}, \mathcal{R}$ for a pair $(\mathcal{E}, \mathcal{R})$ and $\uplus$ denotes disjoint set union:

deduce $\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$ if $s \mathrel{{}_{\mathcal{R}}{\leftarrow}} \cdot \rightarrow_{\mathcal{R}} t$

compose $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$ if $t \to_{\mathcal{R}} u$

orient $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$ if $s > t$ $\qquad \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}$ if $t > s$

simplify $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}}$ if $s \to_{\mathcal{R}} u$ $\qquad \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$

delete $\dfrac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

collapse $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \to_{\mathcal{R}} u$

Here $>$ is a fixed reduction order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

Definition 3.1 differs from most of the inference systems in the literature (like those devised by Bachmair and Dershowitz [Bac91, BD94]) in that we do not impose an encompassment condition on collapse. As long as we only consider *finite* runs (see Definition 3.5 below)—like in Sections 3 to 5—this change is valid (as shown by Sternagel and Thiemann [ST13]).

Concerning notation, we write $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{f}} (\mathcal{E}', \mathcal{R}')$ whenever we can obtain $(\mathcal{E}', \mathcal{R}')$ from $(\mathcal{E}, \mathcal{R})$ by applying one of the inference rules of Definition 3.1. While it is well-known that applying the inference rules of $\mathsf{KB_f}$ does not affect the equational theory induced by $\mathcal{E} \cup \mathcal{R}$, our formulation is new and paves the way for a simple correctness proof.

Lemma 3.2. *Suppose $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{f}} (\mathcal{E}', \mathcal{R}')$. Then, the following two inclusions hold:*

(1) If $s \xrightarrow[\mathcal{E} \cup \mathcal{R}]{} t$ then $s \xrightarrow[\mathcal{R}']{=} \cdot \xrightarrow[\mathcal{E}' \cup \mathcal{R}']{=} \cdot \xleftarrow[\mathcal{R}']{=} t$. ☑

(2) If $s \xrightarrow[\mathcal{E}' \cup \mathcal{R}']{} t$ then $s \xleftrightarrow[\mathcal{E} \cup \mathcal{R}]{*} t$. ☑

Proof. By inspecting the inference rules of $\mathsf{KB_f}$ we easily obtain the following inclusions:

deduce

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \xleftarrow[\mathcal{R}]{} \cdot \xrightarrow[\mathcal{R}]{}$$

orient

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup (\mathcal{R}')^{-1} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \mathcal{E}^{-1}$$

delete

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup = \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R}$$

compose

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup \xrightarrow[\mathcal{R}']{} \cdot \xleftarrow[\mathcal{R}']{} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \xrightarrow[\mathcal{R}]{} \cdot \xrightarrow[\mathcal{R}]{}$$

simplify

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup \xrightarrow[\mathcal{R}']{} \cdot \xrightarrow[\mathcal{E}']{} \cup \xrightarrow[\mathcal{E}']{} \cdot \xleftarrow[\mathcal{R}']{} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \xleftarrow[\mathcal{R}]{} \cdot \xrightarrow[\mathcal{E}]{} \cup \xrightarrow[\mathcal{E}]{} \cdot \xrightarrow[\mathcal{R}]{}$$

collapse

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup \xrightarrow[\mathcal{R}']{} \cdot \xrightarrow[\mathcal{E}']{} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \xleftarrow[\mathcal{R}]{} \cdot \xrightarrow[\mathcal{R}]{}$$

Consider for instance the collapse rule and suppose that $s \approx t \in \mathcal{E} \cup \mathcal{R}$. If $s \approx t \in \mathcal{E}$ then $s \approx t \in \mathcal{E}'$ because $\mathcal{E} \subseteq \mathcal{E}'$. If $s \approx t \in \mathcal{R}$ then either $s \approx t \in \mathcal{R}'$ or $s \to_{\mathcal{R}} u$ with $u \approx t \in \mathcal{E}'$ and thus $s \to_{\mathcal{R}'} \cdot \to_{\mathcal{E}'} t$. This proves the inclusion on the left. For the inclusion on the right the reasoning is similar. Suppose that $s \approx t \in \mathcal{E}' \cup \mathcal{R}'$. If $s \approx t \in \mathcal{R}'$ then $s \approx t \in \mathcal{R}$ because $\mathcal{R}' \subseteq \mathcal{R}$. If $s \approx t \in \mathcal{E}'$ then either $s \approx t \in \mathcal{E}$ or there exists a rule $u \to t \in \mathcal{R}$ with $u \to_{\mathcal{R}} s$ and thus $s\ {}_{\mathcal{R}}\!\leftarrow \cdot \to_{\mathcal{R}} t$.

Since rewrite relations are closed under contexts and substitutions, the inclusions in the right column prove statement (2). Moreover note that each inclusion in the left column is a special case of

$$\mathcal{E} \cup \mathcal{R} \subseteq \xrightarrow[\mathcal{R}']{=} \cdot \xrightarrow[\mathcal{E}' \cup \mathcal{R}']{=} \cdot \xleftarrow[\mathcal{R}']{=}$$

and thus also statement (1) follows from closure under contexts and substitutions of rewrite relations. □

Corollary 3.3. *If $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{f}}^* (\mathcal{E}', \mathcal{R}')$ then the relations $\xleftrightarrow[\mathcal{E} \cup \mathcal{R}]{*}$ and $\xleftrightarrow[\mathcal{E}' \cup \mathcal{R}']{*}$ coincide.*

The next lemma states that termination of $\mathcal{R}$ is preserved by applications of the inference rules of $\mathsf{KB_f}$. It is the final result in this section whose proof refers to the inference rules.

Lemma 3.4. *If $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{f}}^* (\mathcal{E}', \mathcal{R}')$ and $\mathcal{R} \subseteq >$ then $\mathcal{R}' \subseteq >$.*

Proof. We consider a single step $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{f}} (\mathcal{E}', \mathcal{R}')$. The statement of the lemma follows by a straightforward induction proof. Observe that deduce, delete, and simplify do not change the set of rewrite rules and hence $\mathcal{R}' = \mathcal{R} \subseteq >$. For collapse we have $\mathcal{R}' \subsetneq \mathcal{R} \subseteq >$. In the case of orient we have $\mathcal{R}' = \mathcal{R} \cup \{s \to t\}$ with $s > t$ and hence $\mathcal{R}' \subseteq >$ follows from the assumption $\mathcal{R} \subseteq >$. Finally, consider an application of compose. So $\mathcal{R} = \mathcal{R}'' \uplus \{s \to t\}$ and $\mathcal{R}' = \mathcal{R}'' \cup \{s \to u\}$ with $t \to_{\mathcal{R}} u$. We obtain $s > t$ from the assumption $\mathcal{R} \subseteq >$. Since $>$ is a reduction order, $t > u$ follows from $t \to_{\mathcal{R}} u$. Transitivity of $>$ yields $s > u$ and hence $\mathcal{R}' \subseteq >$ as desired. □

To guarantee that the result of a finite $\mathsf{KB_f}$ derivation is a complete TRS equivalent to the initial $\mathcal{E}$, $\mathsf{KB_f}$ derivations must satisfy the *fairness condition* that prime critical pairs of the final TRS $\mathcal{R}_n$ which were not considered during the derivation are joinable in $\mathcal{R}_n$.

Definition 3.5 (Finite Runs and Fairness)**.** A *finite run* for a given ES $\mathcal{E}$ is a finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \ \vdash_{\mathsf{f}} \ \mathcal{E}_1, \mathcal{R}_1 \ \vdash_{\mathsf{f}} \ \cdots \ \vdash_{\mathsf{f}} \ \mathcal{E}_n, \mathcal{R}_n$$

such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \varnothing$. The run is *fair* if $\mathcal{E}_n = \varnothing$ and

$$\mathsf{PCP}(\mathcal{R}_n) \;\subseteq\; \downarrow_{\mathcal{R}_n} \cup \bigcup_{i=0}^{n} \underset{\mathcal{E}_i}{\leftrightarrow}$$

The reason for writing $\leftrightarrow_{\mathcal{E}_i}$ instead of $\mathcal{E}_i$ in the definition of fairness is that critical pairs are ordered, so in a fair run a (prime) critical pair $s \approx t$ of $\mathcal{R}_n$ may be ignored by deduce if $t \approx s$ was generated, or more generally, if $s \leftrightarrow_{\mathcal{E}_i} t$ holds at some point in the run. Non-prime critical pairs can always be ignored. Note that our fairness condition differs from earlier notions by permitting that (prime) critical pairs may be joinable in $\mathcal{R}_n$. This was done to allow for more flexibility in implementations. Our proofs smoothly extend to the relaxed condition.

According to the main result of this section (Theorem 3.8), a completion procedure that produces fair runs is correct. The challenge is the confluence proof of $\mathcal{R}_n$. We show that $\mathcal{R}_n$ is peak decreasing by labeling rewrite steps (not only in $\mathcal{R}_n$) with multisets of terms. As well-founded order on these multisets we take the multiset extension $>_{\mathsf{mul}}$ of the given reduction order $>$.

Definition 3.6 (Labeled Rewriting ☑)**.** Let $\to$ be a rewrite relation and M a finite multiset of terms. We write $s \xrightarrow{M} t$ if $s \to t$ and there exist terms $s', t' \in M$ such that $s' \geqslant s$ and $t' \geqslant t$. Here $\geqslant$ denotes the reflexive closure of the given reduction order $>$.

Since both $\to$ and $\geqslant$ are closed under contexts and substitutions, we have $C[t\sigma] \xrightarrow{M'} C[u\sigma]$ whenever $t \xrightarrow{M} u$ and $M' = \{C[s\sigma] \mid s \in M\}$, for all contexts C and substitutions σ.

Lemma 3.7. *Let $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{f}} (\mathcal{E}', \mathcal{R}')$. If $t \xleftrightarrow[\mathcal{E} \cup \mathcal{R}]{M}{}^{*} u$ and $\mathcal{R}' \subseteq >$ then $t \xleftrightarrow[\mathcal{E}' \cup \mathcal{R}']{M}{}^{*} u$.* ☑

Proof. We consider a single $(\mathcal{E} \cup \mathcal{R})$-step from t to u. The lemma follows then by induction on the length of the conversion between t and u. According to Lemma 3.2(1) there exist terms v and w such that

$$t \xrightarrow[\mathcal{R}']{=} v \xrightarrow[\mathcal{E}' \cup \mathcal{R}']{=} w \xleftarrow[\mathcal{R}']{=} u$$

We claim that the (non-empty) steps can be labeled by M. There exist terms $t', u' \in M$ with $t' \geqslant t$ and $u' \geqslant u$. Since $\mathcal{R}' \subseteq >$, we have $t \geqslant v$ and $u \geqslant w$ and thus also $t' \geqslant v$ and $u' \geqslant w$. Hence

$$t \xrightarrow[\mathcal{R}']{M}{}^{=} v \xrightarrow[\mathcal{E}' \cup \mathcal{R}']{M}{}^{=} w \;{}^{=}\!\xleftarrow[\mathcal{R}']{M} u$$

and thus also $t \xleftrightarrow[\mathcal{E}' \cup \mathcal{R}']{M}{}^{*} u$. □

Theorem 3.8. *For every fair run Γ* ☑

$$\mathcal{E}_0, \mathcal{R}_0 \;\vdash_{\mathsf{f}}\; \mathcal{E}_1, \mathcal{R}_1 \;\vdash_{\mathsf{f}}\; \cdots \;\vdash_{\mathsf{f}}\; \mathcal{E}_n, \mathcal{R}_n$$

the TRS $\mathcal{R}_n$ is a complete presentation of $\mathcal{E}$.

Proof. We have $\mathcal{E}_n = \varnothing$. From Corollary 3.3 we know that $\leftrightarrow_{\mathcal{E}}^{*} = \leftrightarrow_{\mathcal{R}_n}^{*}$. Lemma 3.4 yields $\mathcal{R}_n \subseteq >$ and hence $\mathcal{R}_n$ is terminating. It remains to prove that $\mathcal{R}_n$ is confluent. Let

$$t \xleftarrow[\mathcal{R}_n]{M_1} s \xrightarrow[\mathcal{R}_n]{M_2} u$$

be a labeled local peak in $\mathcal{R}_n$. From Lemma 2.16 we obtain $t \triangledown_s^2 u$. Let $v \triangledown_s w$ appear in this sequence (so $t = v$ or $w = u$). We obtain

$$(v, w) \in {\downarrow_{\mathcal{R}_n}} \cup \bigcup_{i=0}^{n} \underset{\mathcal{E}_i}{\leftrightarrow}$$

from the definition of $\triangledown_s$ and fairness of Γ. We label all steps between v and w with the multiset $\{v, w\}$. Because $s > v$ and $s > w$ we have $M_1 >_{\mathsf{mul}} \{v, w\}$ and $M_2 >_{\mathsf{mul}} \{v, w\}$. Hence by repeated applications of Lemma 3.7 we obtain a conversion in $\mathcal{R}_n$ between v and w in which each step is labeled with a multiset that is smaller than both M_1 and M_2. It follows that $\mathcal{R}_n$ is peak decreasing and thus confluent by Lemma 2.5. □

A completion procedure is a program that generates $\mathsf{KB_f}$ runs. In order to ensure that the final outcome $\mathcal{R}_n$ is a complete presentation of the initial ES, fair runs should be produced. Fairness requires that prime critical pairs of $\mathcal{R}_n$ are considered during the run. Of course, $\mathcal{R}_n$ is not known during the run, so to be on the safe side, prime critical pairs of any $\mathcal{R}$ that appears during the run should be generated by deduce. In particular, there is no need to deduce equations that are not prime critical pairs. So we may strengthen the condition $s\ {}_{\mathcal{R}}\!\leftarrow \cdot \rightarrow_{\mathcal{R}} t$ of deduce to $s \approx t \in \mathsf{PCP}(\mathcal{R})$ without affecting Theorem 3.8.

The following example shows that the success of a run may depend on the order in which inference rules are applied [BDP89].

Example 3.9. Consider the ES $\mathcal{E}$ consisting of the four equations

$$\mathsf{a} \approx \mathsf{b} \qquad\qquad \mathsf{a} \approx \mathsf{c} \qquad\qquad \mathsf{f(b)} \approx \mathsf{b} \qquad\qquad \mathsf{f(a)} \approx \mathsf{d}$$

and the reduction order $>_{\mathsf{lpo}}$ with the partial precedence $\mathsf{a} > \mathsf{b} > \mathsf{d}$ and $\mathsf{a} > \mathsf{c} > \mathsf{d}$ but where b and c are incomparable. One possible run is

$$\begin{aligned}
(\mathcal{E}, \varnothing) &\vdash_{\mathsf{f}}^{\mathsf{orient}^+} (\{\mathsf{a} \approx \mathsf{c}, \mathsf{f(a)} \approx \mathsf{d}\}, \{\mathsf{a} \to \mathsf{b}, \mathsf{f(b)} \to \mathsf{b}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{simplify}^+} (\{\mathsf{b} \approx \mathsf{c}, \mathsf{f(b)} \approx \mathsf{d}\}, \{\mathsf{a} \to \mathsf{b}, \mathsf{f(b)} \to \mathsf{b}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{simplify}} (\{\mathsf{b} \approx \mathsf{c}, \mathsf{b} \approx \mathsf{d}\}, \{\mathsf{a} \to \mathsf{b}, \mathsf{f(b)} \to \mathsf{b}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{orient}} (\{\mathsf{b} \approx \mathsf{c}\}, \{\mathsf{a} \to \mathsf{b}, \mathsf{f(b)} \to \mathsf{b}, \mathsf{b} \to \mathsf{d}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{collapse}} (\{\mathsf{b} \approx \mathsf{c}, \mathsf{f(d)} \approx \mathsf{b}\}, \{\mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{d}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{simplify}^+} (\{\mathsf{d} \approx \mathsf{c}, \mathsf{f(d)} \approx \mathsf{d}\}, \{\mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{d}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{orient}^+} (\varnothing, \{\mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{d}, \mathsf{c} \to \mathsf{d}, \mathsf{f(d)} \to \mathsf{d}\})
\end{aligned}$$

which derives a complete presentation of $\mathcal{E}$. However, the run

$$\begin{aligned}
(\mathcal{E}, \varnothing) &\vdash_{\mathsf{f}}^{\mathsf{orient}} (\{\mathsf{a} \approx \mathsf{b}, \mathsf{f(b)} \approx \mathsf{b}, \mathsf{f(a)} \approx \mathsf{d}\}, \{\mathsf{a} \to \mathsf{c}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{simplify}^+} (\{\mathsf{c} \approx \mathsf{b}, \mathsf{f(b)} \approx \mathsf{b}, \mathsf{f(c)} \approx \mathsf{d}\}, \{\mathsf{a} \to \mathsf{c}\}) \\
&\vdash_{\mathsf{f}}^{\mathsf{orient}^+} (\{\mathsf{c} \approx \mathsf{b}\}, \{\mathsf{a} \to \mathsf{c}, \mathsf{f(b)} \to \mathsf{b}, \mathsf{f(c)} \to \mathsf{d}\})
\end{aligned}$$

cannot be extended to a successful one because the equation $\mathsf{c} \approx \mathsf{b}$ cannot be oriented.

The following example shows that even after a $\mathsf{KB_f}$ run derived a complete system, exponentially many steps might be performed to obtain a canonical TRS.

Example 3.10. Consider the ESs $\mathcal{E}_n = \{\mathsf{f}(\mathsf{g}^i(\mathsf{c})) \approx \mathsf{g}(\mathsf{f}^i(\mathsf{c})) \mid 0 \leqslant i \leqslant n\}$ for $n \geqslant 1$. By taking the Knuth-Bendix order $>_{\mathsf{kbo}}$ with precedence $\mathsf{f} > \mathsf{g}$ and where $w(\mathsf{f}) = w(\mathsf{g})$, all equations can be oriented from left to right. Since there are no critical pairs, the resulting TRSs $\mathcal{R}_n = \{\mathsf{f}(\mathsf{g}^i(\mathsf{c})) \to \mathsf{g}(\mathsf{f}^i(\mathsf{c})) \mid 0 \leqslant i \leqslant n\}$ are complete by Theorem 3.8. However, it is not canonical since right-hand sides are not normal forms. When applying compose steps in a naive way by simplifying the rules in descending order, exponentially many steps are required to obtain a canonical system [PSK96]. However, when processing the rules in reverse order only a polynomial number of steps is necessary.

This section resumes our results on finite runs [HMS14]. The presented correctness proof differs substantially from all earlier proofs in that it does not rely on a proof order [BDH86] but is instead based on peak decreasingness. It supports a relaxed side condition of the collapse rule as first used in [ST13], but in contrast to the latter demands only prime critical pairs to be considered.

4. Canonicity and Normalization Equivalence

A natural question arising in the context of completion concerns uniqueness of resulting systems: Is there a single complete presentation of a given equational theory conforming to a certain reduction order? Métivier [Mét83] showed that for reduced and hence canonical systems this is indeed the case, up to renaming variables. In this section we revisit his work, aiming at generalizing his uniqueness result for canonical TRSs and at establishing a transformation to simplify ground-complete TRSs. A key notion to that end is normalization equivalence.

Definition 4.1 (Conversion/Normalization Equivalence)**.** Two ARSs $\mathcal{A}$ and $\mathcal{B}$ are said to be *(conversion) equivalent* if $\leftrightarrow^*_{\mathcal{A}} = \leftrightarrow^*_{\mathcal{B}}$ and *normalization equivalent* if $\to^!_{\mathcal{A}} = \to^!_{\mathcal{B}}$.

The following example shows that these two equivalence notions do not coincide.

Example 4.2. Consider the four ARSs:

$$\mathcal{A}_1 : \quad \mathsf{a} \longrightarrow \mathsf{b} \qquad\qquad \mathcal{B}_1 : \quad \mathsf{a} \longleftarrow \mathsf{b}$$

$$\begin{tikzcd} \mathcal{A}_2 : & \mathsf{a} \arrow[r] & \mathsf{b} \arrow[loop below] & & \mathcal{B}_2 : & \mathsf{a} \arrow[loop below] & \mathsf{b} \arrow[loop below] \end{tikzcd}$$

While $\mathcal{A}_1$ and $\mathcal{B}_1$ are conversion equivalent but not normalization equivalent, the ARSs $\mathcal{A}_2$ and $\mathcal{B}_2$ are normalization equivalent but not conversion equivalent.

The easy proof (by induction on the length of conversions) of the following result is omitted.

Lemma 4.3. *Normalization equivalent terminating ARSs are equivalent.*

Note that the termination assumption can be weakened to weak normalization. However, the present version suffices to prove the following lemma that we employ in our proof of Métivier's transformation result [Mét83] (Theorem 4.7 below).

Lemma 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be ARSs such that $\mathsf{NF}(\mathcal{B}) \subseteq \mathsf{NF}(\mathcal{A})$ and either*
(1) $\to_{\mathcal{B}} \subseteq \to^+_{\mathcal{A}}$ or

(2) $\to_{\mathcal{B}} \subseteq \leftrightarrow^*_{\mathcal{A}}$ *and* $\mathcal{B}$ *is terminating.*

If $\mathcal{A}$ *is complete then* $\mathcal{B}$ *is complete and normalization equivalent to* $\mathcal{A}$.

Proof. We first show $\to^!_{\mathcal{B}} \subseteq \to^!_{\mathcal{A}}$. In case (1), from the inclusion $\to_{\mathcal{B}} \subseteq \to^+_{\mathcal{A}}$ we infer that $\mathcal{B}$ is terminating. Moreover, $\to^*_{\mathcal{B}} \subseteq \to^*_{\mathcal{A}}$ and, since $\mathsf{NF}(\mathcal{B}) \subseteq \mathsf{NF}(\mathcal{A})$, also $\to^!_{\mathcal{B}} \subseteq \to^!_{\mathcal{A}}$. For case (2), $\to^!_{\mathcal{B}} \subseteq \to^!_{\mathcal{A}}$ holds because $\to^!_{\mathcal{B}} \subseteq \leftrightarrow^*_{\mathcal{A}}$, so by confluence of $\mathcal{A}$ and $\mathsf{NF}(\mathcal{B}) \subseteq \mathsf{NF}(\mathcal{A})$ we obtain $\to^!_{\mathcal{B}} \subseteq \to^!_{\mathcal{A}}$. Next we show that the reverse inclusion $\to^!_{\mathcal{A}} \subseteq \to^!_{\mathcal{B}}$ holds in both cases. Let $a \to^!_{\mathcal{A}} b$. Because $\mathcal{B}$ is terminating, $a \to^!_{\mathcal{B}} c$ for some $c \in \mathsf{NF}(\mathcal{B})$. So $a \to^!_{\mathcal{A}} c$ and thus $b = c$ from the confluence of $\mathcal{A}$. It follows that $\mathcal{A}$ and $\mathcal{B}$ are normalization equivalent. It remains to show that $\mathcal{B}$ is locally confluent. This follows from the sequence of inclusions

$$ {}_{\mathcal{B}}\!\leftarrow \cdot \to_{\mathcal{B}} \;\subseteq\; \leftrightarrow^*_{\mathcal{A}} \;\subseteq\; \to^!_{\mathcal{A}} \cdot {}^{\,!}_{\mathcal{A}}\!\leftarrow \;\subseteq\; \to^!_{\mathcal{B}} \cdot {}^{\,!}_{\mathcal{B}}\!\leftarrow $$

where we obtain the inclusions from $\to_{\mathcal{B}} \subseteq \leftrightarrow^*_{\mathcal{A}}$, confluence of $\mathcal{A}$, termination of $\mathcal{A}$, and normalization equivalence of $\mathcal{A}$ and $\mathcal{B}$, respectively. ☐

In the above lemma, completeness can be weakened to semi-completeness (that is, the combination of confluence and weak normalization), which is not true for Theorem 4.7 as shown by Gramlich [Gra01]. Again, the present version suffices for our purposes. Condition (2) of the lemma can be regarded as a specialization of an abstract result of Toyama [Toy91, Corollary 3.2] to complete systems and will be used in Section 7.

Theorem 4.7 below shows that we can always eliminate redundancy in a complete TRS. This is achieved by the following two-stage transformation, where, given a TRS $\mathcal{R}$, we write $\mathcal{R}_{\doteq}$ for a set of representatives of the equivalence classes of rules in $\mathcal{R}$ with respect to $\doteq$ (that is, $\mathcal{R}_{\doteq}$ is a variant-free version of $\mathcal{R}$).

Definition 4.5. Given a terminating TRS $\mathcal{R}$, the TRSs $\dot{\mathcal{R}}$ and $\ddot{\mathcal{R}}$ are defined as follows:

$$\dot{\mathcal{R}} = \{\ell \to r{\downarrow_{\mathcal{R}}} \mid \ell \to r \in \mathcal{R}\}_{\doteq}$$

$$\ddot{\mathcal{R}} = \{\ell \to r \in \dot{\mathcal{R}} \mid \ell \in \mathsf{NF}(\dot{\mathcal{R}} \setminus \{\ell \to r\})\}$$

Here $t{\downarrow_{\mathcal{R}}}$ stands for an arbitrary but fixed normal form of t.

The TRS $\dot{\mathcal{R}}$ is obtained from $\mathcal{R}$ by first normalizing the right-hand sides and then taking representatives of variants of the resulting rules, thereby making sure that the result does not contain several variants of the same rule. To obtain $\ddot{\mathcal{R}}$ we remove the rules of $\dot{\mathcal{R}}$ whose left-hand sides are reducible with another rule of $\dot{\mathcal{R}}$. (This is the only place in the paper where variant-freeness of TRSs is important.)

The following example shows why the result of $\dot{\mathcal{R}}$ has to be variant-free.

Example 4.6. Consider the TRS $\mathcal{R}$ consisting of the four rules

$$\mathsf{f}(x) \to \mathsf{a} \qquad \mathsf{f}(y) \to \mathsf{b} \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

Then the first transformation without taking representatives of rules would yield $\dot{\mathcal{R}}$

$$\mathsf{f}(x) \to \mathsf{c} \qquad \mathsf{f}(y) \to \mathsf{c} \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

and the second one $\ddot{\mathcal{R}}$

$$\mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

Note that $\ddot{\mathcal{R}}$ is *not* equivalent to $\mathcal{R}$. This is caused by the fact that the result of the first transformation was no longer variant-free.

The following result, due to Métivier [Mét83, Theorem 7], allows us to obtain a canonical representation of any complete TRS.[3] Our proof below proceeds by induction on the well-founded encompassment order $\rhd$.

Theorem 4.7. *If $\mathcal{R}$ is a complete TRS then $\ddot{\mathcal{R}}$ is a normalization and conversion equivalent canonical TRS.*

Proof. Let $\mathcal{R}$ be a complete TRS. The inclusions $\ddot{\mathcal{R}} \subseteq \dot{\mathcal{R}} \subseteq \to_{\mathcal{R}}^{+}$ are obvious from the definitions. Since $\mathcal{R}$ and $\dot{\mathcal{R}}$ have the same left-hand sides, their normal forms coincide. We show that $\mathsf{NF}(\ddot{\mathcal{R}}) \subseteq \mathsf{NF}(\dot{\mathcal{R}})$. To this end we show that $\ell \notin \mathsf{NF}(\ddot{\mathcal{R}})$ whenever $\ell \to r \in \dot{\mathcal{R}}$ by induction on ℓ with respect to the well-founded order $\rhd$. If $\ell \to r \in \ddot{\mathcal{R}}$ then $\ell \notin \mathsf{NF}(\ddot{\mathcal{R}})$ holds. So suppose $\ell \to r \notin \ddot{\mathcal{R}}$. By definition of $\ddot{\mathcal{R}}$, $\ell \notin \mathsf{NF}(\dot{\mathcal{R}} \setminus \{\ell \to r\})$. So there exists a rewrite rule $\ell' \to r' \in \dot{\mathcal{R}}$ different from $\ell \to r$ such that $\ell \trianglerighteq \ell'$. We distinguish two cases.

- If $\ell \rhd \ell'$ then we obtain $\ell' \notin \mathsf{NF}(\ddot{\mathcal{R}})$ from the induction hypothesis and hence $\ell \notin \mathsf{NF}(\ddot{\mathcal{R}})$ as desired.
- If $\ell \doteq \ell'$ then by Lemma 2.1 there exists a renaming σ such that $\ell = \ell'\sigma$. Since $\dot{\mathcal{R}}$ is right-reduced by construction, r and r' are normal forms of $\dot{\mathcal{R}}$. The same holds for $r'\sigma$ because normal forms are closed under renaming. We have $r \,{}_{\dot{\mathcal{R}}}{\leftarrow}\, \ell = \ell'\sigma \to_{\dot{\mathcal{R}}} r'\sigma$. Since $\dot{\mathcal{R}}$ is confluent as a consequence of Lemma 4.4(1), $r = r'\sigma$. Hence $\ell' \to r'$ is a variant of $\ell \to r$, contradicting the fact that $\dot{\mathcal{R}}$ is variant-free (by construction).

From Lemma 4.4(1) we infer that the TRSs $\dot{\mathcal{R}}$ and $\ddot{\mathcal{R}}$ are complete and normalization equivalent to $\mathcal{R}$. The TRS $\ddot{\mathcal{R}}$ is right-reduced because $\ddot{\mathcal{R}} \subseteq \dot{\mathcal{R}}$ and $\dot{\mathcal{R}}$ is right-reduced. From $\mathsf{NF}(\ddot{\mathcal{R}}) = \mathsf{NF}(\dot{\mathcal{R}})$ we easily infer that $\ddot{\mathcal{R}}$ is left-reduced. It follows that $\ddot{\mathcal{R}}$ is canonical. It remains to show that $\ddot{\mathcal{R}}$ is not only normalization equivalent but also (conversion) equivalent to $\mathcal{R}$. This is an immediate consequence of Lemma 4.3. ☐

Before we proceed to show uniqueness of normalization equivalent TRSs, we need the following technical lemma.

Lemma 4.8. *Let $\mathcal{R}$ be a right-reduced TRS and let s be a reducible term which is minimal with respect to $\rhd$. If $s \to_{\mathcal{R}}^{+} t$ then $s \to t$ is a variant of a rule in $\mathcal{R}$.*

Proof. Let $\ell \to r$ be the rewrite rule that is used in the first step from s to t. So $s \trianglerighteq \ell$. By assumption, $s \rhd \ell$ does not hold and thus $s \doteq \ell$ because $\trianglerighteq \,=\, \rhd \cup \doteq$. According to Lemma 2.1 there exists a renaming σ such that $s = \ell\sigma$. We have $s \to_{\mathcal{R}} r\sigma \to_{\mathcal{R}}^{*} t$. Because $\mathcal{R}$ is right-reduced, $r \in \mathsf{NF}(\mathcal{R})$. Since normal forms are closed under renaming, also $r\sigma \in \mathsf{NF}(\mathcal{R})$ and thus $r\sigma = t$. It follows that $s \to t$ is a variant of $\ell \to r$. ☐

In our formalization, the above proof is the first spot of this section where we actually need that $\mathcal{R}$ satisfies the variable condition (more precisely, only the part of it that right-hand sides of rules do not introduce fresh variables). We are now in a position to present the main result of this section.

Theorem 4.9. *Normalization equivalent reduced TRSs are unique up to literal similarity.*

[3]We were not able to reconstruct enough detail for an Isabelle/HOL formalization from its original proof. Another textbook proof [Ter03, Exercise 7.4.7] involves 13 steps with lots of redundancy.

Proof. Let $\mathcal{R}$ and $\mathcal{S}$ be normalization equivalent reduced TRSs. Suppose $\ell \to r \in \mathcal{R}$. Because $\mathcal{R}$ is right-reduced, $r \in \mathsf{NF}(\mathcal{R})$ and thus $\ell \neq r$. Hence $\ell \to_{\mathcal{S}}^{+} r$ by normalization equivalence. Because $\mathcal{R}$ is left-reduced, ℓ is a minimal (with respect to $\rhd$) $\mathcal{R}$-reducible term. Another application of normalization equivalence yields that ℓ is minimal $\mathcal{S}$-reducible. Hence $\ell \to r$ is a variant of a rule in $\mathcal{S}$ by Lemma 4.8. $\square$

Example 4.10. Consider the rewrite system $\mathcal{R}$ of combinatory logic with equality test, studied by Klop [Klo80]:

$$@(@(@(\mathsf{S},x),y),z) \to @(x,@(z,@(y,z))) \qquad @(@(\mathsf{K},x),y) \to x$$
$$@(\mathsf{I},x) \to x \qquad @(@(\mathsf{D},x),x) \to \mathsf{E}$$

The rewrite system $\mathcal{R}$ is reduced, but neither terminating nor confluent. One might ask whether there is another reduced rewrite system that computes the same normal forms for every starting term. Theorem 4.9 shows that $\mathcal{R}$ is unique up to variable renaming.

As the final result of this section, we prove this result of Métivier [Mét83, Theorem 8] to be an easy consequence of Theorem 4.9. Here a TRS $\mathcal{R}$ is said to be compatible with a reduction order $>$ if $\ell > r$ for every rewrite rule $\ell \to r$ of $\mathcal{R}$.

Theorem 4.11. *Let $\mathcal{R}$ and $\mathcal{S}$ be equivalent canonical TRSs. If $\mathcal{R}$ and $\mathcal{S}$ are compatible with the same reduction order then $\mathcal{R} \doteq \mathcal{S}$.*

Proof. Suppose $\mathcal{R}$ and $\mathcal{S}$ are compatible with the reduction order $>$. We show that $\to_{\mathcal{R}}^{!} \subseteq \to_{\mathcal{S}}^{!}$. Let $s \to_{\mathcal{R}}^{!} t$. We show that $t \in \mathsf{NF}(\mathcal{S})$. Let u be the unique $\mathcal{S}$-normal form of t. We have $t \to_{\mathcal{S}}^{!} u$ and thus $t \leftrightarrow_{\mathcal{R}}^{*} u$ because $\mathcal{R}$ and $\mathcal{S}$ are equivalent. Since $t \in \mathsf{NF}(\mathcal{R})$, we have $u \to_{\mathcal{R}}^{!} t$. If $t \neq u$ then both $t > u$ (as $t \to_{\mathcal{S}}^{!} u$) and $u > t$ (as $u \to_{\mathcal{R}}^{!} t$), which is impossible. Hence $t = u$ and thus $t \in \mathsf{NF}(\mathcal{S})$. Together with $s \leftrightarrow_{\mathcal{S}}^{*} t$, which follows from the equivalence of $\mathcal{R}$ and $\mathcal{S}$, we conclude that $s \to_{\mathcal{S}}^{!} t$. We obtain $\to_{\mathcal{S}}^{!} \subseteq \to_{\mathcal{R}}^{!}$ by symmetry. Hence $\mathcal{R}$ and $\mathcal{S}$ are normalization equivalent and the result follows from Theorem 4.9. $\square$

This section resumes our results on canonicity [HMSW17]. While the results of Theorem 4.7 and Theorem 4.11 are due to Métivier [Mét83], we present novel and simpler proofs based on the (new) auxiliary results Lemma 4.4 and Theorem 4.9.

5. Ground Completion

In this section we focus on the special case of ground equations, that is, equations where both sides are ground terms.

Definition 5.1 (Ground Completion). The inference system $\mathsf{KB_g}$ consists of the inference rules of $\mathsf{KB_f}$ except for deduce.

Snyder [Sny93] proved that sets of ground equations can always be completed by $\mathsf{KB_g}$, provided a *ground-total* reduction order $>$ is used, that is, for all ground terms $s, t \in \mathcal{T}(\mathcal{F})$ either $s > t$, $t > s$, or $s = t$. He further proved that every reduced ground rewrite system is canonical and can be obtained by completion from any equivalent set of ground equations. Below, we present the proofs of these results that we formalized in Isabelle/HOL.

The following example illustrates the inference system $\mathsf{KB_g}$ on a set of ground equations.

Example 5.2. Consider the ES $\mathcal{E}$ consisting of the ground equations

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \approx \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \approx \mathsf{a} \qquad \mathsf{f}(\mathsf{a}) \approx \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

As reduction order we take LPO induced by the total precedence $\mathsf{a} > \mathsf{b} > \mathsf{c} > \mathsf{f}$. We start by applying orient to the last two equations:

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \approx \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \leftarrow \mathsf{a} \qquad \mathsf{f}(\mathsf{a}) \rightarrow \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

An application of collapse produces

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \approx \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \leftarrow \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{c})) \approx \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

Next we orient the second equation:

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \rightarrow \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \leftarrow \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{c})) \approx \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

Two applications of simplify produce

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{c})))) \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \rightarrow \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \leftarrow \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{c})) \approx \mathsf{c}$$

We continue by orienting the last equation:

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{c})))) \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \rightarrow \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \leftarrow \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{c})) \rightarrow \mathsf{c}$$

Two applications of simplify produce

$$\mathsf{c} \approx \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{f}(\mathsf{b})) \rightarrow \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \leftarrow \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{c})) \rightarrow \mathsf{c}$$

Orienting the remaining equation followed by a collapse step produces

$$\mathsf{c} \leftarrow \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{c}) \approx \mathsf{c} \qquad \mathsf{f}(\mathsf{c}) \leftarrow \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{c})) \rightarrow \mathsf{c}$$

Finally, we orient the only remaining equation and collapse, compose, simplify, and delete exhaustively, thereby obtaining the TRS $\mathcal{R}$

$$\mathsf{c} \leftarrow \mathsf{f}(\mathsf{b}) \qquad \mathsf{f}(\mathsf{c}) \rightarrow \mathsf{c} \qquad \mathsf{c} \leftarrow \mathsf{a}$$

which constitutes a canonical presentation of $\mathcal{E}$.

The absence of deduce from $\mathsf{KB_g}$ does not hurt for ground systems. If $s \leftarrow \cdot \rightarrow t$ and the two contracted redexes are at parallel positions then trivially $s \rightarrow \cdot \leftarrow t$. If the steps are identical then $s = t$. In the remaining case one of the contracted redexes is a subterm of the other contracted redex, and the effect of deduce is achieved by the collapse inference rule. On the contrary, the absence of deduce is crucial to conclude that $\mathsf{KB_g}$ derivations are always finite, as illustrated by the following simple example.

Example 5.3. Consider the ground ES $\mathcal{E}$ consisting of the single equation $\mathsf{a} \approx \mathsf{b}$ and LPO induced by the precedence $\mathsf{a} > \mathsf{b}$. Using $\mathsf{KB_f}$ (i.e., $\mathsf{KB_g}$ with deduce) the following infinite run is possible:

$$\begin{aligned}(\mathcal{E}, \varnothing) &\vdash_{\mathsf{f}}^{\mathsf{orient}} (\varnothing, \{\mathsf{a} \rightarrow \mathsf{b}\}) \\ &\vdash_{\mathsf{f}}^{\mathsf{deduce}} (\{\mathsf{b} \approx \mathsf{b}\}, \{\mathsf{a} \rightarrow \mathsf{b}\}) \\ &\vdash_{\mathsf{f}}^{\mathsf{delete}} (\varnothing, \{\mathsf{a} \rightarrow \mathsf{b}\}) \\ &\vdash_{\mathsf{f}}^{\mathsf{deduce}} \ldots\end{aligned}$$

Lemma 5.4. *There are no infinite runs $\mathcal{E}_0, \varnothing \vdash_{\mathsf{g}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathsf{g}} \cdots$ for finite ground ES $\mathcal{E}_0$.* ☑

Proof. Let $\succ$ denote the lexicographic combination of the multiset extension $>_{\mathsf{mul}}$ of the reduction order $>$ with the standard order on natural numbers $>_{\mathbb{N}}$. Furthermore let $M(\mathcal{E},\mathcal{R})$ denote the (finite) multiset of left-hand sides and right-hand sides occurring in $\mathcal{E}$ and $\mathcal{R}$

$$M(\mathcal{E},\mathcal{R}) = \bigcup\{\{s,t\} \mid (s,t) \in \mathcal{E}\} \ \cup \ \bigcup\{\{s,t\} \mid (s,t) \in \mathcal{R}\}$$

and consider the function P that maps the pair $(\mathcal{E},\mathcal{R})$ to $(M(\mathcal{E},\mathcal{R}), |\mathcal{E}|)$. Now it is straightforward to verify that any infinite $\vdash_{\mathsf{g}}$-sequence would give rise to an infinite sequence $P(\mathcal{E}_0,\varnothing) \succ P(\mathcal{E}_1,\mathcal{R}_1) \succ \cdots$, contradicting the well-foundedness of $\succ$. □

Theorem 5.5. *If $>$ is total on $\mathcal{E}$-equivalent ground terms then every maximal* $\mathsf{KB_g}$ *run produces an equivalent canonical presentation for every ground ES $\mathcal{E}$.*

Proof. Consider a maximal $\mathsf{KB_g}$ run $\mathcal{E}_0,\varnothing \vdash_{\mathsf{g}} \mathcal{E}_1,\mathcal{R}_1 \vdash_{\mathsf{g}} \cdots \vdash_{\mathsf{g}} \mathcal{E}_n,\mathcal{R}_n$ where $\mathcal{E}_0 = \mathcal{E}$ is a ground ES. Because the run is maximal, no inference rule of $\mathsf{KB_g}$ is applicable to the final pair $(\mathcal{E}_n,\mathcal{R}_n)$. In particular, compose and collapse are not applicable and hence the final TRS $\mathcal{R}_n$ is reduced. Since $\mathcal{R}_n$ is ground, this means in particular that there are no critical pairs. Moreover, termination of $\mathcal{R}_n$ follows from Lemma 3.4 (since any $\mathsf{KB_g}$ run is also a $\mathsf{KB_f}$ run), so $\mathcal{R}_n$ is canonical. From Corollary 3.3 and the inclusion $\mathsf{KB_g} \subseteq \mathsf{KB_f}$ we infer that $\mathcal{E}$ and $\mathcal{E}_n \cup \mathcal{R}_n$ are equivalent. It follows that $>$ is total on $\mathcal{E}_n$-equivalent ground terms and thus $\mathcal{E}_n = \varnothing$, for otherwise the run could be extended with an application of delete or orient. Hence $\mathcal{R}_n$ and $\mathcal{E}$ are equivalent. □

The restriction on the reduction order $>$ in the above correctness theorem is easy to satisfy. In particular, it holds for any LPO or KBO based on a total precedence.

Next we consider completeness of ground completion. Our proof makes use of the following concept.

Definition 5.6 (Random Descent ☑)**.** An ARS $\mathcal{A}$ has *random descent* if for every conversion $a \leftrightarrow^* b$ with normal form b we have $a \to^n b$ with $n + l = r$. Here l (r) denotes the number of $\leftarrow$ $(\to)$ steps in the conversion $a \leftrightarrow^* b$.

Random descent is useful in the analysis of rewrite strategies [vOT16]. It generalizes a number of earlier concepts, including the property $\leftarrow \cdot \to \subseteq (\to \cdot \leftarrow) \cup =$ which is known as WCR1 and holds for left-reduced ground TRSs. We formalized a new, short and direct proof of the following result due to van Oostrom [vO07]. Here an element a is said to be complete if it is both terminating (there are no infinite rewrite sequences starting at a) and confluent (if $b \mathrel{{}^*{\leftarrow}} a \to^* c$ then $b \downarrow c$).

Theorem 5.7. *Let $\mathcal{A}$ be an ARS with random descent. If $a \leftrightarrow^* b$ with normal form b then a is complete and all rewrite sequences from a to b have the same length.*

Proof. Let l (r) be the number of $\leftarrow$ $(\to)$ steps in the conversion from a to b. We have $l \leqslant r$ since $n + l = r$ for some n by random descent. First we prove termination of a. For a proof by contradiction, suppose the existence of an infinite rewrite sequence

$$a = a_0 \to a_1 \to a_2 \to \cdots$$

Clearly, $a \to^{r-l} a_{r-l}$ and thus there exists a conversion $a_{r-l} \mathrel{{}^*{\leftarrow}} a \leftrightarrow^* b$ with r backwards and r forwards steps. Hence $a_{r-l} = b$ by another application of random descent and therefore $b \to a_{r-l+1}$, contradicting the fact that b is a normal form. Next we prove confluence of a. Suppose $c \mathrel{{}^*{\leftarrow}} a \to^* d$. We obtain the two conversions $c \leftrightarrow^* b$ and $d \leftrightarrow^* b$, which are transformed into $c \downarrow d$ by two applications of random descent. Finally, assume there are two rewrite sequences $a \to^m b$ and $a \to^n b$ from a to b of length m and n. Reversing the first sequence and appending the second one yields a conversion $b \leftrightarrow^* b$ with m backwards and n forwards steps. A final application of random descent yields $b \to^k b$ for some k with $k + m = n$. Since b is a normal form, $k = 0$ and thus $m = n$ as desired. □

In the series of lemmas below, we establish that reduced ground TRSs are canonical and have random descent.

Lemma 5.8. *Left-reduced TRSs enjoy the* $\mathsf{WCR1}$ *property.*

Proof. This follows from a straightforward case analysis on the relative positions of the two redexes that are part of a peak together with the fact that for left-reduced TRSs the left-hand side alone uniquely determines the employed rewrite rule. □

Lemma 5.9. *Left-reduced ground TRSs have random descent.*

Proof. Let $\mathcal{R}$ be a left-reduced TRS and $s \leftrightarrow^* t$ a conversion between two arbitrary but fixed terms s and t such that t is a normal form. We proceed by induction on the length of this conversion. If it is empty or the first step is to the right, we are done. Otherwise, we have $s \leftarrow u \leftrightarrow^* t$ where the conversion between u and t has l (r) $\leftarrow$ ($\to$) steps and obtain $u \to^k t$ with $k + l = r$ by the induction hypothesis. The remainder of the proof proceeds by induction on k together with Lemma 5.8. □

Lemma 5.10. *Right-reduced ground TRSs are terminating.*

Proof. Let $\mathcal{R}$ be a right-reduced ground TRS. For the sake of a contradiction, assume that $\mathcal{R}$ is non-terminating. Then there is a minimal non-terminating term t (that is, all its proper subterms are terminating). This means that after a finite number of non-root steps $t \to^* u$ there will be a root step $u \to v$ such that v is non-terminating. But since $\mathcal{R}$ is right-reduced and ground, v is a ground normal form, deriving the desired contradiction. □

Corollary 5.11. *Reduced ground TRSs are canonical and have random descent.*

Proof. Let $\mathcal{R}$ be a reduced ground TRS. Then, by Lemma 5.9, $\mathcal{R}$ has random descent. Moreover, by Lemma 5.10, $\mathcal{R}$ is terminating. Finally, since all terms are $\mathcal{R}$-terminating, confluence of $\mathcal{R}$ is an immediate consequence of the definition of random descent. □

Theorem 5.12. *For every ground ES* $\mathcal{E}$ *and every equivalent reduced ground TRS* $\mathcal{R}$ *there exist a reduction order* $>$ *and a derivation* $\mathcal{E}, \varnothing \vdash_{\mathsf{g}} \cdots \vdash_{\mathsf{g}} \varnothing, \mathcal{R}$.

Proof. Let $>$ be a reduction order that contains $\mathcal{R}$ and is total on $\mathcal{E}$-equivalent ground terms. Consider a maximal $\mathsf{KB_g}$ run starting from $\mathcal{E}$ and using $>$. According to Theorem 5.5, the run produces an equivalent reduced TRS $\mathcal{R}'$. Since $\mathcal{R} \subseteq >$ and $\mathcal{R}' \subseteq >$, we obtain $\mathcal{R} = \mathcal{R}'$ from Theorem 4.11. It remains to show that $>$ exists. Let $\sqsupset$ be a total precedence and define $s > t$ if and only if $s \leftrightarrow^*_{\mathcal{E}} t$ and either $d_{\mathcal{R}}(s) > d_{\mathcal{R}}(t)$ or both $d_{\mathcal{R}}(s) = d_{\mathcal{R}}(t)$ and $s \sqsupset_{\mathsf{lpo}} t$.[4] Here $d_{\mathcal{R}}(u)$ is the number of rewrite steps in $\mathcal{R}$ to normalize the term u, which is well-defined since all normalizing sequences in a reduced ground TRS have the same length as a consequence of Corollary 5.11 and Theorem 5.7. It is easy to show that $>$ has the required properties. The only interesting cases are closure under contexts and substitutions. Both are basically handled by the following observation: $d_{\mathcal{R}}(C[t\sigma]) = d_{\mathcal{R}}(C[t{\downarrow}\sigma]) + d_{\mathcal{R}}(t)$ for any term t (which holds due to random descent together with termination). This allows us to lift $d_{\mathcal{R}}(s) = d_{\mathcal{R}}(t)$ and $d_{\mathcal{R}}(s) > d_{\mathcal{R}}(t)$ into arbitrary contexts and substitutions. □

[4] In the formalization we actually use $\sqsupset_{\mathsf{kbo}}$ with all weights set to 1, since in contrast to LPO, for KBO ground-totality for total precedences has already been formalized before.

The above result cannot be generalized to left-linear right-ground systems, as shown in the following example due to Dominik Klein (personal communication).

Example 5.13. Consider the ES $\mathcal{E}$ consisting of the two equations $\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b}) \approx \mathsf{b}$. Let $>$ be a reduction order. If $\mathsf{f}(\mathsf{b}) > \mathsf{b}$ does not hold, no inference rule of $\mathsf{KB_g}$ is applicable to $(\mathcal{E}, \varnothing)$. If $\mathsf{f}(\mathsf{b}) > \mathsf{b}$ then the second equation can be oriented

$$(\mathcal{E}, \varnothing) \ \vdash_{\mathsf{g}} \ (\{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a})\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}\})$$

but no further inference steps of $\mathsf{KB_g}$ are possible. Hence completion will fail on $\mathcal{E}$. Nevertheless, the TRS $\mathcal{R}$ consisting of the rewrite rule $\mathsf{f}(x) \to \mathsf{b}$ constitutes a canonical presentation of $\mathcal{E}$.

The correctness result of ground completion (Theorem 5.5) is due to Snyder [Sny93], and our formalized proof basically follows his approach. In addition, we present a new completeness proof based on random descent (Theorem 5.12).

6. Correctness for Infinite Runs

Completion as presented in the preceding sections does not always succeed in producing a finite complete presentation. It may fail because an unorientable equation is encountered or it may run forever. In the latter case it is possible that in the limit a possibly infinite complete presentation is obtained. In this case, completion can serve as a semi-decision procedure for the validity problem of the initial equations [Hue81]. In this section we give a new proof that fair infinite runs produce complete presentations of the initial equations, provided the collapse rule is restored to its original formulation (cf. Definition 6.2 below).

The reason why this restriction is necessary is provided by the following example (due to Baader and Nipkow [BN98]), which shows that the correctness result (Theorem 3.8) of Section 3 does not extend to infinite runs without further ado.

Example 6.1. Consider the ES $\mathcal{E}$ consisting of the equations

$$\mathsf{aba} \approx \mathsf{ab} \qquad\qquad \mathsf{bb} \approx \mathsf{b}$$

and LPO with precedence $\mathsf{a} > \mathsf{b}$ as reduction order. After two orient steps, we apply deduce to generate the two critical pairs:

$$\mathsf{aba} \to \mathsf{ab} \qquad \mathsf{bb} \to \mathsf{b} \qquad \mathsf{abab} \approx \mathsf{abba} \qquad \mathsf{bb} \approx \mathsf{bb}$$

The second one is immediately deleted and the first one is simplified:

$$\mathsf{aba} \to \mathsf{ab} \qquad \mathsf{bb} \to \mathsf{b} \qquad \mathsf{abb} \approx \mathsf{aba}$$

and subsequently oriented:

$$\mathsf{aba} \to \mathsf{ab} \qquad \mathsf{bb} \to \mathsf{b} \qquad \mathsf{aba} \to \mathsf{abb}$$

At this point we use the third rule to collapse the first rule:

$$\mathsf{abb} \approx \mathsf{ab} \qquad \mathsf{bb} \to \mathsf{b} \qquad \mathsf{aba} \to \mathsf{abb}$$

An application of simplify followed by delete results in:

$$\mathsf{bb} \to \mathsf{b} \qquad \mathsf{aba} \to \mathsf{abb}$$

Repeating the above process produces

$$\mathsf{bb} \to \mathsf{b} \qquad \mathsf{aba} \to \mathsf{abbb}$$

and then

$$\mathsf{bb} \to \mathsf{b} \qquad\qquad \mathsf{aba} \to \mathsf{abbbb}$$

ad infinitum. Since none of the rules $\mathsf{aba} \to \mathsf{ab}^n$ survives, in the limit we obtain the TRS consisting of the single rule $\mathsf{bb} \to \mathsf{b}$. This TRS is complete but not equivalent to $\mathcal{E}$ as witnessed by non-joinability of aba and ab.

Definition 6.2 (Knuth-Bendix Completion)**.** The inference system $\mathsf{KB_i}$ consists of the inference rules deduce, orient, delete, $\mathsf{compose}$, and $\mathsf{simplify}$ of $\mathsf{KB_f}$ together with the following modified collapse rule:

$$\mathsf{collapse}_{\rhd} \qquad \frac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}} \qquad \text{if } t \xrightarrow{\rhd}_{\mathcal{R}} u$$

Here the condition $t \xrightarrow{\rhd}_{\mathcal{R}} u$ is defined as $t \to u$ using some rule $\ell \to r \in \mathcal{R}$ such that $t \rhd \ell$.

Note that the collapse step in Example 6.1 does not satisfy the encompassment condition from the previous definition.

We write $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{i}} (\mathcal{E}', \mathcal{R}')$ if $(\mathcal{E}', \mathcal{R}')$ can be reached from $(\mathcal{E}, \mathcal{R})$ by employing one of the inference rules of Definition 6.2.

Definition 6.3. An *infinite run* is a maximal sequence of the form

$$\Gamma\colon (\mathcal{E}_0, \mathcal{R}_0) \;\vdash_{\mathsf{i}}\; (\mathcal{E}_1, \mathcal{R}_1) \;\vdash_{\mathsf{i}}\; (\mathcal{E}_2, \mathcal{R}_2) \;\vdash_{\mathsf{i}}\; \cdots$$

with $\mathcal{R}_0 = \varnothing$. We define

$$\mathcal{E}_\infty = \bigcup_{i \geqslant 0} \mathcal{E}_i \qquad \mathcal{R}_\infty = \bigcup_{i \geqslant 0} \mathcal{R}_i \qquad \mathcal{E}_\omega = \bigcup_{i \geqslant 0} \bigcap_{j \geqslant i} \mathcal{E}_j \qquad \mathcal{R}_\omega = \bigcup_{i \geqslant 0} \bigcap_{j \geqslant i} \mathcal{R}_j$$

Equations in $\mathcal{E}_\omega$ and rules in $\mathcal{R}_\omega$ are called *persistent*. The run Γ is called *fair* if $\mathcal{E}_\omega = \varnothing$ and the inclusion $\mathsf{PCP}(\mathcal{R}_\omega) \subseteq {\downarrow_{\mathcal{R}_\omega}} \cup {\leftrightarrow_{\mathcal{E}_\infty}}$ holds.

Bachmair et al. [BDH86] proved that for every fair run satisfying $\mathcal{E}_\omega = \varnothing$ the TRS $\mathcal{R}_\omega$ constitutes a complete presentation of $\mathcal{E}_0$. The remainder of this section is dedicated to establish the same result, but on a different route without encountering proof orders.

Compared to our proofs for finite runs from Section 3, in the following we will disentangle our reasoning about rules from our reasoning about equations and furthermore replace peak decreasingness by the slightly simpler concept of source decreasingness. So why not use this more modular and simpler approach also in our earlier proofs for finite runs? The main difference between the two situations is the encompassment condition of deduce. Unfortunately, without the encompassment condition the equivalent of Lemma 6.10 below for finite runs breaks down and we are forced to reason about rules and equations simultaneously (Lemma 3.7). Nevertheless, it seems useful to also have a correctness proof for $\mathsf{KB_f}$ (lacking the encompassment condition), since out of the four completion tools we are aware of (C*i*ME 3 [CCF+11], KBCV [SZ12], mkbTT [WSMK10], Slothrop [WSW06]), only C*i*ME 3 actually implements the encompassment condition.

Lemma 6.4. *If* $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{i}} (\mathcal{E}', \mathcal{R}')$ *then the following inclusions hold:*

(1) $\mathcal{E}' \cup \mathcal{R}' \;\subseteq\; \overset{*}{\underset{\mathcal{E} \cup \mathcal{R}}{\longleftrightarrow}}$ ☑

(2) $\mathcal{E} \setminus \mathcal{E}' \;\subseteq\; (\underset{\mathcal{R}'}{\longrightarrow} \cdot \mathcal{E}') \;\cup\; (\mathcal{E}' \cdot \underset{\mathcal{R}'}{\longleftarrow}) \;\cup\; \mathcal{R}' \;\cup\; \mathcal{R}'^{-1} \;\cup\; =$ ☑

(3) $\mathcal{R} \setminus \mathcal{R}' \subseteq (\xrightarrow[\mathcal{R}']{\rhd} \cdot \mathcal{E}') \cup (\mathcal{R}' \cdot \xleftarrow[\mathcal{R}']{})$

Together these properties reveal that inference steps do not change the conversion relation.

Corollary 6.5. *If $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{i}}^{*} (\mathcal{E}', \mathcal{R}')$ then the relations $\xleftrightarrow[\mathcal{E} \cup \mathcal{R}]{*}$ and $\xleftrightarrow[\mathcal{E}' \cup \mathcal{R}']{*}$ coincide.*

Below, we consider an infinite run $\Gamma\colon (\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathsf{i}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathsf{i}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathsf{i}} \cdots$ such that $\mathcal{E}_\omega = \varnothing$. First we show that all rewrite rules are compatible with the reduction order $>$.

Lemma 6.6. *The inclusions $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty \subseteq >$ hold.*

Next, we verify that every equality in $\mathcal{E}_i$ can be turned into a valley in $\mathcal{R}_\infty$. Note that in contrast to the proof order approach [BDH86] and to the correctness proof for finite runs given in Section 3 we reason separately about equations and rules. This more local rationale simplifies the analysis as we can use different well-founded induction arguments for the two cases, rather than synthesizing an order that covers both.

Lemma 6.7. *The inclusion $\mathcal{E}_\infty \subseteq {\downarrow_{\mathcal{R}_\infty}}$ holds.*

Proof. Let $s \approx t \in \mathcal{E}_i$ for some $i \geqslant 0$. By induction on $\{s, t\}$ with respect to $>_{\mathsf{mul}}$ we show $s \downarrow_{\mathcal{R}_\infty} t$. Because $\mathcal{E}_\omega = \varnothing$, $s \approx t \in \mathcal{E}_{j-1} \setminus \mathcal{E}_j$ for some $j > i$. Following Lemma 6.4(2), we distinguish three cases.

- If $s \approx t \in \mathcal{R}_j \cup \mathcal{R}_j^{-1} \cup =$ then the claim trivially holds.
- If $s \to_{\mathcal{R}_j} u$ and $u \approx t \in \mathcal{E}_j$ for some term u then $\{s, t\} >_{\mathsf{mul}} \{u, t\}$ and thus $u \downarrow_{\mathcal{R}_\infty} t$ by the induction hypothesis. Hence also $s \downarrow_{\mathcal{R}_\infty} t$.
- Similarly, if $s \approx u \in \mathcal{E}_j$ and $u \mathrel{{}_{\mathcal{R}_j}{\leftarrow}} t$ for some term u then $\{s, t\} >_{\mathsf{mul}} \{s, u\}$ and we obtain $s \downarrow_{\mathcal{R}_\infty} t$ as in the preceding case. ☐

Corollary 6.8. *The inclusion $\xrightarrow[\mathcal{E}_\infty]{} \subseteq \xleftrightarrow[\mathcal{R}_\infty]{*}$ holds.*

In order to show confluence of $\mathcal{R}_\omega$ we use source decreasingness as defined in Section 2, employing the following extension of the reduction order $>$.

Definition 6.9. We define $\succ = ((> \cup \rhd) \,/\, \unrhd)^{+}$.

According to Lemma 2.3, $\succ$ is a well-founded order. The next lemma allows us to transform every non-persistent rule $\ell \to r$ into an $\mathcal{R}_\omega$-conversion below ℓ.

Lemma 6.10. *The inclusion $\xrightarrow[\mathcal{R}_\infty]{s} \subseteq \xleftrightarrow[\mathcal{R}_\omega]{\curlyvee s}{}^{*}$ holds for all terms s.*

Proof. Let $s \xrightarrow{s}_{\mathcal{R}_\infty} t$ by employing the rewrite rule $\ell \to r$. We prove $s \xleftrightarrow{\curlyvee s}{}^{*}_{\mathcal{R}_\omega} t$ by induction on (ℓ, r) with respect to $\succ_{\mathsf{lex}}$. If $\ell \to r \in \mathcal{R}_\omega$ then the claim trivially holds. Otherwise, $\ell \to r \in \mathcal{R}_{i-1} \setminus \mathcal{R}_i$ for some $i > 0$. Using Lemma 6.4(3), we distinguish two cases.

- Suppose $\ell \xrightarrow{\rhd}_{\ell' \to r'} u$ and $u \approx r \in \mathcal{E}_i$ for some term u and rule $\ell' \to r' \in \mathcal{R}_i$. We obtain $\ell \xrightarrow{\rhd}_{\ell' \to r'} u \downarrow_{\mathcal{R}_\infty} r$ from Lemma 6.7. We have $\ell \rhd \ell'$ and both $\ell > u$ and $\ell > r$. It follows that all rewrite rules $\ell'' \to r''$ employed in $\ell \xrightarrow{\rhd} u \downarrow_{\mathcal{R}_\infty} r$ satisfy $(\ell, r) \succ_{\mathsf{lex}} (\ell'', r'')$. Moreover, all steps in $\ell \downarrow_{\mathcal{R}_\infty} r$ are labeled with a term $\leqslant \ell$. Hence we obtain $\ell \xleftrightarrow{\curlyvee \ell}{}^{*}_{\mathcal{R}_\omega} r$ from the induction hypothesis.
- Suppose $\ell \to u \in \mathcal{R}_i$ and $u \leftarrow_{\ell' \to r'} r$ for some term u and rewrite rule $\ell' \to r' \in \mathcal{R}_i$. We have $(\ell, r) \succ_{\mathsf{lex}} (\ell, u)$ and $(\ell, r) \succ_{\mathsf{lex}} (\ell', r')$ because $r > u$ and $\ell > r \unrhd \ell' > r'$. Moreover, both steps are labeled with a term $\leqslant \ell$ and thus we obtain $\ell \xleftrightarrow{\curlyvee \ell}{}^{*}_{\mathcal{R}_\omega} r$ from the induction hypothesis.

So in both cases we have $\ell \xleftrightarrow{\vee\ell}{}^{*}_{\mathcal{R}_\omega} r$ and thus also $s \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\omega} t$. ☐

Corollary 6.11. *The relations* $\xleftrightarrow[\mathcal{R}_\infty]{*}$ *and* $\xleftrightarrow[\mathcal{R}_\omega]{*}$ *coincide.*

We arrive at the main theorem of this section. Note that Bachmair's correctness proof [Bac91] uses induction with respect to a well-founded order on conversions to directly show that any conversion of $\mathcal{E}_\infty \cup \mathcal{R}_\infty$ can be transformed into a joining sequence of $\mathcal{R}_\omega$. In contrast, we prove confluence via source decreasingness. This allows us to concentrate on *local* peaks.

Theorem 6.12. *If* Γ *is fair then* $\mathcal{R}_\omega$ *is a complete presentation of* $\mathcal{E}_0$*.*

Proof. We have $\mathcal{E}_\omega = \varnothing$ because Γ is non-failing. The TRS $\mathcal{R}_\omega$ is terminating by Lemma 6.6. We show source decreasingness of labeled $\mathcal{R}_\omega$ reduction with respect to the reduction order $>$. So let $t \;{}_{\mathcal{R}_\omega}\!\xleftarrow{s}\; s \xrightarrow{s}_{\mathcal{R}_\omega} u$. From Lemma 2.16 we obtain $t \;\triangledown_s^2\; u$. Let $v \;\triangledown_s\; w$ appear in this sequence (so $t = v$ or $w = u$). We have $s > v$, $s > w$, and $(v,w) \in {\downarrow_{\mathcal{R}_\omega}} \cup {\leftrightarrow_{\mathcal{E}_\infty}}$ by the definition of $\triangledown_s$ and fairness of Γ.

- If $v \downarrow_{\mathcal{R}_\omega} w$ then $v \xrightarrow{\vee v}{}^{*}_{\mathcal{R}_\omega} \cdot {}_{\mathcal{R}_\omega}^{\;*}\!\xleftarrow{\vee w} w$ and thus $v \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\omega} w$.
- If $v \leftrightarrow_{\mathcal{E}_\infty} w$ then $v \leftrightarrow_{\mathcal{E}_i} w$ for some $i \geqslant 0$ then $v \downarrow_{\mathcal{R}_\infty} w$ by Lemma 6.7. We obtain $v \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\infty} w$ as in the previous case and thus $v \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\omega} w$ by Lemma 6.10.

Hence $t \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\omega} u$. Confluence of $\mathcal{R}_\omega$ now follows from Lemmata 2.9 and 2.5. It remains to show $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R}_\omega}$. Using Corollary 6.5 we obtain $\to_{\mathcal{E}_i \cup \mathcal{R}_i} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ by a straightforward induction on i. This in turn yields $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty}$. From Corollary 6.8 we infer $\leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty} = \leftrightarrow^*_{\mathcal{R}_\infty}$ and we conclude by an appeal to Corollary 6.11. ☐

Example 6.13. Consider the ES $\mathcal{E}$ and the KBO $>$ from Example 1.1. Let $\mathcal{P}_n$ for $n \geqslant 1$ denote the TRS $\{\mathsf{ab}^{i+1}\mathsf{ab} \to \mathsf{babba}^i \mid 1 \leqslant i \leqslant n\}$. One possible infinite completion run is the following:

$$\begin{aligned}
(\mathcal{E}, \varnothing) \;&\vdash_{\mathsf{i}}^{\mathsf{orient}} (\varnothing, \{\mathsf{aba} \to \mathsf{bab}\}) && \vdash_{\mathsf{i}}^{\mathsf{deduce}} (\{\mathsf{abbab} \approx \mathsf{babba}\}, \{\mathsf{aba} \to \mathsf{bab}\}) \\
&\vdash_{\mathsf{i}}^{\mathsf{orient}} (\varnothing, \{\mathsf{aba} \to \mathsf{bab}\} \cup \mathcal{P}_1) && \vdash_{\mathsf{i}}^{\mathsf{deduce}} (\{\mathsf{abbbab} \approx \mathsf{babbaa}\}, \{\mathsf{aba} \to \mathsf{bab}\} \cup \mathcal{P}_1) \\
&\vdash_{\mathsf{i}}^{\mathsf{orient}} (\varnothing, \{\mathsf{aba} \to \mathsf{bab}\} \cup \mathcal{P}_2) && \vdash_{\mathsf{i}} \quad \cdots
\end{aligned}$$

If this run is continued in a fair way we subsequently construct the TRSs $\mathcal{P}_n$ and can in the limit obtain the result $\mathcal{R}_\omega = \{\mathsf{aba} \to \mathsf{bab}\} \cup \{\mathsf{ab}^{i+1}\mathsf{ab} \to \mathsf{babba}^i \mid i \geqslant 1\}$, which is complete according to Theorem 6.12.

This section recapitulates our results on infinite runs [HMSW17]. Our correctness proof (Theorem 6.12) differs substantially from earlier proofs in the literature. Due to a less monolithic structure we consider this proof to be more formalization friendly: Instead of lexicographically combining several orders into a single proof reduction relation, we use source decreasingness together with different orders as necessary to prove auxiliary results. In particular, our approach naturally supports prime critical pairs.

7. Ordered Completion

Completion may fail to construct a complete system if unorientable equations are encountered. For example, the ES $\mathcal{E}$ consisting of the two equations $0 + x \approx x$ and $x + y \approx y + x$ admits no complete presentation. (We will prove it in Section 8.) This can happen even if a finite complete system exists, as illustrated by the following example.

Example 7.1. Consider the ES $\mathcal{E}$ [BD94] consisting of the three equations

$$1 \cdot (-x + x) \approx 0 \qquad 1 \cdot (x + -x) \approx x + -x \qquad -x + x \approx y + -y$$

Any run of standard Knuth-Bendix completion will fail on this input system; the first two equations may be oriented from left to right if a suitable order is employed but no further steps are possible. However, the TRS $\mathcal{R}$ consisting of the rules

$$1 \cdot 0 \to 0 \qquad x + -x \to 0 \qquad -x + x \to 0$$

constitutes a canonical presentation of $\mathcal{E}$.

Ordered completion was developed to remedy this shortcoming. In contrast to completion as presented in the preceding section it never fails, though the resulting system is in general only ground complete.

For an ES $\mathcal{E}$, an ordered rewrite step is a rewrite step using a rule from $\mathcal{E}^{>}$, which is the infinite set of rewrite rules $\ell\sigma \to r\sigma$ such that $\ell \approx r \in \mathcal{E}^{\pm}$ and $\ell\sigma > r\sigma$ for some substitution σ.

The following inference rules for ordered completion are due to Bachmair, Dershowitz, and Plaisted [BDP89]. In order to simplify the notation, we abbreviate $\mathcal{E}_\omega^{>} \cup \mathcal{R}_\omega$ to $\mathcal{S}$, and use the following shorthands. We write $t \xrightarrow{\rhd}_{\mathcal{E}^>} u$ if there exist an equation $\ell \approx r \in \mathcal{E}^{\pm}$, a context C, and a substitution σ such that $t = C[\ell\sigma]$, $u = C[r\sigma]$, $\ell\sigma > r\sigma$, and $t \rhd \ell$. The union of $\to_{\mathcal{R}}$ and $\xrightarrow{\rhd}_{\mathcal{E}^>}$ is denoted by $\xrightarrow{\rhd_1}_{\mathcal{S}}$ and we write $\xrightarrow{\rhd_2}_{\mathcal{S}}$ for the union of $\xrightarrow{\rhd}_{\mathcal{R}}$ and $\xrightarrow{\rhd}_{\mathcal{E}^>}$.

Definition 7.2 (Ordered Completion ☑)**.** The inference system $\mathsf{KB_o}$ of ordered completion operates on pairs $(\mathcal{E}, \mathcal{R})$ of equations $\mathcal{E}$ and rules $\mathcal{R}$ over a common signature $\mathcal{F}$. It consists of the following inference rules:

deduce $\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$ if $s \xleftarrow[\mathcal{R} \cup \mathcal{E}^{\pm}]{} \cdot \xrightarrow[\mathcal{R} \cup \mathcal{E}^{\pm}]{} t$

compose $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$ if $t \to_{\mathcal{S}} u$

orient $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$ if $s > t$ $\qquad \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}$ if $t > s$

simplify $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}}$ if $s \xrightarrow{\rhd_1}_{\mathcal{S}} u$ $\qquad \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$ if $t \xrightarrow{\rhd_1}_{\mathcal{S}} u$

delete $\dfrac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

collapse $\dfrac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$ if $t \xrightarrow{\rhd_2}_{\mathcal{S}} u$

The deduce rule may be applied to any peak, though in practice it is typically limited to the addition of extended critical pairs (which are defined in Definition 7.11 below). We write $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{o}} (\mathcal{E}', \mathcal{R}')$ if $(\mathcal{E}', \mathcal{R}')$ can be reached from $(\mathcal{E}, \mathcal{R})$ by employing one of the inference rules of Definition 7.2. We start by stating the equivalents of Lemma 6.4 and Corollary 6.5 for ordered completion.

Lemma 7.3. *If* $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{o}} (\mathcal{E}', \mathcal{R}')$ *then the following inclusions hold:*

(1) $\mathcal{E}' \cup \mathcal{R}' \subseteq \xleftrightarrow[\mathcal{E} \cup \mathcal{R}]{*}$ ☑

(2) $\mathcal{E} \setminus \mathcal{E}' \subseteq (\xrightarrow[\mathcal{S}']{\rhd_1} \cdot \mathcal{E}'^{\pm})^{\pm} \cup \mathcal{R}'^{\pm} \cup =$ ☑

(3) $\mathcal{R} \setminus \mathcal{R}' \subseteq (\xrightarrow[\mathcal{S}']{\rhd_2} \cdot \mathcal{E}') \cup (\mathcal{R}' \cdot \xleftarrow[\mathcal{S}']{})$

Corollary 7.4. *If $(\mathcal{E},\mathcal{R}) \vdash_{\mathsf{o}} (\mathcal{E}',\mathcal{R}')$ then the relations $\xleftrightarrow[\mathcal{E}\cup\mathcal{R}]{*}$ and $\xleftrightarrow[\mathcal{E}'\cup\mathcal{R}']{*}$ coincide.*

We illustrate $\mathsf{KB_o}$ by means of an example.

Example 7.5. Consider the ES $\mathcal{E}$ consisting of the following three equations:

$$\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a}) \qquad\qquad \mathsf{f}(\mathsf{b}) \approx \mathsf{b} \qquad\qquad \mathsf{g}(\mathsf{f}(\mathsf{b}),x) \approx \mathsf{g}(x,\mathsf{b})$$

By taking the Knuth-Bendix order $>_{\mathsf{kbo}}$ with precedence $\mathsf{f} > \mathsf{b}$ and where all function symbols are assigned weight 1, the following $\mathsf{KB_o}$ inference sequence can be obtained:

$$\begin{aligned}
(\mathcal{E},\varnothing) \;&\vdash_{\mathsf{o}}^{\mathsf{orient}^+} (\{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a})\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}, \mathsf{g}(\mathsf{f}(\mathsf{b}),x) \to \mathsf{g}(x,\mathsf{b})\})\\
&\vdash_{\mathsf{o}}^{\mathsf{deduce}} (\{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}) \approx \mathsf{f}(\mathsf{a})\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}, \mathsf{g}(\mathsf{f}(\mathsf{b}),x) \to \mathsf{g}(x,\mathsf{b})\})\\
&\vdash_{\mathsf{o}}^{\mathsf{simplify}} (\{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a}), \mathsf{b} \approx \mathsf{f}(\mathsf{a})\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}, \mathsf{g}(\mathsf{f}(\mathsf{b}),x) \to \mathsf{g}(x,\mathsf{b})\})\\
&\vdash_{\mathsf{o}}^{\mathsf{orient}} (\{\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a})\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}, \mathsf{g}(\mathsf{f}(\mathsf{b}),x) \to \mathsf{g}(x,\mathsf{b}), \mathsf{f}(\mathsf{a}) \to \mathsf{b}\})\\
&\vdash_{\mathsf{o}}^{\mathsf{simplify}} (\{\mathsf{f}(x) \approx \mathsf{b}\}, \{\mathsf{f}(\mathsf{b}) \to \mathsf{b}, \mathsf{g}(\mathsf{f}(\mathsf{b}),x) \to \mathsf{g}(x,\mathsf{b}), \mathsf{f}(\mathsf{a}) \to \mathsf{b}\})\\
&\vdash_{\mathsf{o}}^{\mathsf{collapse}^+} (\{\mathsf{f}(x) \approx \mathsf{b}, \mathsf{b} \approx \mathsf{b}, \mathsf{g}(\mathsf{b},x) \approx \mathsf{g}(x,\mathsf{b})\}, \varnothing)\\
&\vdash_{\mathsf{o}}^{\mathsf{orient}} (\{\mathsf{b} \approx \mathsf{b}, \mathsf{g}(\mathsf{b},x) \approx \mathsf{g}(x,\mathsf{b})\}, \{\mathsf{f}(x) \to \mathsf{b}\})\\
&\vdash_{\mathsf{o}}^{\mathsf{delete}} (\{\mathsf{g}(\mathsf{b},x) \approx \mathsf{g}(x,\mathsf{b})\}, \{\mathsf{f}(x) \to \mathsf{b}\})\\
&\vdash_{\mathsf{o}}^{\mathsf{deduce}} (\{\mathsf{g}(\mathsf{b},x) \approx \mathsf{g}(x,\mathsf{b}), \mathsf{b} \approx \mathsf{b}\}, \{\mathsf{f}(x) \to \mathsf{b}\})
\end{aligned}$$

This sequence can be extended to an (infinite) run by repeating the last two steps. Then we have $\mathcal{R}_\omega = \{\mathsf{f}(x) \to \mathsf{b}\}$ and $\mathcal{E}_\omega = \{\mathsf{g}(\mathsf{b},x) \approx \mathsf{g}(x,\mathsf{b})\}$.

Below, we consider an arbitrary run $\Gamma\colon (\mathcal{E}_0,\mathcal{R}_0) \vdash_{\mathsf{o}} (\mathcal{E}_1,\mathcal{R}_1) \vdash_{\mathsf{o}} (\mathcal{E}_2,\mathcal{R}_2) \vdash_{\mathsf{o}} \cdots$. In general $\mathcal{E}_\infty \subseteq {\downarrow_{\mathcal{R}_\infty}}$ does not hold, as Example 7.5 illustrates. So unlike in the preceding section we now omit the condition $\mathcal{E}_\omega = \varnothing$. However, this comes at the price of weaker properties of the resulting system, as the remainder of this section shows.

Lemma 7.6. *The inclusions $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty \subseteq >$ and $\mathcal{E}_\omega \subseteq \mathcal{E}_\infty$ hold.*

We use the relation $\xrightarrow{M}$ from Definition 3.6 to show that any equation step below a term set M eventually turns into a conversion over $\mathcal{E}_\omega \cup \mathcal{R}_\infty$ that is still below M. Note that just like in Section 6 we avoid the use of a synthesized termination argument by handling equations and rules separately.

Lemma 7.7. *The inclusion $\xrightarrow[\mathcal{E}_\infty]{S} \subseteq \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\infty]{S}{}^*$ holds for all sets S of terms.*

Proof. Let $t \approx u \in \mathcal{E}_\infty$. We prove

$$\xrightarrow[t \approx u]{M} \subseteq \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\infty]{M}{}^*$$

by induction on $\{t,u\}$ with respect to the well-founded order $\succ_{\mathsf{mul}}$. If $t \approx u \in \mathcal{E}_\omega^{\pm}$ then the claim follows trivially. Otherwise, $t \approx u \in (\mathcal{E}_{i-1} \setminus \mathcal{E}_i)^{\pm}$ for some $i > 0$. Using Lemma 7.3(2), we distinguish two subcases.

- Suppose $t \approx u \in (\xrightarrow{\rhd_1}_{\mathcal{S}_i} \cdot \mathcal{E}_i^{\pm})^{\pm}$. There exist a term t' and an equation $v' \approx u' \in \mathcal{E}_i^{\pm}$ such that $\{t, u\} = \{t', u'\}$ and $t' \xrightarrow{\rhd_1}_{\mathcal{S}_i} v'$. It is sufficient to show
$$t' \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\infty]{\{t',u'\}}{}^{*} v' \quad \text{and} \quad v' \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\infty]{\{t',u'\}}{}^{*} u'$$
The second conversion follows from $t' > v'$ and the induction hypothesis for $v' \approx u' \in \mathcal{E}_i^{\pm}$, which is applicable as $\{t, u\} = \{t', u'\} \succ_{\mathsf{mul}} \{v', u'\}$. The first conversion is obtained as follows. Because of $t' \xrightarrow{\rhd_1}_{\mathcal{S}_i} v'$, we have $t' \to_{\mathcal{R}_i} v'$ or $t' \xrightarrow{\rhd}_{\mathcal{E}_i^{>}} v'$. If $t' \to_{\mathcal{R}_i} v'$ then this step can be labeled with $\{t', u'\}$ as $t' > v'$. Otherwise, there exist an equation $\ell \approx r \in \mathcal{E}_i^{\pm}$, a context C, and a substitution σ such that $t' = C[\ell\sigma]$, $v' = C[r\sigma]$, $\ell\sigma > r\sigma$, and $t' \rhd \ell$. We have $t' \succ \ell$ and $t' \succ r$ as $t' \trianglerighteq \ell\sigma > r\sigma \trianglerighteq r$. Therefore $\{t', u'\} \succ_{\mathsf{mul}} \{\ell, r\}$ holds, so
$$\ell \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\infty]{\{\ell,r\}}{}^{*} r$$
follows from the induction hypothesis. Closure under contexts and substitutions now yields $t \xleftrightarrow{\{t,u\}}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\infty} u$.
- If $t \approx u \in \mathcal{R}_i^{\pm} \cup =$ then $t \xleftrightarrow[\mathcal{R}_\infty]{\{t,u\}}{}^{=} u$.

In both cases $t \xleftrightarrow{\{t,u\}}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\infty} u$ holds. Since M contains upper bounds of t and u with respect to $\geqslant$, the desired inclusion follows from the closure under contexts and substitutions of $\to_{\mathcal{E}_\omega \cup \mathcal{R}_\infty}$ and $\geqslant$. □

Next, we show that a rewrite step that uses a rule in $\mathcal{R}_\infty$ and is below a multiset of terms M eventually turns into a conversion over persistent rules and equations that is still below M. To this end we write $\curlyvee t$ for the set $\{u \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid t \succ u\}$.

Lemma 7.8. *The inclusion* $\xrightarrow[\mathcal{R}_\infty]{M} \subseteq \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\omega]{M}{}^{*}$ *holds for all multisets M of terms.*

Proof. Let $\ell \approx r \in \mathcal{R}_\infty$. We prove
$$\xrightarrow[\ell \to r]{M} \subseteq \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\omega]{M}{}^{*}$$
by induction on (ℓ, r) with respect to the well-founded order $\succ_{\mathsf{lex}}$. If $\ell \to r \in \mathcal{R}_\omega$ then the claim trivially holds. Otherwise, there is some $i > 0$ such that $\ell \to r \in \mathcal{R}_{i-1} \setminus \mathcal{R}_i$. From Lemma 7.7 and the induction hypothesis the inclusions
$$\xrightarrow[\mathcal{E}_\infty \cup \mathcal{R}_\infty]{N} \subseteq \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\infty]{N}{}^{*} \subseteq \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\omega]{N}{}^{*} \tag{7.1}$$
are obtained for every set $N \subseteq \curlyvee\ell$. Using Lemma 7.3, we distinguish two cases.

- Suppose $\ell \xrightarrow{\rhd_2}_{\mathcal{S}_i} u$ and $u \approx r \in \mathcal{E}_i$ for some term u. There exist an equation $\ell' \approx r' \in \mathcal{E}_\infty^{\pm} \cup \mathcal{R}_\infty$, a context C and a substitution σ such that $\ell = C[\ell'\sigma]$, $u = C[r'\sigma]$, $\ell\sigma > r\sigma$, and $\ell \rhd \ell'$. We have $\ell \succ \ell', r'$ as $\ell \rhd \ell'$ and $\ell \trianglerighteq \ell'\sigma > r'\sigma \trianglerighteq r'$ and thus
$$\ell' \xleftrightarrow[\mathcal{E}_\infty \cup \mathcal{R}_\infty]{\{\ell'\}} r'$$
Since $\{\ell', r'\} \subseteq \curlyvee\ell$ we obtain $\ell' \xleftrightarrow{\{\ell',r'\}}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} r'$ from (7.1). Therefore, $\ell \xleftrightarrow{\{\ell\}}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} u$ follows from closure under contexts and substitutions and $\ell > u$. Again from $\ell > u, r$ we obtain $u \xleftrightarrow{\curlyvee\ell}_{\mathcal{E}_\infty \cup \mathcal{R}_\infty} r$ and thus $u \xleftrightarrow{\curlyvee\ell}_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} r$ follows from (7.1).
- Suppose $\ell \to u \in \mathcal{R}_i$ and $u \;{}_{\mathcal{S}_i}\!\!\leftarrow r$ for some term u. We have $r > u$ and thus $(\ell, r) \succ_{\mathsf{lex}} (\ell, u)$. Hence we can apply the induction hypothesis to $\ell \xrightarrow{\{\ell\}}_{\ell \to u} u$, yielding $\ell \xleftrightarrow{\{\ell\}}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} u$. From $\ell > r > u$ we obtain $u \xleftrightarrow{\curlyvee\ell}_{\mathcal{E}_\infty \cup \mathcal{R}_\infty} r$ and thus $u \xleftrightarrow{\curlyvee\ell}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} r$ follows by (7.1).

In both cases $\ell \xleftrightarrow{\{\ell\}}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} r$ holds. Since $\to_{\mathcal{E}_\omega \cup \mathcal{R}_\omega}$ and $\geqslant$ are closed under contexts and substitutions, the desired inclusion on steps using $\ell \to r$ follows. $\square$

We can combine the preceding lemmata to obtain an inclusion in conversions over persistent equations and rules.

Corollary 7.9. *The inclusion* $\xrightarrow[\mathcal{E}_\infty \cup \mathcal{R}_\infty]{M} \subseteq \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\omega]{M}{}^{*}$ *holds for all multisets of terms M.*

For instance, in Example 7.5 we have $\mathsf{f}(x) \xleftrightarrow{M}_{\mathcal{E}_\infty} \mathsf{f}(\mathsf{a})$ for $M = \{\mathsf{f}(x), \mathsf{f}(\mathsf{a})\}$ and the conversion $\mathsf{f}(x) \leftrightarrow \mathsf{b} \leftrightarrow \mathsf{f}(\mathsf{a})$ in $\mathcal{E}_\omega \cup \mathcal{R}_\omega$ clearly satisfies $\mathsf{f}(x) \xleftrightarrow{M}{}^{*}_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} \mathsf{f}(\mathsf{a})$.

The results obtained so far are sufficient to show that ordered completion can produce a complete system.

Theorem 7.10. *If Γ satisfies $\mathsf{PCP}(\mathcal{R}_\omega) \subseteq {\downarrow_{\mathcal{R}_\omega}} \cup {\leftrightarrow_{\mathcal{E}_\infty}}$ and $\mathcal{E}_\omega = \varnothing$ then $\mathcal{R}_\omega$ is a complete presentation of $\mathcal{E}_0$.*

Proof. We prove that $\mathcal{R}_\omega$ is confluent by showing that labeled $\mathcal{R}_\omega$ reduction on arbitrary terms is source decreasing. Consider $t \,{}_{\mathcal{R}_\omega}\!\xleftarrow{s}\, s \xrightarrow{s}_{\mathcal{R}_\omega} u$. From Lemma 2.16 we obtain $t \;\triangledown_s^2\; u$ (where $\mathcal{R}_\omega$ takes the place of $\mathcal{R}$ in the definition of $\triangledown_s$). Let $v \;\triangledown_s\; w$ appear in this sequence (so $t = v$ or $w = u$). We have $s > v$, $s > w$, and $v \downarrow_{\mathcal{R}_\omega} w$ or $v \leftrightarrow_{\mathcal{E}_\infty} w$ by the definition of $\triangledown_s$ and the assumption $\mathsf{PCP}(\mathcal{R}_\omega) \subseteq {\leftrightarrow_{\mathcal{E}_\infty}}$.

- If $v \downarrow_{\mathcal{R}_\omega} w$ then $v \xrightarrow{\mathbb{V} v}{}^{*}_{\mathcal{R}_\omega} \cdot {}_{\mathcal{R}_\omega}^{\;*}\!\xleftarrow{\mathbb{V} w} w$ and thus $v \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\omega} w$.
- If $v \leftrightarrow_{\mathcal{E}_\infty} w$ then $v \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\omega} w$ by Corollary 7.9.

Hence $t \xleftrightarrow{\vee s}{}^{*}_{\mathcal{R}_\omega} u$. Confluence of $\mathcal{R}_\omega$ follows from Lemmata 2.9 and 2.5. Termination of $\mathcal{R}_\omega$ holds by Lemma 7.6. We have $\leftrightarrow^{*}_{\mathcal{E}_0} = \leftrightarrow^{*}_{\mathcal{R}_\omega}$ by an easy induction argument using Corollary 7.4, so $\mathcal{R}_\omega$ is a complete presentation of $\mathcal{E}_0$. $\square$

From now on we specialize our results to ground terms. In the remainder of this section we therefore assume that $>$ is a ground-total reduction order. Before continuing with results on ordered completion, we define extended critical pairs.

Definition 7.11 (Extended Overlaps). An *extended overlap* of a given ES $\mathcal{E}$ is a triple $\langle \ell_1 \approx r_1, p, \ell_2 \approx r_2 \rangle$ satisfying the following properties:

- there are renamings π_1 and π_2 such that $\pi_1(\ell_1 \approx r_1), \pi_2(\ell_2 \approx r_2) \in \mathcal{E}^{\pm}$ (that is, the equations are variants of equations in $\mathcal{E}^{\pm}$),
- $\mathcal{V}\mathsf{ar}(\ell_1 \approx r_1) \cap \mathcal{V}\mathsf{ar}(\ell_2 \approx r_2) = \varnothing$,
- $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$,
- ℓ_1 and $\ell_2|_p$ are unifiable with some mgu μ, and
- $r_1\mu \not> \ell_1\mu$ and $r_2\mu \not> \ell_2\mu$.

An extended overlap gives rise to the *extended critical pair* $\ell_2[r_1]_p\mu \approx r_2\mu$. An extended critical pair is called *prime* if all proper subterms of $\ell_1\mu$ are $\mathcal{E}^{>}$-normal forms. The set of extended prime critical pairs among equations in $\mathcal{E}$ is denoted by $\mathsf{PCP}_{>}(\mathcal{E})$.

For example, the equations $\mathsf{1}\cdot(x+-x) \approx x+-x$ and $y+-y \approx -z+z$ are variable-disjoint variants of equations in Example 7.1. Neither of them can be oriented from right to left (independent of the choice of $>$). Because of the peak $\mathsf{1}\cdot(-z+z) \leftrightarrow \mathsf{1}\cdot(x+-x) \leftrightarrow x+-x$ they admit the extended overlap $\langle y+-y \approx -z+z, 1, \mathsf{1}\cdot(x+-x) \approx x+-x \rangle$ which gives rise to the extended critical pair $\mathsf{1}\cdot(-z+z) \approx x+-x$. Note that since the second equation is unorientable, a run of a standard completion procedure will not encounter this critical pair.

Extended critical pairs are important due to the Extended Critical Pair Lemma [BDP89], according to which these are the only peaks relevant for ground confluence. In our formalization we use the following variant. The proof employs a similar peak analysis as in Lemma 2.12.

Lemma 7.12. *Let $\mathcal{E}$ be an ES and consider a peak*

$$t \xleftarrow[r_1 \approx \ell_1]{pq,\,\sigma_1} s \xrightarrow[\ell_2 \approx r_2]{p,\,\sigma_2} u$$

involving ground terms s, t, and u such that $\ell_1\sigma_1 > r_1\sigma_1$ and $\ell_2\sigma_2 > r_2\sigma_2$. If $\ell_1 \approx r_1, \ell_2 \approx r_2 \in \mathcal{E}$ do not form an extended overlap at position q then $t \downarrow_{\mathcal{E}^>} u$.

In the sequel, we write $\mathcal{S}_\omega$ for the TRS $\mathcal{E}_\omega^> \cup \mathcal{R}_\omega$.

Corollary 7.13. *If $s \xleftrightarrow[\mathcal{E}_\infty]{} t$ for ground terms s and t then $s \xleftrightarrow[\mathcal{S}_\omega]{\{s,t\}}{}^* t$.*

Proof. We obtain $s \xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\omega]{\{s,t\}}{}^* t$ from Corollary 7.9. Since $>$ is ground-total, all $\mathcal{E}_\omega$ steps in this conversion are $(\mathcal{E}_\omega^>)^\pm$ steps or trivial steps between identical terms. Hence $s \xleftrightarrow[\mathcal{S}_\omega]{\{s,t\}}{}^* t$ as desired. □

Definition 7.14. A run $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathsf{o}} (\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathsf{o}} (\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathsf{o}} \cdots$ is called *fair* if the inclusion $\mathsf{PCP}_>(\mathcal{E}_\omega \cup \mathcal{R}_\omega) \subseteq {\downarrow_{\mathcal{S}_\omega}} \cup {\leftrightarrow_{\mathcal{E}_\infty}}$ holds.

The following lemma links extended prime critical pairs to standard critical pairs and hence allows us to reuse results from Section 2.3 for our main correctness result (Theorem 7.16 below).

Lemma 7.15. *For a TRS $\mathcal{R}$ and an ES $\mathcal{E}$, the inclusion $\xleftrightarrow[\mathsf{PCP}(\mathcal{S})]{} \subseteq \xleftrightarrow[\mathsf{PCP}_>(\mathcal{E}\cup\mathcal{R})]{} \cup \downarrow_{\mathcal{S}}$ holds on ground terms.*

Proof. Suppose $s \leftrightarrow_e t$ for ground terms s and t and a prime critical pair $e\colon \ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ generated from the overlap $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ in $\mathcal{S}$. Let $u_i \approx v_i$ be the equation $\ell_i \approx r_i$ if $\ell_i \to r_i \in \mathcal{R}$ and the equation in $\mathcal{E}^\pm$ such that $\ell_i = u_i\tau_i$ and $r_i = v_i\tau_i$ for some substitution τ_i if $\ell_i \to r_i \in \mathcal{E}^>$. In the former case we let τ_i be the empty substitution. Since the equations $u_1 \approx v_1$ and $u_2 \approx v_2$ are assumed to be variable-disjoint, the substitution $\tau = \tau_1 \cup \tau_2$ is well-defined. We distinguish two cases.

- If $p \notin \mathcal{P}\mathsf{os}_\mathcal{F}(u_2)$ then $\langle u_1 \approx v_1, p, u_2 \approx v_2 \rangle$ is not an overlap and hence $s \downarrow_\mathcal{S} t$ by Lemma 7.12.
- Suppose $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(u_2)$. Since $u_2|_p\tau\sigma = \ell_2|_p\sigma = \ell_1\sigma = u_1\tau\sigma$ there exist an mgu μ of $u_2|_p$ and u_1, and a substitution ρ such that $\mu\rho = \tau\sigma$. Because $u_i\mu\rho = \ell_i\sigma > r_i\sigma = v_i\mu\rho$, $v_i\mu > u_i\mu$ is impossible. Hence $e'\colon u_2\mu[v_1\mu]_p \approx v_2\mu \in \mathsf{CP}_>(\mathcal{E} \cup \mathcal{R})$ and

  $$\ell_2\sigma[r_1\sigma]_p = u_2\mu\rho[v_1\mu\rho]_p = u_2\mu[v_1\mu]_p\rho \xleftrightarrow[e']{} v_2\mu\rho = r_2\sigma$$

  Since e is prime, proper subterms of $\ell_2\sigma|_p = u_2\mu\rho|_p$ are irreducible with respect to $\mathcal{S}$, and hence the same holds for proper subterms of $u_2\mu$. It follows that $e' \in \mathsf{PCP}_>(\mathcal{E} \cup \mathcal{R})$ and thus $\ell_2\sigma[r_1\sigma]_p \xleftrightarrow[\mathsf{PCP}_>(\mathcal{E}\cup\mathcal{R})]{} r_2\sigma$. Hence also $s \xleftrightarrow[\mathsf{PCP}_>(\mathcal{E}\cup\mathcal{R})]{} t$. □

This relationship between extended critical pairs among $\mathcal{E} \cup \mathcal{R}$ and critical pairs among $\mathcal{S}$ is the final ingredient for the main result of this section. As in the preceding section, we establish correctness of ordered completion via source decreasingness.

Theorem 7.16. *If Γ is fair then $\mathcal{S}_\omega$ is ground-complete and $\xleftrightarrow[\mathcal{E}_0]{*}$ and $\xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\omega]{*}$ coincide.*

Proof. Termination of $\mathcal{S}_\omega$ is a consequence of Lemma 7.6 and the definition of $\mathcal{E}_\omega^>$. Next we show that $\mathcal{S}_\omega$ is ground-confluent. To this end, we show that labeled $\mathcal{S}_\omega$ reduction is source decreasing on ground terms. So let s, t, and u be ground terms such that

$$t \xleftarrow[\mathcal{S}_\omega]{s} s \xrightarrow[\mathcal{S}_\omega]{s} u$$

Lemma 2.16 yields $t \triangledown_s^2 u$ (where $\mathcal{S}_\omega$ takes the place of $\mathcal{R}$ in the definition of $\triangledown_s$). Let $v \triangledown_s w$ appear in this sequence (so $t = v$ or $w = u$ and both terms are ground). We have $s > v$, $s > w$, and $(v, w) \in \downarrow_{\mathcal{S}_\omega} \cup \leftrightarrow_{\mathcal{E}_\infty}$ by the definition of $\triangledown_s$, Lemma 7.15, and fairness of Γ.

- If $v \downarrow_{\mathcal{S}_\omega} w$ then $v \xrightarrow{\curlyvee v}{}^*_{\mathcal{S}_\omega} \cdot {}_{\mathcal{S}_\omega}^{*}\xleftarrow{\curlyvee w} w$ and thus $v \xleftrightarrow{\vee s}{}^*_{\mathcal{S}_\omega} w$.
- If $v \leftrightarrow_{\mathcal{E}_\infty} w$ then $v \leftrightarrow_{\mathcal{E}_i} w$ for some $i \geqslant 0$ and thus $v \xleftrightarrow{\vee s}{}^*_{\mathcal{S}_\omega} w$ by Corollary 7.13.

Hence $t \xleftrightarrow{\vee s}{}^*_{\mathcal{S}_\omega} u$. Confluence of the ARS that is obtained by restricting $\mathcal{S}_\omega$ to ground terms now follows from Lemmata 2.9 and 2.5. It remains to show $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{E}_\omega \cup \mathcal{R}_\omega}$. Using Corollary 7.4 we obtain $\rightarrow_{\mathcal{E}_i \cup \mathcal{R}_i} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ for all $i \geqslant 0$ by a straightforward induction argument. This in turn yields $\leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ and in particular $\leftrightarrow^*_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$. The reverse inclusion follows from Corollary 7.9 and the inclusion $\leftrightarrow^*_{\mathcal{E}_0} \subseteq \leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty}$. □

If $\mathcal{E}_\omega$ is empty, the TRS $\mathcal{R}_\omega$ is not only ground-confluent but actually confluent on all terms. Even though this result is not surprising, we did not find it explicitly stated in the literature.

Theorem 7.17. *If Γ is fair and $\mathcal{E}_\omega = \varnothing$ then $\mathcal{R}_\omega$ is a complete presentation of $\mathcal{E}_0$.*

Proof. We have $\mathsf{PCP}(\mathcal{R}_\omega) \subseteq \mathsf{PCP}_>(\mathcal{E}_\omega \cup \mathcal{R}_\omega)$ since $\mathcal{R}_\omega \subseteq >$. Hence the result follows from fairness and Theorem 7.10. □

Example 7.18. Consider the ES $\mathcal{E}$ from Example 7.1 and $>_{\mathsf{lpo}}$ with precedence $+ > \mathsf{0}$. After two orient steps, we apply deduce:

$$\mathsf{1} \cdot (-x + x) \to \mathsf{0} \quad \mathsf{1} \cdot (x + -x) \to x + -x \quad -x + x \approx y + -y \quad \mathsf{1} \cdot (-z + z) \approx x + -x$$

The newly added equation is simplified and then oriented:

$$\mathsf{1} \cdot (-x + x) \to \mathsf{0} \quad \mathsf{1} \cdot (x + -x) \to x + -x \quad -x + x \approx y + -y \qquad x + -x \to \mathsf{0}$$

Using the new rewrite rule, the remaining equation is simplified, the second rule is subjected to compose and subsequently to collapse:

$$\mathsf{1} \cdot (-x + x) \to \mathsf{0} \qquad \mathsf{1} \cdot \mathsf{0} \approx \mathsf{0} \qquad -x + x \approx \mathsf{0} \qquad x + -x \to \mathsf{0}$$

Orienting both equations results in:

$$\mathsf{1} \cdot (-x + x) \to \mathsf{0} \qquad \mathsf{1} \cdot \mathsf{0} \to \mathsf{0} \qquad -x + x \to \mathsf{0} \qquad x + -x \to \mathsf{0}$$

At this point the first rule is collapsed using the third rule, and subsequently oriented (into an existing rule):

$$\mathsf{1} \cdot \mathsf{0} \to \mathsf{0} \qquad -x + x \to \mathsf{0} \qquad x + -x \to \mathsf{0}$$

This sequence can be extended to an infinite run by repeatedly adding (using deduce) and deleting the trivial equation $\mathsf{0} \approx \mathsf{0}$. Then the set of persistent rules $\mathcal{R}_\omega$ coincides with the TRS $\mathcal{R}$ from Example 7.1, and $\mathcal{E}_\omega = \varnothing$.

The final result in this section is in the spirit of Theorem 4.7 but for ordered completion, showing that a ground-complete system can be interreduced to some extent.

Definition 7.19. Given a ground-complete system $\mathcal{S} = \mathcal{E}^{>} \cup \mathcal{R}$, we define

$$\mathcal{R}' = \{\ell \to r \mid \ell \to r \in \dot{\mathcal{Q}} \text{ and } \ell \in \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})\}$$
$$\mathcal{E}' = \{s{\downarrow_{\mathcal{R}'}} \approx t{\downarrow_{\mathcal{R}'}} \mid s \approx t \in \mathcal{E}\} \setminus =$$

where $\mathcal{Q} = \mathcal{R} \cup (\mathcal{E}^{\pm} \cap >)$ and $\dot{\mathcal{Q}}$ is defined in Definition 4.5.

Here we write $t \xrightarrow{\rhd}_{\mathcal{S}} u$ if there are a rule $\ell \to r \in \mathcal{S}$, a context C, and a substitution σ such that $t = C[\ell\sigma]$, $u = C[r\sigma]$, and $t \rhd \ell$. For example, if $\mathcal{E}$ is empty and $\mathcal{R}$ consists of the single rule $\mathsf{f}(x,y) \to \mathsf{g}(x)$ we have $\mathsf{f}(y,z) \in \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$, but $\mathsf{f}(\mathsf{g}(x),y) \notin \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$ and $\mathsf{f}(x,x) \notin \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$.

Theorem 7.20. *If $\mathcal{S} = \mathcal{E}^{>} \cup \mathcal{R}$ is ground-complete then $\mathcal{S}' = \mathcal{E}'^{>} \cup \mathcal{R}'$ is ground-complete and normalization and conversion equivalent on ground terms.*

Proof. We first show $\mathsf{NF}(\mathcal{S}') \subseteq \mathsf{NF}(\mathcal{S})$. For a rule $\ell \to r \in \mathcal{S}$, let $b_{\ell\to r}$ be $\bot$ if $\ell \to r \in \mathcal{Q}$ and $\top$ otherwise. We prove $\ell \notin \mathsf{NF}(\mathcal{S}')$ for every rule $\ell \to r \in \mathcal{S}$, by induction on $(\ell, b_{\ell\to r})$ with respect to the lexicographic combination of $\rhd$ and the order where $\top > \bot$.

- If $\ell \to r \in \mathcal{Q}$ two cases can be distinguished. If $\ell \notin \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$ then $\ell \rhd \ell'$ for some rule $\ell' \to r' \in \mathcal{S}$ and thus $\ell' \notin \mathsf{NF}(\mathcal{S}')$ by the induction hypothesis. Hence also $\ell \notin \mathsf{NF}(\mathcal{S}')$. If $\ell \in \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$ then, by construction of $\mathcal{R}'$, there is some rule $\ell \to r' \in \mathcal{R}'$ (modulo renaming), so $\ell \notin \mathsf{NF}(\mathcal{S}')$.
- If $\ell \to r \notin \mathcal{Q}$ then $\ell = u\sigma$ and $r = v\sigma$ for some equation $u \approx v \in \mathcal{E}^{\pm}$ and substitution σ such that $\ell > r$. We distinguish two cases. First, if $u \in \mathsf{NF}(\mathcal{R}')$ then $u = u{\downarrow_{\mathcal{R}'}}$. We have $\ell > r \geqslant v{\downarrow_{\mathcal{R}'}}\sigma$ because $\mathcal{R}' \subseteq >$ and hence $u \neq v{\downarrow_{\mathcal{R}'}}$. It follows that $u \approx v{\downarrow_{\mathcal{R}'}} \in \mathcal{E}'^{\pm}$ and thus $\ell \to v{\downarrow_{\mathcal{R}'}}\sigma \in \mathcal{E}'^{\succ}$. Hence $\ell \notin \mathsf{NF}(\mathcal{S}')$. Second, if $u \notin \mathsf{NF}(\mathcal{R}')$ then $u \notin \mathsf{NF}(\dot{\mathcal{Q}})$ since $\mathcal{R}' \subseteq \dot{\mathcal{Q}}$. So there exists a rule $\ell' \to r' \in \mathcal{Q}$ such that $u \unrhd \ell'$. Clearly $\ell \unrhd \ell'$. Since $\ell \to r \notin \mathcal{Q}$, the induction hypothesis yields $\ell' \notin \mathsf{NF}(\mathcal{S}')$. Hence also $\ell \notin \mathsf{NF}(\mathcal{S}')$.

We next establish the inclusion $\to_{\mathcal{S}'} \subseteq \leftrightarrow^*_{\mathcal{S}}$ on ground terms. We have $\mathcal{E}' \cup \mathcal{R}' \subseteq \leftrightarrow^*_{\mathcal{E}\cup\mathcal{R}}$ by construction. For ground terms s and t, a step $s \to_{\mathcal{S}'} t$ implies $s \leftrightarrow_{\mathcal{E}'\cup\mathcal{R}'} t$ and hence existence of a conversion $s \leftrightarrow^*_{\mathcal{E}\cup\mathcal{R}} t$. We can also obtain such a conversion where all intermediate terms are ground by replacing every variable with some ground term. Since the reduction order $>$ is ground-total, $\to_{\mathcal{E}\cup\mathcal{R}} \subseteq \leftrightarrow^{=}_{\mathcal{S}}$ holds on ground terms. Hence there is a conversion $s \leftrightarrow^*_{\mathcal{S}} t$.

Moreover, the system $\mathcal{S}'$ is clearly terminating as it is included in $>$. Thus the result follows from Lemma 4.4(2), viewing $\mathcal{S}$ and $\mathcal{S}'$ as ARSs on ground terms. □

We illustrate the transformation of Definition 7.19 on a concrete example.

Example 7.21. Consider the following system with $\mathcal{R}$ consisting of one rule and $\mathcal{E}$ consisting of three equations:

$$\mathsf{s}(\mathsf{s}(x)) + \mathsf{s}(x) \to \mathsf{s}(x) + \mathsf{s}(\mathsf{s}(x)) \qquad x + \mathsf{s}(y) \approx \mathsf{s}(x+y) \qquad x + y \approx y + x$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x+y)$$

It is ground-complete for the lexicographic path order [KL80] with $+ > \mathsf{s}$ as precedence. We have $\mathcal{Q} = \mathcal{R} \cup \{x + \mathsf{s}(y) \to \mathsf{s}(x+y),\ \mathsf{s}(x) + y \to \mathsf{s}(x+y)\}$. Since the term $\mathsf{s}(\mathsf{s}(x)) + \mathsf{s}(x)$

is reducible by the rule $\mathsf{s}(x) + x \to x + \mathsf{s}(x) \in \mathcal{S}$ and $\mathsf{s}(\mathsf{s}(x)) + \mathsf{s}(x) \rhd \mathsf{s}(x) + x$, the rule of $\mathcal{R}$ does not remain in $\mathcal{R}'$. Hence, $\mathcal{R}' = \{x + \mathsf{s}(y) \to \mathsf{s}(x + y),\ \mathsf{s}(x) + y \to \mathsf{s}(x + y)\}$ and $\mathcal{E}' = \{x + y \approx y + x\}$.

One may wonder whether $\mathcal{R}'$ can simply be defined as $\ddot{\mathcal{Q}}$ instead of imposing a strict encompassment condition. The following example shows that this destroys reducibility.

Example 7.22. Consider the following system where $\mathcal{R}$ consists of two rules and $\mathcal{E}$ consists of one equation:

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad \mathsf{f}(x, y) \to \mathsf{g}(y) \qquad \mathsf{g}(x) \approx \mathsf{g}(y)$$

Then $\mathcal{E}^{>} \cup \mathcal{R}$ is ground-complete if $>$ is the lexicographic path order with $\mathsf{f} > \mathsf{g}$ as precedence. We have $\mathcal{R}' = \dot{\mathcal{Q}} = \mathcal{Q} = \mathcal{R}$ and $\mathcal{E}' = \mathcal{E}$ but $\ddot{\mathcal{Q}} = \varnothing$.

Note that we obtain an equivalent ground-complete system if we add, for instance, an equation $\mathsf{g}(\mathsf{g}(x)) \approx \mathsf{g}(y)$. This shows that even systems which are simplified with respect to the procedure suggested by Theorem 7.20 are not unique.

This section resumes our results on ordered completion [HMSW17]. Like in Sections 3 and 6, our proofs deviate from the standard approach [BDP89] in that we avoid proof orders in favor of different, simpler orderings as required, together with source decreasingness. Again, we also support prime critical pairs. For Theorem 7.17 and the interreduction result of Theorem 7.20 we are not aware of earlier references in the literature.

8. Completeness Results for Ordered Completion

Ordered completion never fails and its limit always constitutes a ground-complete system. On the other hand, if there is a *complete* presentation that is compatible with the employed reduction order, does ordered completion also produce a complete presentation, ending with $\mathcal{E}_\omega = \varnothing$? In this section we revisit two results from the literature which provide sufficient conditions for ordered completion to always derive a complete system, independent of the strategy employed by a completion procedure. In Section 8.1 we reprove the result by Bachmair, Dershowitz, and Plaisted for the case where the reduction order is ground total [BDP89]. The corresponding result by Devie [Dev91] for linear systems is considered in Section 8.2.

8.1. Ground-Total Orders.

In this subsection we consider a fair run Γ of ordered completion

$$(\mathcal{E}_0, \mathcal{R}_0) \ \vdash_{\mathsf{o}} \ (\mathcal{E}_1, \mathcal{R}_1) \ \vdash_{\mathsf{o}} \ (\mathcal{E}_2, \mathcal{R}_2) \ \vdash_{\mathsf{o}} \ \cdots$$

with respect to a ground-total reduction order $>$. If $\mathcal{E}_\omega = \varnothing$ then the TRS $\mathcal{R}_\omega$ is a complete presentation of $\mathcal{E}_0$ by Theorem 7.17. According to Bachmair et al. [BDP89, Theorem 2], under certain conditions fair runs always conclude with $\mathcal{E}_\omega = \varnothing$ whenever there exists a complete presentation of $\mathcal{E}_0$ compatible with $>$. In the remainder of this subsection we give a formalized proof of this result. Like the original proof, it is based on the idea that ground-completeness of $\mathcal{R}_\omega$ is preserved under signature extension with constants. Let $\mathcal{K}$ be a set of different fresh constants $\hat{x}$ for every variable $x \in \mathcal{V}$. We first show that the reduction order $>$ can be extended to a ground-total order on the signature augmented by $\mathcal{K}$ such that minimum constants are preserved.

Lemma 8.1. *There exists a ground-total reduction order* $>^{\mathcal{K}}$ *on* $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$ *such that* $> \subseteq >^{\mathcal{K}}$ *and the minimum constant with respect to* $>$ *is also minimum in* $>^{\mathcal{K}}$*.*

Proof. Let $\bot \in \mathcal{F}$ be the minimum constant with respect to $>$. We consider the KBO $\sqsupset_{\mathsf{kbo}}$ with weights $w_0 = 1$ and $w(f) = 1$ for all $f \in \mathcal{F} \cup \mathcal{K}$ together with a precedence $\sqsupset$ which is total on $\mathcal{F} \cup \mathcal{K}$, has $\bot$ as the minimum element, and satisfies $\hat{x} \sqsupset f$ for all $f \in \mathcal{F}$ and $\hat{x} \in \mathcal{K}$. Given a term $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$, we write $t_\bot$ for the term obtained from t by replacing every constant in $\mathcal{K}$ with $\bot$. Furthermore, we define $s >^{\mathcal{K}} t$ as $s_\bot > t_\bot$, or both $s_\bot = t_\bot$ and $s \sqsupset_{\mathsf{kbo}} t$. We show that $>^{\mathcal{K}}$ is a ground-total reduction order with the stated properties. Ground totality of $>^{\mathcal{K}}$ follows from ground totality of $\sqsupset_{\mathsf{kbo}}$ given the total precedence. Well-foundedness holds by construction as a lexicographic combination of well-founded relations. Closure under substitutions is satisfied because it holds for both $>$ and $\sqsupset_{\mathsf{kbo}}$, and $s_\bot = t_\bot$ implies $s\sigma_\bot = t\sigma_\bot$. Similar arguments apply to closure under contexts and transitivity. By construction of $\sqsupset$ and the definition of $>^{\mathcal{K}}$, the constant $\bot$ is still minimal. Moreover $>^{\mathcal{K}}$ extends $>$ because $s > t$ implies $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, so $s_\bot = s > t = t_\bot$ and hence $s >^{\mathcal{K}} t$. □

We write $\hat{t}$ for the ground term that is obtained from t by replacing every variable x by the constant $\hat{x}$. In the next lemma we verify some basic properties related to this *grounding* operation.

Lemma 8.2. *Let* $\mathcal{R}$ *be a TRS over a signature* $\mathcal{F}$ *and let* $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$*.*

(1) If $s > t$ *then* $\hat{s} >^{\mathcal{K}} \hat{t}$*.*

(2) Suppose $s \neq t$*. Then* $s \to_{\mathcal{R}} t$ *if and only if* $\hat{s} \to_{\mathcal{R}} \hat{t}$*.*

Proof.

(1) Suppose $s > t$. Lemma 8.1 yields $s >^{\mathcal{K}} t$ and, because $>^{\mathcal{K}}$ is closed under substitutions, $\hat{s} >^{\mathcal{K}} \hat{t}$.

(2) We consider the two implications separately.
- If $s \to_{\mathcal{R}} t$ then $\mathcal{V}\mathsf{ar}(t) \subseteq \mathcal{V}\mathsf{ar}(s)$. Let σ be a substitution such that $\hat{s} = s\sigma$. We have $\hat{t} = t\sigma$ and thus $\hat{s} = s\sigma \to_{\mathcal{R}} t\sigma = \hat{t}$.
- Conversely, if $\hat{s} \to_{\mathcal{R}} \hat{t}$ then $\hat{s}|_p = \ell\sigma$ and $\hat{t} = \hat{s}[r\sigma]_p$ for some rule $\ell \to r \in \mathcal{R}$, position p, and substitution σ. We denote the substitution $\{x \mapsto \phi(\sigma(x)) \mid x \in \mathcal{V}\}$ by σ_ϕ. Here $\phi(u)$ denotes the term obtained from u after replacing every constant $\hat{x}$ of $\mathcal{K}$ by x. Because $s|_p = \phi(\hat{s}|_p) = \phi(\ell\sigma) = \ell\sigma_\phi$ and $t = \phi(\hat{t}) = \phi(\hat{s}[r\sigma]_p) = s[r\sigma_\phi]_p$, we obtain $s \to_{\mathcal{R}} t$ as desired. □

It is not hard to see that the TRS $\mathcal{S}_\omega$ still constitutes a ground-complete presentation of $\mathcal{E}_0$ when considered over the extended signature, as shown below.

Lemma 8.3. *The TRS* $\mathcal{S}_\omega$ *is ground-complete over* $\mathcal{F} \cup \mathcal{K}$ *and* $\leftrightarrow^*_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} = \leftrightarrow^*_{\mathcal{E}_0}$*.*

Proof. Since $>^{\mathcal{K}}$ contains $>$ by Lemma 8.1, the run Γ is also a valid run with respect to $>^{\mathcal{K}}$. It is moreover fair since $> \subseteq >^{\mathcal{K}}$ implies $\mathsf{PCP}_{>^{\mathcal{K}}}(\mathcal{E}) \subseteq \mathsf{PCP}_{>}(\mathcal{E})$ for any set of equations $\mathcal{E}$, by Definition 7.11. Hence the result follows from Theorem 7.16. □

An important observation for the completeness proof below is that normal forms with respect to the final system $\mathcal{S}_\omega$ and with respect to the union $\mathcal{S}_\infty$ of intermediate systems coincide, as shown below.

Lemma 8.4. *The inclusion* $\mathsf{NF}(\mathcal{S}_\omega) \subseteq \mathsf{NF}(\mathcal{S}_\infty)$ *holds.*

Proof. The result is an immediate consequence of the following two claims:

(a) If $\ell \approx r \in \mathcal{E}_\infty^\pm$, $\ell \in \mathsf{NF}(\mathcal{R}_\infty)$, and $\ell\sigma > r\sigma$ then $\ell\sigma \notin \mathsf{NF}(\mathcal{E}_\omega^>)$.

(b) If $\ell \to r \in \mathcal{R}_\infty$ then $\ell \notin \mathsf{NF}(\mathcal{S}_\omega)$.

For claim (a) we use induction on $\{\ell, r\}$ with respect to $\succ_{\mathsf{mul}}$. If $\ell \approx r \in \mathcal{E}_\omega^\pm$ the result is immediate. Otherwise, $\ell \approx r \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}$ or $r \approx \ell \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}$ for some $i \geqslant 0$. Without loss of generality we assume the former since the latter case is similar. From Lemma 7.3(2) we obtain $\ell\ (\to_{\mathcal{S}_{i+1}}^{\rhd_1} \cdot \mathcal{E}_{i+1}{}^\pm)^\pm\ r$, $\ell \to r \in \mathcal{R}_{i+1}$, $r \to \ell \in \mathcal{R}_{i+1}$, or $\ell = r$. The latter two cases are impossible because of the assumption $\ell\sigma > r\sigma$ and the inclusion $\mathcal{R}_{i+1} \subseteq \mathcal{R}_\infty \subseteq >$. Also $\ell \to r \in \mathcal{R}_{i+1}$ is impossible because of the assumption $\ell \in \mathsf{NF}(\mathcal{R}_\infty)$.

- Suppose $\ell \to_{\mathcal{S}_{i+1}}^{\rhd_1} u$ and $u \approx r \in \mathcal{E}_{i+1}^\pm$ for some term u. The step $\ell \to_{\mathcal{S}_{i+1}} u$ cannot use a rule in $\mathcal{R}_{i+1}$ because $\ell \in \mathsf{NF}(\mathcal{R}_\infty)$. So there must be an equation $\ell' \approx r' \in \mathcal{E}_{i+1}^\pm$, a substitution τ, and a position p in ℓ such that $\ell|_p = \ell'\tau$, $u|_p = r'\tau$, $\ell'\tau > r'\tau$, and $\ell \rhd \ell'$. Because of $\ell \trianglerighteq \ell'\tau > r'\tau \trianglerighteq r'$ we have $\ell \succ r'$, and therefore $\{\ell, r\} \succ_{\mathsf{mul}} \{\ell', r'\}$. Moreover, $\ell' \in \mathsf{NF}(\mathcal{R}_\infty)$. The induction hypothesis yields $\ell'\tau \notin \mathsf{NF}(\mathcal{E}_\omega^>)$. Since $\ell \trianglerighteq \ell'\tau$, we have $\ell \notin \mathsf{NF}(\mathcal{E}_\omega^>)$ and thus also $\ell\sigma \notin \mathsf{NF}(\mathcal{E}_\omega^>)$.
- In the remaining case we have $r \to_{\mathcal{S}_{i+1}}^{\rhd_1} u$ and $u \approx \ell \in \mathcal{E}_{i+1}^\pm$ for some term u. We have $r > u$ and thus also $r \succ u$ and $\{\ell, r\} \succ_{\mathsf{mul}} \{\ell, u\}$. Because $\ell\sigma > r\sigma > u\sigma$, the result follows from the induction hypothesis.

For claim (b) we use induction on (ℓ, r) with respect to $\succ_{\mathsf{lex}}$. If $\ell \to r \in \mathcal{R}_\omega$ then $\ell \notin \mathsf{NF}(\mathcal{S}_\omega)$ trivially holds. Otherwise, $\ell \to r \in \mathcal{R}_i \setminus \mathcal{R}_{i+1}$ for some $i \geqslant 0$. From Lemma 7.3(3) we obtain $\ell \to_{\mathcal{S}_{i+1}}^{\rhd_2} \cdot \mathcal{E}_{i+1}\ r$ or $\ell\ \mathcal{R}_{i+1} \cdot {}_{\mathcal{S}_{i+1}}\!\leftarrow r$. In the latter case there is a term u such that $\ell \to u \in \mathcal{R}_{i+1}$ and $r \to_{\mathcal{S}_{i+1}} u$. Since this implies $r > u$ and thus $(\ell, r) \succ_{\mathsf{lex}} (\ell, u)$, we obtain $\ell \notin \mathsf{NF}(\mathcal{S}_\omega)$ from the induction hypothesis. In the former case there is a term u such that $\ell \to_{\mathcal{S}_{i+1}}^{\rhd_2} u$ and $u \approx r \in \mathcal{E}_{i+1}$. If the step $\ell \to_{\mathcal{S}_{i+1}} u$ uses a rule $\ell' \to r' \in \mathcal{R}_{i+1}$ then the result follows from the induction hypothesis because $\ell \rhd \ell'$ implies $(\ell, r) \succ_{\mathsf{lex}} (\ell', r')$, and $\ell' \notin \mathsf{NF}(\mathcal{S}_\omega)$ implies $\ell \notin \mathsf{NF}(\mathcal{S}_\omega)$. Otherwise, there exist an equation $\ell' \approx r' \in \mathcal{E}_{i+1}$, a position p in ℓ, and a substitution σ such that $\ell|_p = \ell'\sigma$, $r|_p = r'\sigma$, $\ell'\sigma > r'\sigma$, and $\ell \rhd \ell'$. If $\ell' \in \mathsf{NF}(\mathcal{R}_\infty)$ then we obtain $\ell'\sigma \notin \mathsf{NF}(\mathcal{E}_\omega^>)$ from claim (a) and thus $\ell \notin \mathsf{NF}(\mathcal{S}_\omega)$ because $\ell \trianglerighteq \ell'\sigma$. If $\ell' \notin \mathsf{NF}(\mathcal{R}_\infty)$ then there exists some rule $\ell'' \to r'' \in \mathcal{R}_\infty$ such that $\ell' \trianglerighteq \ell''$. In this case we have $\ell \succ \ell''$ and thus $(\ell, r) \succ_{\mathsf{lex}} (\ell'', r'')$. We obtain $\ell'' \notin \mathsf{NF}(\mathcal{S}_\omega)$ from the induction hypothesis. Hence also $\ell \notin \mathsf{NF}(\mathcal{S}_\omega)$. ☐

Corollary 8.5. *The identity* $\mathsf{NF}(\mathcal{S}_\omega) = \mathsf{NF}(\mathcal{S}_\infty)$ *holds.* ☑

Proof. We obtain $\mathsf{NF}(\mathcal{S}_\infty) \subseteq \mathsf{NF}(\mathcal{S}_\omega)$ from the inclusion $\to_{\mathcal{S}_\omega} \subseteq \to_{\mathcal{S}_\infty}$ and hence the result follows from Lemma 8.4. ☐

Hereafter we assume that there is a complete presentation $\mathcal{R}$ of $\mathcal{E}_0$ with $\mathcal{R} \subseteq >$. We next show that grounded terms which are $\mathcal{S}_\omega$-normal forms are also $\mathcal{R}$-normal forms.

Lemma 8.6. *If* $\hat{t} \in \mathsf{NF}(\mathcal{S}_\omega)$ *then* $\hat{t} \in \mathsf{NF}(\mathcal{R})$. ☑

Proof. Suppose $\hat{t} \in \mathsf{NF}(\mathcal{S}_\omega)$ but $\hat{t} \notin \mathsf{NF}(\mathcal{R})$, so $\hat{t} \to_{\mathcal{R}} u$ for some term u. Since $\hat{t}$ is ground and $\mathcal{R}$ is terminating, also u is ground. We obtain $\hat{t} \downarrow_{\mathcal{S}_\omega} u$ from the ground-completeness of $\mathcal{S}_\omega$ (Lemma 8.3). Since $\hat{t} >^{\mathcal{K}} u$ by the global assumption $\mathcal{R} \subseteq >$ and Lemma 8.2(1), the joining sequence cannot be of the form $\hat{t}\ {}_{\mathcal{S}_\omega}^{\;*}\!\leftarrow u$ as this would imply $u \geqslant \hat{t}$ and thus $u \geqslant^{\mathcal{K}} \hat{t}$, contradicting the well-foundedness of $>^{\mathcal{K}}$. Therefore we must have $\hat{t} \to_{\mathcal{S}_\omega}^+ \cdot {}_{\mathcal{S}_\omega}^{\;*}\!\leftarrow u$ which means that $\hat{t}$ is reducible in $\mathcal{S}_\omega$, contradicting the assumption $\hat{t} \in \mathsf{NF}(\mathcal{S}_\omega)$. ☐

The preliminary results collected so far now lead to the following key observation: If a grounded term $\hat{s}$ is reducible then so is its (possibly non-ground) counterpart s. In Lemma 8.8 below we can then connect $\mathcal{R}$-reducibility to $\mathcal{S}_\omega$-reducibility for terms over the original signature.

Lemma 8.7. *If $\hat{s} \to_{\mathcal{S}_\infty} \hat{t}$ then $s \notin \mathsf{NF}(\mathcal{S}_\infty)$.*

Proof. There exist an equation $\ell \approx r \in \mathcal{E}_\infty^\pm \cup \mathcal{R}_\infty$, a position p, and a substitution σ such that $\hat{s}|_p = \ell\hat{\sigma}$, $\hat{t} = \hat{s}[r\hat{\sigma}]_p$, and $\ell\hat{\sigma} >^{\mathcal{K}} r\hat{\sigma}$. We perform induction on $\hat{t}$ with respect to $>^{\mathcal{K}}$. If $p \neq \epsilon$ then $\hat{t} \rhd r\hat{\sigma}$ and thus $\hat{t} >^{\mathcal{K}} r\hat{\sigma}$ because $>^{\mathcal{K}}$ is a ground-total reduction order. The induction hypothesis yields $\ell\sigma \notin \mathsf{NF}(\mathcal{S}_\infty)$, which implies $s \notin \mathsf{NF}(\mathcal{S}_\infty)$. So in the following we assume that the step $\hat{s} \to_{\mathcal{S}_\infty} \hat{t}$ takes place at the root position. If $s > t$ then $s \to t \in \mathcal{S}_\infty$, from which the claim is immediate. This covers the case $\ell \approx r \in \mathcal{R}_\infty$, so if $s \not> t$ then $\ell \approx r \in \mathcal{E}_\infty^\pm$. We distinguish two cases, $\hat{t} \in \mathsf{NF}(\mathcal{S}_\infty)$ and $\hat{t} \notin \mathsf{NF}(\mathcal{S}_\infty)$.

- If $\hat{t} \in \mathsf{NF}(\mathcal{S}_\infty)$ then $\hat{t} \in \mathsf{NF}(\mathcal{S}_\omega)$ by Corollary 8.5 and thus $\hat{t} \in \mathsf{NF}(\mathcal{R})$ by Lemma 8.6. From Lemma 8.3 and the fact that $\mathcal{R}$ is a complete presentation of $\mathcal{E}_0$ we obtain $\hat{s} \to_{\mathcal{R}}^+ \hat{t}$. The latter implies $s \to_{\mathcal{R}}^+ t$ by Lemma 8.2(2) and thus $s > t$, which is a contradiction.
- Suppose $\hat{t} \notin \mathsf{NF}(\mathcal{S}_\infty)$. We distinguish two further cases, depending on whether or not $\ell \approx r$ belongs to $\mathcal{E}_\omega^\pm$.

  If $\ell \approx r \notin \mathcal{E}_\omega^\pm$ then $\ell \approx r \in (\mathcal{E}_i \setminus \mathcal{E}_{i+1})^\pm$ for some $i \geqslant 0$. From Lemma 7.3(2) we obtain $\ell\ (\to_{\mathcal{S}_{i+1}} \cdot \mathcal{E}_{i+1}^\pm)^\pm\ r$, $\ell \to r \in \mathcal{R}_{i+1}$, $r \to \ell \in \mathcal{R}_{i+1}$, or $\ell = r$. The last two cases contradict $\hat{s} >^{\mathcal{K}} \hat{t}$. If $\ell \to_{\mathcal{S}_{i+1}} \cdot \mathcal{E}_{i+1}^\pm\ r$ or $\ell \to r \in \mathcal{R}_{i+1}$ then $\ell \notin \mathsf{NF}(\mathcal{S}_\infty)$ and thus $s = \ell\sigma \notin \mathsf{NF}(\mathcal{S}_\infty)$. Otherwise, $r \to_{\mathcal{S}_{i+1}} u$ for some term u with $u \approx \ell \in \mathcal{E}_{i+1}^\pm$. We have $s = \ell\sigma \leftrightarrow_{\mathcal{E}_\infty} u\sigma\ {}_{\mathcal{S}_\infty}\!\leftarrow r\sigma = t$ and thus $\hat{s} = \ell\hat{\sigma} \to_{\mathcal{S}_\infty} u\hat{\sigma} = \widehat{u\sigma}$ and $\hat{t} = r\hat{\sigma} >^{\mathcal{K}} u\hat{\sigma}$. The induction hypothesis yields $s \notin \mathsf{NF}(\mathcal{S}_\infty)$.

  In the second case we assume $\ell \approx r \in \mathcal{E}_\omega^\pm$. From the assumption $\hat{t} \notin \mathsf{NF}(\mathcal{S}_\infty)$ we obtain a term u such that $\hat{t} \to_{\mathcal{S}_\infty} \hat{u}$. We have $\hat{t} >^{\mathcal{K}} \hat{u}$ and thus $t \notin \mathsf{NF}(\mathcal{S}_\omega)$ by the induction hypothesis. Consider an innermost $\mathcal{S}_\omega$-step starting from t, say $t \xrightarrow{\mathsf{i}}_{\mathcal{S}_\omega} v$, such that there exists a peak

  $$s = \ell\sigma \leftrightarrow_{\ell \approx r} r\sigma = t \to_{\mathcal{S}_\omega}^{q} v \qquad (\star)$$

  with $l\sigma = s \not> t = r\sigma$. If the two steps form an overlap then we have $s \leftrightarrow_{\mathsf{PCP}_>(\mathcal{E}_\omega \cup \mathcal{R}_\omega)} v$ since $t \xrightarrow{\mathsf{i}}_{\mathcal{S}_\omega} v$ is innermost, and thus $s \leftrightarrow_{\mathcal{E}_\infty} v$ or $s \downarrow_{\mathcal{S}_\omega} v$ is obtained from the fairness of the run. In the former case, since $\hat{s} >^{\mathcal{K}} \hat{t} >^{\mathcal{K}} \hat{v}$, we have $\hat{s} \to_{\mathcal{S}_\infty} \hat{v}$ and thus the induction hypothesis applies. If on the other hand $s \downarrow_{\mathcal{S}_\omega} v$ then we cannot have $v \to_{\mathcal{S}_\omega}^* s$ as this would imply $v \geqslant s$, contradicting $\hat{s} >^{\mathcal{K}} \hat{v}$ because $>$ and $>^{\mathcal{K}}$ are compatible by Lemma 8.2(1). So s must be $\mathcal{S}_\omega$-reducible.

  Otherwise, the peak $(\star)$ constitutes a variable overlap, so there is some variable $x \in \mathcal{V}\mathsf{ar}(r)$ and positions q_1 and q_2 such that $r|_{q_1} = x$ and $q = q_1q_2$. If $x \notin \mathcal{V}\mathsf{ar}(\ell)$ then $s \leftrightarrow_{\mathcal{E}_\infty} v$ and the induction hypothesis applies as before. Otherwise, $s = \ell\sigma$ is reducible in $\mathcal{S}_\infty$. □

Lemma 8.8. *The inclusion $\mathsf{NF}(\mathcal{S}_\omega) \subseteq \mathsf{NF}(\mathcal{R})$ holds.*

Proof. Suppose $t \to_{\mathcal{R}} u$, so $t > u$ and thus also $\hat{t} >^{\mathcal{K}} \hat{u}$. From Lemma 8.3 we obtain $\hat{t} \downarrow_{\mathcal{S}_\omega} \hat{u}$. Like in the proof of Lemma 8.6, $\hat{u} \to_{\mathcal{S}_\omega}^* \hat{t}$ would imply $\hat{u} \geqslant^{\mathcal{K}} \hat{t}$, contradicting well-foundedness of $>^{\mathcal{K}}$. Therefore the joining sequence must be of the shape $\hat{t} \to_{\mathcal{S}_\omega}^+ \cdot\ {}_{\mathcal{S}_\omega}^{\ *}\!\leftarrow \hat{u}$ and thus $\hat{t}$ is reducible in $\mathcal{S}_\omega$ and also in $\mathcal{S}_\infty$ because $\mathcal{S}_\omega \subseteq \mathcal{S}_\infty$. Lemma 8.7 yields $t \notin \mathsf{NF}(\mathcal{S}_\infty)$ and thus $t \notin \mathsf{NF}(\mathcal{S}_\omega)$ by Corollary 8.5. □

We call $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ *simplified* if $\mathcal{R}_\omega$ is reduced and all equations in $\mathcal{E}_\omega$ are irreducible with respect to $\mathcal{R}_\omega$, unorientable with respect to $>$, and non-trivial. We use the following auxiliary result before proving the main completeness theorem.

Lemma 8.9. *If $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified then $\mathsf{NF}(\mathcal{R}_\omega) \subseteq \mathsf{NF}(\mathcal{S}_\omega)$.*

Proof. Suppose $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified and let $s \in \mathsf{NF}(\mathcal{R}_\omega)$. We prove $s \in \mathsf{NF}(\mathcal{S}_\omega)$ by induction on s with respect to $\succ$. If $s \notin \mathsf{NF}(\mathcal{S}_\omega)$ then there exist an equation $\ell \approx r \in \mathcal{E}_\omega^{\pm}$, a context C, and a substitution σ such that $s = C[\ell\sigma]$ and $\ell\sigma > r\sigma$. Since $s \trianglerighteq \ell\sigma > r\sigma \trianglerighteq r$ implies $s \succ r$ and $r \in \mathsf{NF}(\mathcal{R}_\omega)$ by the assumption that $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified, the induction hypothesis yields $r \in \mathsf{NF}(\mathcal{S}_\omega)$. Hence $r \in \mathsf{NF}(\mathcal{R})$ by Lemma 8.8. Since $\mathcal{R}$ is a complete presentation of $\mathcal{E}_0$, we have $\ell \downarrow_{\mathcal{R}} r$ and thus $\ell \to_{\mathcal{R}}^* r$. From $\mathcal{R} \subseteq >$ we infer $\ell > r$ or $\ell = r$, contradicting the assumption that $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified. □

Theorem 8.10. *If $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified then $\mathcal{E}_\omega = \varnothing$ and $\mathcal{R}_\omega$ is literally similar to $\mathcal{R}$.*

Proof. Suppose $\mathcal{E}_\omega \neq \varnothing$ and let $s \approx t$ be an equation in $\mathcal{E}_\omega$. The terms s and t are $\mathcal{R}$-normal forms by the assumption that $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified in combination with Lemmata 8.8 and 8.9. Since $\mathcal{R}$ is a complete presentation of $\mathcal{E}_0$, we obtain $s = t$, contradicting the assumption that $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified. Hence, $\mathcal{E}_\omega = \varnothing$ and therefore $\mathcal{R}_\omega$ is a complete presentation of $\mathcal{E}_0$ by Theorem 7.17. Since $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified, $\mathcal{R}_\omega$ is even a canonical presentation of $\mathcal{E}_0$. As $\mathcal{R}_\omega \subseteq >$, literal similarity of $\mathcal{R}_\omega$ and $\mathcal{R}$ is concluded by Theorem 4.11. □

A run of ordered completion is called *simplifying* if its limit $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified.

Example 8.11. Consider again the ES $\mathcal{E}$ from Example 7.1 and its complete presentation $\mathcal{R}$, which cannot be derived using standard completion. Termination of $\mathcal{R}$ can be shown by a suitable KBO. Thus, by Theorem 8.10 any fair and simplifying run of ordered completion on $\mathcal{E}$ using the same order will succeed with a variant of $\mathcal{R}$, independent of the employed strategy.

The results in this subsection are due to Bachmair, Dershowitz, and Plaisted [BDP89]. However, our proof is structured into many preliminary results, as opposed to the monolithic original version, and we fill in numerous details omitted in the original version.

8.2. Linear Systems.

The previously presented correctness and completeness results (Theorems 7.16 and 8.10) do not state any properties of the system obtained when running $\mathsf{KB_o}$ with a reduction order that is not ground-total. The following example from Devie [Dev91] shows that the restriction to ground-total orders can actually be severe.

Example 8.12. Consider the ES $\mathcal{E}$ consisting of the following equations:

$$
\begin{array}{lll}
\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_2(x))))) & \mathsf{h}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(x)) & \mathsf{f}_1(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_1(x))))) & \mathsf{h}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(x)) & \mathsf{f}_2(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{g}_1(\mathsf{a}) \approx \mathsf{a} & \mathsf{h}_1(\mathsf{a}) \approx \mathsf{a} & \mathsf{i}_1(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{g}_2(\mathsf{a}) \approx \mathsf{a} & \mathsf{h}_2(\mathsf{a}) \approx \mathsf{a} & \mathsf{i}_2(\mathsf{a}) \approx \mathsf{a}
\end{array}
$$

When orienting all equations from left to right we obtain a TRS $\mathcal{R}$ which is easily shown to be terminating by automatic tools. As all critical pairs are joinable it is confluent, and thus canonical since it is also reduced. However, $\mathcal{R}$ cannot be oriented by any ground-total reduction order $>$. We have $\mathsf{i}_1(\mathsf{a}) \leftrightarrow^*_{\mathcal{E}} \mathsf{i}_2(\mathsf{a})$ but neither $\mathsf{i}_2(\mathsf{a}) > \mathsf{i}_1(\mathsf{a})$ nor $\mathsf{i}_1(\mathsf{a}) > \mathsf{i}_2(\mathsf{a})$ can hold; using the rule $\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \to \mathsf{g}_1(\mathsf{i}_1(\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_2(x)))))$, the former would imply

$$\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_2(\mathsf{a}))) > \mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_1(\mathsf{a}))) > \mathsf{g}_1(\mathsf{i}_1(\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_2(\mathsf{a})))))$$

which contradicts well-foundedness, and for the latter a similar argument applies. As a matter of fact, in [Dev91] it is shown that any $\mathsf{KB_o}$ run starting from $\mathcal{E}$ and using a *ground-total* reduction order will fail to generate a finite result.

Devie [Dev91] gives a second sufficient condition for an ordered completion procedure to compute a canonical result whenever such a presentation exists, without imposing any restriction on the reduction order. Instead, the set of input equalities $\mathcal{E}_0$ is required to be linear, and Devie considers an ordered completion inference system with a modified deduction rule to ensure that linearity is preserved. He moreover shows that under these circumstances a relaxed fairness condition is sufficient. In this section we give a new proof of this result which has been formalized. First we recall Devie's inference system.

Definition 8.13 (Linear Ordered Completion ☑)**.** The inference system $\mathsf{KB_l}$ of linear ordered completion consists of the rules orient, delete, compose, simplify, and collapse$_\rhd$ of $\mathsf{KB_i}$ (Definition 6.2) together with the following modified deduction rule:

$$\mathsf{deduce_l} \qquad \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \qquad \text{if } s \xleftarrow[\mathcal{E}^{\pm}\cup\mathcal{R}]{} \cdot \xrightarrow[\mathcal{E}^{\pm}\cup\mathcal{R}]{} t \text{ and } s \approx t \text{ is linear}$$

We write $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{l}} (\mathcal{E}', \mathcal{R}')$ if $(\mathcal{E}', \mathcal{R}')$ can be reached from $(\mathcal{E}, \mathcal{R})$ by employing one of the inference rules of Definition 8.13.

Lemma 8.14. *The inclusion* $\mathsf{KB_l} \subseteq \mathsf{KB_o}$ *holds.* ☑

Note that in contrast to the ordered completion system $\mathsf{KB_o}$, ordered rewriting using orientable instances of $\mathcal{E}$ is not permitted in compose, simplify, and collapse$_\rhd$. This is because ordered rewrite steps need not preserve linearity as stated in Lemma 8.15 below. For example, a compose step in $\mathsf{KB_o}$ on the linear rule $\mathsf{g}(x) \to \mathsf{f}(\mathsf{f}(x))$ using the linear equation $\mathsf{f}(x) \approx \mathsf{f}(y)$ may result in the nonlinear rule $\mathsf{g}(x) \to \mathsf{f}(\mathsf{h}(x,x))$ when $\mathsf{h}(x,x)$ is substituted for the variable y and a reduction order $>$ is used such that $\mathsf{f}(\mathsf{f}(x)) > \mathsf{f}(\mathsf{h}(x,x))$.

With these restrictions, it is not hard to prove that inference steps preserve linearity.

Lemma 8.15. *If* $\mathcal{E} \cup \mathcal{R}$ *is linear and* $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{l}} (\mathcal{E}', \mathcal{R}')$ *then* $\mathcal{E}' \cup \mathcal{R}'$ *is linear.* ☑

From now on we consider $\mathcal{E}_0 \cup \mathcal{R}_0$ to be linear.

Definition 8.16. An extended overlap (Definition 7.11) which satisfies $\ell_1 > r_1$ and $r_2 \not> \ell_2$, or $\ell_2 > r_2$ and $r_1 \not> \ell_1$ gives rise to a *linear critical pair* [Dev91]. The set of all linear critical pairs originating from equations in $\mathcal{E}$ is denoted $\mathsf{LCP}_>(\mathcal{E})$. An infinite run

$$(\mathcal{E}_0, \mathcal{R}_0) \;\vdash_{\mathsf{l}}\; (\mathcal{E}_1, \mathcal{R}_1) \;\vdash_{\mathsf{l}}\; (\mathcal{E}_2, \mathcal{R}_2) \;\vdash_{\mathsf{l}}\; \cdots$$

is *fair* if the inclusion $\mathsf{LCP}_>(\mathcal{E}_\omega \cup \mathcal{R}_\omega) \subseteq \downarrow_{\mathcal{R}_\omega} \cup \leftrightarrow_{\mathcal{E}_\infty}$ holds.

Below, we consider an infinite fair run Γ. We next show that the result of a fair run without persistent equations is indeed complete.

Theorem 8.17. *If* Γ *is fair and* $\mathcal{E}_\omega = \varnothing$ *then* $\mathcal{R}_\omega$ *is a complete presentation of* $\mathcal{E}_\omega$*.* ☑

Proof. The run Γ is also a valid $\mathsf{KB_o}$ run by Lemma 8.14. We moreover have $\mathsf{PCP}(\mathcal{R}_\omega) \subseteq \mathsf{LCP}(\mathcal{E}_\omega \cup \mathcal{R}_\omega)$ since $\mathcal{R}_\omega \subseteq >$, and hence $\mathsf{PCP}(\mathcal{R}_\omega) \subseteq {\downarrow_{\mathcal{R}_\omega}} \cup {\leftrightarrow_{\mathcal{E}_\infty}}$ by fairness. So the result follows from Theorem 7.10. □

The following result relates equations in $\mathcal{E}_\infty$ and rules in $\mathcal{R}_\infty$ to persistent equations and rules, respectively.

Lemma 8.18.

(1) $\mathcal{E}_\infty \subseteq \xrightarrow[\mathcal{R}_\infty]{*} \cdot \xleftrightarrow[\mathcal{E}_\omega]{=} \cdot \xleftarrow[\mathcal{R}_\infty]{*}$ ☑

(2) If $\ell \to r \in \mathcal{R}_\infty$ *then* $\ell \xrightarrow[\mathcal{R}_\omega]{} \cdot (\xleftrightarrow[\mathcal{E}_\omega \cup \mathcal{R}_\omega]{<\ell})^* \; r$. ☑

Proof.

(1) For an equation $s \approx t \in \mathcal{E}_\infty$ we prove the desired inclusion by induction on $\{s, t\}$ with respect to $>_{\mathsf{mul}}$.

(2) By induction on (ℓ, r) with respect to $>_{\mathsf{lex}}$. □

In order to show that $\mathcal{R}_\omega$ is Church-Rosser modulo $\mathcal{E}_\omega$, we need a result about joinability of critical peaks modulo persistent equations.

Lemma 8.19. *If there is an equation* $\ell \approx r \in \mathcal{E}_\omega^\pm \cup \mathcal{R}_\omega$ *with* $r \not> \ell$ *that is involved in a peak* $s \xleftarrow[r\approx\ell]{} \cdot \xrightarrow[\mathcal{R}_\omega]{} t$ *then* $s \xrightarrow[\mathcal{R}_\infty]{*} \cdot \xleftrightarrow[\mathcal{E}_\omega]{=} \cdot \xleftarrow[\mathcal{R}_\infty]{*} t$. ☑

Proof. If the two steps occur at parallel positions then they commute and thus $s \to_{\mathcal{R}_\omega} \cdot {}_{r\approx\ell}{\leftarrow}\; t$. If the peak constitutes an overlap then $s \leftrightarrow_{\mathsf{LCP}(\mathcal{E}_\omega\cup\mathcal{R}_\omega)} t$ since $\mathcal{R}_\omega \subseteq >$ and $r \not> \ell$ by assumption. We thus have $s \leftrightarrow_{\mathcal{E}_\infty} t$ or $s \downarrow_{\mathcal{R}_\omega} t$ by fairness such that the claim follows from Lemma 8.18(1) and $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty$. Otherwise, we have a variable overlap. By Lemma 8.15 both $\mathcal{E}_\omega$ and $\mathcal{R}_\omega$ are linear. This implies $s \to^{=}_{\mathcal{R}_\omega} \cdot {}_{r\approx\ell}{\leftarrow^{=}}\; t$, so the claim follows from the inclusion $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty$. □

Lemma 8.20. *The TRS* $\mathcal{R}_\omega$ *is Church-Rosser modulo* $\mathcal{E}_\omega$. ☑

Proof. Define the ARSs $\mathcal{A}$ and $\mathcal{B}$ with multiset labeling as follows:

- $s \xrightarrow{M}_{\mathcal{A}} t$ if $s \xrightarrow{\{s'\}}_{\mathcal{R}_\omega} t$ and $M = \{s'\}$ for some term $s' \geqslant s$.
- $s \xrightarrow{M}_{\mathcal{B}} t$ if $s \xleftrightarrow{\{s',t'\}}_{\mathcal{E}_\omega} t$ and $M = \{s', t'\}$ for some terms $s' \geqslant s$ and $t' \geqslant t$.

By equipping them with the well-founded order $>_{\mathsf{mul}}$ Lemmata 8.19 and 8.18 imply the condition of peak decreasingness modulo. Hence, Lemma 2.7 applies. □

For a run of $\mathsf{KB_l}$ we call $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ *simplified* if $\mathcal{R}_\omega$ is reduced and $\mathcal{E}_\omega$ is irreducible with respect to $\mathcal{R}_\omega$ and does not contain trivial equations. From now on we assume that $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified. This allows us to establish relationships between $\mathcal{R}$-normal forms and normal forms with respect to the result of the linear completion run.

Lemma 8.21. *The inclusion* $\mathsf{NF}(\mathcal{R}) \subseteq \mathsf{NF}(\mathcal{E}_\omega^\pm) \cap \mathsf{NF}(\mathcal{R}_\omega)$ *holds.* ☑

Proof. Let $t \in \mathsf{NF}(\mathcal{R})$. Assume to the contrary that $t \to u$ for some term u by applying an equation $\ell \approx r \in \mathcal{E}_\omega^\pm \cup \mathcal{R}_\omega$ from left to right. Because $\mathcal{R}$ is a complete presentation of $\mathcal{E}_\omega \cup \mathcal{R}_\omega$, we have $\ell \downarrow_{\mathcal{R}} r$. Since $t \in \mathsf{NF}(\mathcal{R})$ implies $\ell \in \mathsf{NF}(\mathcal{R})$, we obtain $r \to^*_{\mathcal{R}} \ell$. If $\ell \approx r \in \mathcal{R}_\omega$ this contradicts $\mathcal{R}_\omega \subseteq >$, otherwise $\ell \approx r \in \mathcal{E}_\omega^\pm$ and $r \to^*_{\mathcal{R}} \ell$ contradict unorientability and non-triviality of $\mathcal{E}_\omega$, which hold by the assumption that $\mathcal{E}_\omega$ is simplified. □

Lemma 8.22. *The inclusion* $\mathsf{NF}(\mathcal{R}_\omega) \subseteq \mathsf{NF}(\mathcal{R})$ *holds.*

Proof. We show that $\ell \to^+_{\mathcal{R}_\omega} r$ for every $\ell \to r \in \mathcal{R}$, which is sufficient to prove the claim. Let $\ell \to r \in \mathcal{R}$. By Lemma 8.20 we have

$$\ell \xrightarrow[\mathcal{R}_\omega]{*} u \xleftrightarrow[\mathcal{E}_\omega]{}^* v \xleftarrow[\mathcal{R}_\omega]{*} r$$

for some terms u and v. Since $\mathcal{R}$ is reduced, $r \in \mathsf{NF}(\mathcal{R})$. According to Lemma 8.21, both $r \in \mathsf{NF}(\mathcal{E}^\pm_\omega)$ and $r \in \mathsf{NF}(\mathcal{R}_\omega)$ hold. Hence $r = v$ follows from $r \to^*_{\mathcal{R}_\omega} v$ and $u \leftrightarrow^*_{\mathcal{E}_\omega} v = r$ implies $u = v$. Therefore $\ell \to^*_{\mathcal{R}_\omega} r$. Since $\mathcal{R}$ is terminating, $\ell = r$ is impossible and thus $\ell \to^+_{\mathcal{R}_\omega} r$ as desired. ☐

As in the previous section, the last result allows us to establish the main completeness theorem.

Theorem 8.23. *If* $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ *is simplified then* $\mathcal{E}_\omega = \varnothing$ *and* $\mathcal{R}_\omega$ *is literally similar to* $\mathcal{R}$.

Proof. The TRS $\mathcal{R}$ is complete, the TRS $\mathcal{R}_\omega$ is terminating, and the inclusion $\to_{\mathcal{R}_\omega} \subseteq \leftrightarrow^*_{\mathcal{R}}$ holds because $\mathcal{R}$ is a complete presentation. Moreover, $\mathsf{NF}(\mathcal{R}_\omega) \subseteq \mathsf{NF}(\mathcal{R})$ by Lemma 8.22. Hence, Lemma 4.4(2) applies. Since $\mathcal{R}_\omega$ is a complete presentation, $\mathcal{E}_\omega = \varnothing$ by the assumption of a simplified system. ☐

Example 8.24. By Theorem 8.23 any simplifying $\mathsf{KB_l}$ run on the equational system $\mathcal{E}$ and the reduction order $>_{\mathsf{lpo}}$ from Example 3.9 will result in a canonical presentation, independent of the order in which inference steps are applied. Note that Theorem 8.10 does not apply since the given order $>_{\mathsf{lpo}}$ is not ground total.

We conclude the subsection by showing the absence of a complete presentation for the equational system mentioned in the first paragraph of Section 7.

Example 8.25. Let $\mathcal{E}$ be the ES consisting of the two equations $\mathsf{0} + x \approx x$ and $x + y \approx y + x$. We show that $\mathcal{E}$ admits no complete presentation. Assume to the contrary that $\mathcal{R}$ is a complete presentation of $\mathcal{E}$. We use $\to^+_{\mathcal{R}}$ as the reduction order $>$ for $\mathsf{KB_l}$. Because $\mathsf{0}+x \leftrightarrow^*_{\mathcal{E}} x$ implies $\mathsf{0} + x \downarrow_{\mathcal{R}} x$ and $x \in \mathsf{NF}(\mathcal{R})$, we have $\mathsf{0} + x > x$. In the same way $x + \mathsf{0} > x$ is derived. Therefore, the following fair run of $\mathsf{KB_l}$ is constructed:

$$\begin{aligned}
(\mathcal{E}, \varnothing) &\vdash^{\mathsf{orient}}_{\mathsf{l}} (\{x + y \approx y + x\}, \{\mathsf{0} + x \to x\}) \\
&\vdash^{\mathsf{deduce}}_{\mathsf{l}} (\{x + y \approx y + x, x + \mathsf{0} \approx x\}, \{\mathsf{0} + x \to x\}) \\
&\vdash^{\mathsf{orient}}_{\mathsf{l}} (\{x + y \approx y + x\}, \{\mathsf{0} + x \to x, x + \mathsf{0} \to x\}) \\
&\vdash^{\mathsf{deduce}}_{\mathsf{l}} (\{x + y \approx y + x, \mathsf{0} \approx \mathsf{0}\}, \{\mathsf{0} + x \to x, x + \mathsf{0} \to x\}) \\
&\vdash^{\mathsf{delete}}_{\mathsf{l}} (\{x + y \approx y + x\}, \{\mathsf{0} + x \to x, x + \mathsf{0} \to x\}) \\
&\vdash^{\mathsf{deduce}}_{\mathsf{l}} (\{x + y \approx y + x, \mathsf{0} \approx \mathsf{0}\}, \{\mathsf{0} + x \to x, x + \mathsf{0} \to x\}) \\
&\vdash^{\mathsf{delete}}_{\mathsf{l}} \cdots
\end{aligned}$$

It is easy to check that this run is fair; the two non-trivial critical pairs $x + \mathsf{0} \approx x$ and $\mathsf{0} + x \approx x$ belong to $\mathcal{E}^\pm_\infty$. We have $\mathcal{E}_\omega = \{x + y \approx y + x\}$ and $\mathcal{R}_\omega = \{\mathsf{0} + x \to x, x + \mathsf{0} \to x\}$. Note that $(\mathcal{E}_\omega, \mathcal{R}_\omega)$ is simplified. According to Theorem 8.23, the persistent set $\mathcal{E}_\omega$ must be empty. This is a contradiction and thus $\mathcal{R}$ does not exist.

In summary, our proof of Theorem 7.10 resembles the approach by Devie [Dev91], though our version is structured along several preliminary results that are of independent interest, such as Lemmata 8.20, 8.21, and 8.22.

9. Conclusion

In this paper we have presented new and formalized proofs for a number of correctness and completeness results for abstract completion, ranging from the decidable case of ground-completion to completeness results for ordered completion. By using modern abstract confluence criteria, we could avoid the use of proof orders, which had a positive effect on the Isabelle/HOL formalization.

We mention some topics for future work. Concerning completion of ground systems, the literature contains other interesting results that we might consider as target for future formalization efforts. Gallier et al. [GNP+93] showed that every ground ES $\mathcal{E}$ can be transformed into an equivalent canonical TRS in $O(n^3)$ time, where n is the combined size of the terms appearing in $\mathcal{E}$. Snyder [Sny93] improved this result to an $O(n \log n)$ time algorithm. Moreover, his algorithm can enumerate all canonical presentations, of which there are at most 2^k [Sny93, Theorem 4.7], where k is the number of equations in $\mathcal{E}$. Furthermore, all canonical presentations have the same number of rules.

In the context of ordered completion, completeness remains an open problem in the general case: It is unknown whether an ordered completion run can find a complete system $\mathcal{R}$ for a set of input equations $\mathcal{E}$ if neither $\mathcal{E}$ is linear (Theorem 8.23) nor $\mathcal{R}$ is compatible with a ground-total reduction order (Theorem 8.10).

There are several important extensions of completion that we did not consider in this paper. We mention completion in the presence of associative and commutative (AC) symbols [PS81], normalized completion [Mar96, WM13], as well as maximal completion [KH11]. They are natural candidates for future formalization efforts.

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