universität innsbruck

Certified Equational Reasoning via Ordered Completion

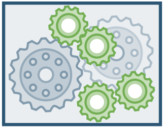
Christian Sternagel and Sarah Winkler

27th International Conference on Automated Deduction
28 August 2019, Natal

Motivation

Automated Reasoning Systems

- sophisticated pieces of software



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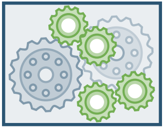

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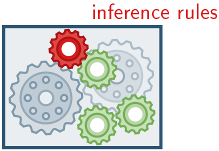

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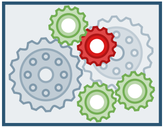

heuristics

Automated Reasoning Systems

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$$x - 0 \approx x$$
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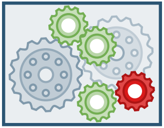

term indexing

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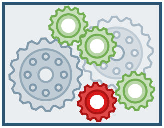

optimizations

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- producing complex derivations: trustworthy?

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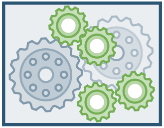

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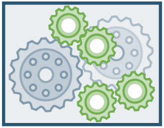

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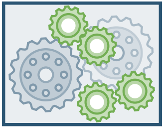

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Ordered Completion

Input: set of input equalities $\mathcal{E}_0$

Output: ground complete TRS $\mathcal{E}^{>} \cup \mathcal{R}$

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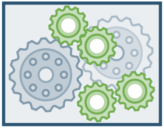

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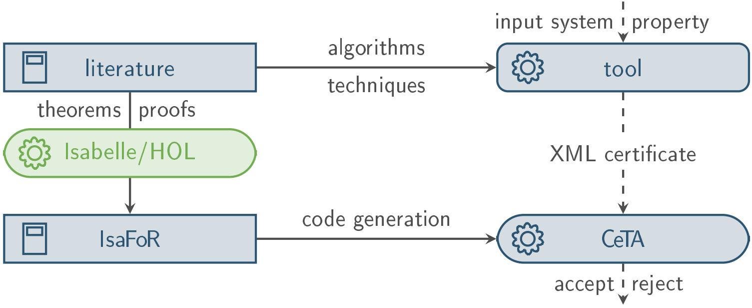
Applications

- decide ground equational theory
- used by confluence tool ConCon to decide infeasibility of CPs

The IsaFoR/CeTA Framework



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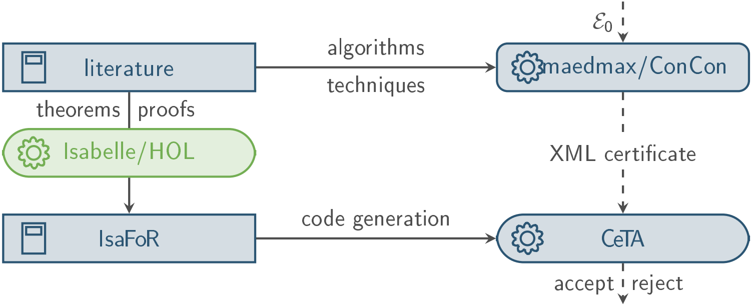

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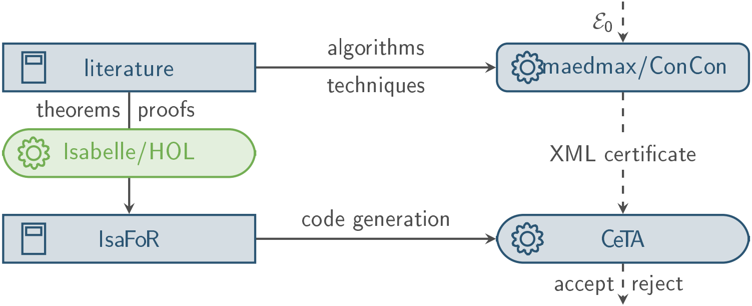

Contributions

- extend formal library IsaFoR with
  - finite ordered completion runs
  - ground joinability criteria

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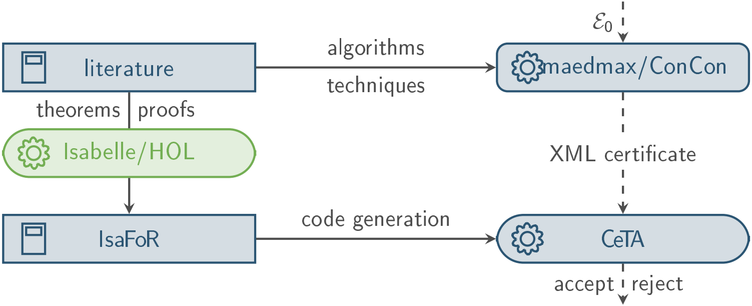

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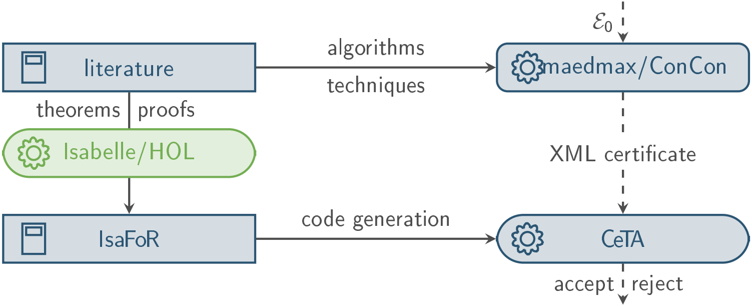

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  - satisfiability (TPTP) proofs in equational logic
  - infeasibility of conditional critical pairs
- respective output in equational theorem prover MædMax and ConCon

Outline

Preliminaries

Ordered Completion

Ground Joinability Criteria

Proof Checking

Conclusion

Definition

term rewrite system $\mathcal{R}$ is

- terminating if $\nexists\ t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$

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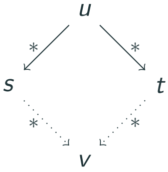

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Definition (Ordered Rewriting)

$\mathcal{E}_{>} = \{s\sigma \to t\sigma \mid s \approx t \in \mathcal{E}^{\pm} \text{ and } s\sigma > t\sigma\}$

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Definition (oKB)
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Ordered Completion

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if $s \ {}_{\mathcal{R}\cup\mathcal{E}}\!\leftrightarrow \cdot \leftrightarrow_{\mathcal{R}\cup\mathcal{E}} \ t$

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orient
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compose
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{\ s \to u\ \}}$$
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ordered rewriting

Definition (oKB)

for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $>$ have six inference rules:

deduce $\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{\ s \approx t\ \}, \mathcal{R}}$ if $s\ {}_{\mathcal{R}\cup\mathcal{E}}\!\leftrightarrow \cdot \leftrightarrow_{\mathcal{R}\cup\mathcal{E}}\ t$

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simplify $\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{\ u \approx t\ \}, \mathcal{R}}$ if $s \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

delete $\dfrac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

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$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{\ u \approx t\ \}, \mathcal{R}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{\ t \approx u\ \}, \mathcal{R}}$$
if $s \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

delete
$$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

Definition (oKB)

for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $>$ have six inference rules:

deduce
$$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{\ s \approx t\ \}, \mathcal{R}}$$
if $s\ {}_{\mathcal{R}\cup\mathcal{E}}\!\leftrightarrow \cdot \leftrightarrow_{\mathcal{R}\cup\mathcal{E}}\ t$

compose
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{\ s \to u\ \}}$$
if $t \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

orient
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{\ s \to t\ \}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{\ s \to t\ \}}$$
if $s > t$

simplify
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{\ u \approx t\ \}, \mathcal{R}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{\ t \approx u\ \}, \mathcal{R}}$$
if $s \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

delete
$$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

collapse
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{\ u \approx s\ \}, \mathcal{R}}$$
if $t \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

Ordered Completion

Definition (oKB)

for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $>$ have six inference rules:

deduce $$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s\pi \approx t\pi\}, \mathcal{R}}$$ if $s \ {}_{\mathcal{R}\cup\mathcal{E}}\!\leftrightarrow \cdot \leftrightarrow_{\mathcal{R}\cup\mathcal{E}}\ t$

compose $$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to u\pi\}}$$ if $t \to_{\mathcal{R}\cup\mathcal{E}^>} u$

orient $$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to t\pi\}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to t\pi\}}$$ if $s > t$

simplify $$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u\pi \approx t\pi\}, \mathcal{R}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{t\pi \approx u\pi\}, \mathcal{R}}$$

delete $$\frac{\mathcal{E} \uplus \{s \approx s\}}{\mathcal{E}, \mathcal{R}}$$

if $t \to_{\mathcal{R}\cup\mathcal{E}^>} u$

L. Bachmair, N. Dershowitz, and D. Plaisted.
Completion Without Failure.
In *Resolution of Equations in Algebraic Structures*, 1989.

Relaxations

- allow variants for renaming π

Ordered Completion

Definition (oKB)

for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $>$ have six inference rules:

deduce
$$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s\pi \approx t\pi\}, \mathcal{R}}$$
if $s \;{}_{\mathcal{R}\cup\mathcal{E}}\!\leftrightarrow \cdot \leftrightarrow_{\mathcal{R}\cup\mathcal{E}}\; t$

compose
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to u\pi\}}$$
if $t \to_{\mathcal{R}\cup\mathcal{E}_>} u$

orient
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to t\pi\}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to t\pi\}}$$
if $s > t$

simplify
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u\pi \approx t\pi\}, \mathcal{R}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{t\pi \approx u\pi\}, \mathcal{R}}$$
if $s \to_{\mathcal{R}\cup\mathcal{E}_>} u$

delete
$$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

collapse
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u\pi \approx s\pi\}, \mathcal{R}}$$
if $t \to_{\mathcal{R}\cup\mathcal{E}_>} u$

Relaxations

- allow variants for renaming π
- no encompassment condition

Ordered Completion

Definition (oKB)

for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $>$ have six inference rules:

deduce
$$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s\pi \approx t\pi\}, \mathcal{R}}$$
if $s \;{}_{\mathcal{R}\cup\mathcal{E}}\!\leftrightarrow \cdot \leftrightarrow_{\mathcal{R}\cup\mathcal{E}}\; t$

compose
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to u\pi\}}$$
if $t \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

orient
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to t\pi\}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s\pi \to t\pi\}}$$
if $s > t$

simplify
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u\pi \approx t\pi\}, \mathcal{R}} \qquad \frac{\mathcal{E} \uplus \{t \approx s\}, \mathcal{R}}{\mathcal{E} \cup \{t\pi \approx u\pi\}, \mathcal{R}}$$
if $s \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

delete
$$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

collapse
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u\pi \approx s\pi\}, \mathcal{R}}$$
if $t \to_{\mathcal{R}\cup\mathcal{E}^{>}} u$

Notation

write $(\mathcal{E}, \mathcal{R}) \vdash_{\mathsf{oKB}} (\mathcal{E}', \mathcal{R}')$ if there is oKB step from $(\mathcal{E}, \mathcal{R})$ to $(\mathcal{E}', \mathcal{R}')$

Example (Ordered Completion)

$\mathcal{E}:$

$$
\begin{aligned}
x - 0 &\approx x \\
\mathsf{s}(x) - \mathsf{s}(y) &\approx x - y \\
0 - y &\approx 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}:$

Remark

generated by conditional confluence tool ConCon from Cops #361:

ground complete system used to show infeasibility of critical pairs

Example (Ordered Completion)

$\mathcal{E}:$

$$
\begin{aligned}
x - 0 &\approx x \\
\mathsf{s}(x) - \mathsf{s}(y) &\approx x - y \\
0 - y &\approx 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}:$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}:$

$$
\begin{aligned}
x - 0 &\approx x \\
\mathsf{s}(x) - \mathsf{s}(y) &\approx x - y \\
0 - y &\approx 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}:$

- fix some KBO (...)
- orient $x - 0 >_{\mathsf{kbo}} x$

Example (Ordered Completion)

$\mathcal{E}$:

$$
\begin{aligned}
\mathsf{s}(x) - \mathsf{s}(y) &\approx x - y \\
0 - y &\approx 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}$:

$$x - 0 \rightarrow x$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$\begin{aligned}
\mathsf{s}(x) - \mathsf{s}(y) &\approx x - y \\
0 - y &\approx 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}$$

$\mathcal{R}$:

$$x - 0 \rightarrow x$$

- fix some KBO (...)
- orient $\mathsf{s}(x) - \mathsf{s}(y) >_{\mathsf{kbo}} x - y$

Example (Ordered Completion)

$\mathcal{E}:$

$$
\begin{aligned}
0 - y &\approx 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}:$

$$
\begin{aligned}
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y
\end{aligned}
$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$
\begin{aligned}
0 - y &\approx 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}$:

$$
\begin{aligned}
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y
\end{aligned}
$$

- fix some KBO (...)
- orient $0 - y >_{\mathsf{kbo}} 0$

Example (Ordered Completion)

$\mathcal{E}$:

$$\begin{aligned} \mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\ x \div y &\approx \langle 0, y \rangle \\ x \div y &\approx \langle \mathsf{s}(q), r \rangle \\ \mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\ \mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\ 0 \preccurlyeq x &\approx \mathsf{true} \end{aligned}$$

$\mathcal{R}$:

$$\begin{aligned} x - 0 &\to x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ 0 - y &\to 0 \end{aligned}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$\begin{aligned}
\mathsf{s}(x) \succ \mathsf{s}(y) &\approx x \succ y \\
x \div y &\approx \langle 0, y \rangle \\
x \div y &\approx \langle \mathsf{s}(q), r \rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}$$

$\mathcal{R}$:

$$\begin{aligned}
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
0 - y &\to 0
\end{aligned}$$

- fix some KBO (...)
- orient $\mathsf{s}(x) \succ \mathsf{s}(y) >_{\mathsf{kbo}} x \succ y$

Example (Ordered Completion)

$\mathcal{E}:$

$$\begin{aligned} x \div y &\approx \langle 0, y\rangle \\ x \div y &\approx \langle \mathsf{s}(q), r\rangle \\ \mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\ \mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\ 0 \preccurlyeq x &\approx \mathsf{true} \end{aligned}$$

$\mathcal{R}:$

$$\begin{aligned} x - 0 &\to x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ 0 - y &\to 0 \\ \mathsf{s}(x) \succ \mathsf{s}(y) &\to x \succ y \end{aligned}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$
\begin{aligned}
x \div y &\approx \langle 0, y\rangle \\
x \div y &\approx \langle \mathsf{s}(q), r\rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}$:

$$
\begin{aligned}
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
0 - y &\to 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\to x \succ y
\end{aligned}
$$

- fix some KBO (...)
- simplify $x \div y \to_{\mathcal{E}^>} \langle 0, y\rangle$ (no encompassment!)

Example (Ordered Completion)

$\mathcal{E}$:

$$
\begin{aligned}
x \div y &\approx \langle 0, y\rangle \\
\langle 0, y\rangle &\approx \langle \mathsf{s}(q), r\rangle \\
\mathsf{s}(x) \succ 0 &\approx \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\approx x \preccurlyeq y \\
0 \preccurlyeq x &\approx \mathsf{true}
\end{aligned}
$$

$\mathcal{R}$:

$$
\begin{aligned}
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
0 - y &\to 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\to x \succ y
\end{aligned}
$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$x \div y \approx \langle 0, y \rangle$$
$$\langle 0, y \rangle \approx \langle \mathsf{s}(q), r \rangle$$
$$\mathsf{s}(x) \succ 0 \approx \mathsf{true}$$
$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \approx x \preccurlyeq y$$
$$0 \preccurlyeq x \approx \mathsf{true}$$

$\mathcal{R}$:

$$x - 0 \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$0 - y \to 0$$
$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

- fix some KBO (...)
- orient $\mathsf{s}(x) \succ 0 >_{\mathsf{kbo}} \mathsf{true}$

Example (Ordered Completion)

$\mathcal{E}:$

$$x \div y \approx \langle 0, y\rangle$$
$$\langle 0, y\rangle \approx \langle s(q), r\rangle$$

$$s(x) \preccurlyeq s(y) \approx x \preccurlyeq y$$
$$0 \preccurlyeq x \approx \mathsf{true}$$

$\mathcal{R}:$

$$x - 0 \to x$$
$$s(x) - s(y) \to x - y$$
$$0 - y \to 0$$
$$s(x) \succ s(y) \to x \succ y$$

$$s(x) \succ 0 \to \mathsf{true}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$x \div y \approx \langle 0, y \rangle$$
$$\langle 0, y \rangle \approx \langle s(q), r \rangle$$
$$s(x) \preccurlyeq s(y) \approx x \preccurlyeq y$$
$$0 \preccurlyeq x \approx \mathsf{true}$$

$\mathcal{R}$:

$$x - 0 \to x$$
$$s(x) - s(y) \to x - y$$
$$0 - y \to 0$$
$$s(x) \succ s(y) \to x \succ y$$
$$s(x) \succ 0 \to \mathsf{true}$$

- fix some KBO (...)
- orient $s(x) \preccurlyeq s(y) >_{\mathsf{kbo}} x \preccurlyeq y$

Example (Ordered Completion)

$\mathcal{E}$:

$$x \div y \approx \langle 0, y \rangle$$

$$\langle 0, y \rangle \approx \langle \mathsf{s}(q), r \rangle$$

$$0 \preccurlyeq x \approx \mathsf{true}$$

$\mathcal{R}$:

$$x - 0 \to x$$

$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

$$0 - y \to 0$$

$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$

$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}:$

$$x \div y \approx \langle 0, y \rangle$$

$$\langle 0, y \rangle \approx \langle \mathsf{s}(q), r \rangle$$

$$0 \preccurlyeq x \approx \mathsf{true}$$

$\mathcal{R}:$

$$x - 0 \to x$$

$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

$$0 - y \to 0$$

$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$

$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$

- fix some KBO (...)
- orient $0 \preccurlyeq x >_{\mathsf{kbo}} \mathsf{true}$

Example (Ordered Completion)

$\mathcal{E}$:

$$x \div y \approx \langle 0, y \rangle$$
$$\langle 0, y \rangle \approx \langle \mathsf{s}(q), r \rangle$$

$\mathcal{R}$:

$$x - 0 \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$0 - y \to 0$$
$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$
$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$
$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$
$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$x \div y \approx \langle 0, y \rangle$$

$$\langle 0, y \rangle \approx \langle \mathsf{s}(q), r \rangle$$

$\mathcal{R}$:

$$x - 0 \to x$$

$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

$$0 - y \to 0$$

$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$

$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$

$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)
- deduce $\langle 0, x \rangle \leftarrow \langle \mathsf{s}(u), v \rangle \to \langle 0, y \rangle$

Example (Ordered Completion)

$\mathcal{E}:$

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$

$$x \div y \approx \langle 0, y\rangle$$

$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$\mathcal{R}:$

$$x - 0 \to x$$

$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

$$0 - y \to 0$$

$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$

$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$

$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}:$

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$

$$x \div y \approx \langle 0, y\rangle$$

$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$\mathcal{R}:$

$$\begin{aligned} x - 0 &\to x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ 0 - y &\to 0 \\ \mathsf{s}(x) \succ \mathsf{s}(y) &\to x \succ y \\ \mathsf{s}(x) \succ 0 &\to \mathsf{true} \\ \mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\to x \preccurlyeq y \\ 0 \preccurlyeq x &\to \mathsf{true} \end{aligned}$$

- fix some KBO (...)
- deduce $\langle \mathsf{s}(x), y\rangle \leftarrow \langle 0, u\rangle \to \langle \mathsf{s}(q), r\rangle$

Example (Ordered Completion)

$\mathcal{E}$:

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$
$$x \div y \approx \langle 0, y\rangle$$
$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$\mathcal{R}$:

$$x - 0 \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$0 - y \to 0$$
$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$
$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$
$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$
$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}:$

$$\begin{aligned} \langle 0, x\rangle &\approx \langle 0, y\rangle \\ \langle \mathsf{s}(x), y\rangle &\approx \langle \mathsf{s}(q), r\rangle \end{aligned}$$

$$\begin{aligned} x \div y &\approx \langle 0, y\rangle \\ \langle 0, y\rangle &\approx \langle \mathsf{s}(q), r\rangle \end{aligned}$$

$\mathcal{R}:$

$$\begin{aligned} x - 0 &\to x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ 0 - y &\to 0 \\ \mathsf{s}(x) \succ \mathsf{s}(y) &\to x \succ y \\ \mathsf{s}(x) \succ 0 &\to \mathsf{true} \\ \mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\to x \preccurlyeq y \\ 0 \preccurlyeq x &\to \mathsf{true} \end{aligned}$$

- fix some KBO (...)
- orient $\quad x \div y >_{\mathsf{kbo}} \langle 0, y\rangle$

Example (Ordered Completion)

$\mathcal{E}:$

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$
$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$\mathcal{R}:$

$$x - 0 \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$0 - y \to 0$$
$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$
$$x \div y \to \langle 0, y\rangle$$
$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$
$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$
$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}:$

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$
$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$\mathcal{R}:$

$$x - 0 \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$0 - y \to 0$$
$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$
$$x \div y \to \langle 0, y\rangle$$
$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$
$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$
$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)
- deduce $\mathsf{s}(\mathsf{s}(x)) \succ \mathsf{s}(0) \leftarrow \mathsf{s}(x) \succ 0 \to \mathsf{true}$

Example (Ordered Completion)

$\mathcal{E}$:

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$

$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$$\mathsf{s}(\mathsf{s}(x)) \succ \mathsf{s}(0) \approx \mathsf{true}$$

$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$\mathcal{R}$:

$$x - 0 \to x$$

$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

$$0 - y \to 0$$

$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

$$x \div y \to \langle 0, y\rangle$$

$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$

$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$

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- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}:$

$$\begin{aligned}
\langle 0, x\rangle &\approx \langle 0, y\rangle \\
\langle \mathsf{s}(x), y\rangle &\approx \langle \mathsf{s}(q), r\rangle \\
\mathsf{s}(\mathsf{s}(x)) \succ \mathsf{s}(0) &\approx \mathsf{true} \\
\langle 0, y\rangle &\approx \langle \mathsf{s}(q), r\rangle
\end{aligned}$$

$\mathcal{R}:$

$$\begin{aligned}
x - 0 &\to x \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\
0 - y &\to 0 \\
\mathsf{s}(x) \succ \mathsf{s}(y) &\to x \succ y \\
x \div y &\to \langle 0, y\rangle \\
\mathsf{s}(x) \succ 0 &\to \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\to x \preccurlyeq y \\
0 \preccurlyeq x &\to \mathsf{true}
\end{aligned}$$

- fix some KBO (...)
- orient $\mathsf{s}(\mathsf{s}(x)) \succ \mathsf{s}(0) >_{\mathsf{kbo}} \mathsf{true}$

Example (Ordered Completion)

$\mathcal{E}$:

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$
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$\mathcal{R}$:

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x - 0 &\to x \\
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\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\to x \preccurlyeq y \\
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Example (Ordered Completion)

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$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
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$$x - 0 \to x$$
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$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$
$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)
- collapse $\mathsf{s}(\mathsf{s}(x)) \succ \mathsf{s}(0) \to_{\mathcal{R}} \mathsf{s}(x) \succ 0$

Example (Ordered Completion)

$\mathcal{E}$:

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$
$$\mathsf{s}(x) \succ 0 \approx \mathsf{true}$$

$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

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$$x - 0 \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
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- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}$:

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$

$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$$\mathsf{s}(x) \succ 0 \approx \mathsf{true}$$

$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

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$$x - 0 \to x$$

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$$0 - y \to 0$$

$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

$$x \div y \to \langle 0, y\rangle$$

$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$

$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$

$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)
- simplify $\mathsf{s}(x) \succ 0 \to_{\mathcal{R}} \mathsf{true}$

Example (Ordered Completion)

$\mathcal{E}:$

$$\begin{aligned} \langle 0, x\rangle &\approx \langle 0, y\rangle \\ \langle \mathsf{s}(x), y\rangle &\approx \langle \mathsf{s}(q), r\rangle \\ \mathsf{true} &\approx \mathsf{true} \\ \langle 0, y\rangle &\approx \langle \mathsf{s}(q), r\rangle \end{aligned}$$

$\mathcal{R}:$

$$\begin{aligned} x - 0 &\to x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \\ 0 - y &\to 0 \\ \mathsf{s}(x) \succ \mathsf{s}(y) &\to x \succ y \\ x \div y &\to \langle 0, y\rangle \\ \mathsf{s}(x) \succ 0 &\to \mathsf{true} \\ \mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\to x \preccurlyeq y \\ 0 \preccurlyeq x &\to \mathsf{true} \end{aligned}$$

- fix some KBO (...)

Example (Ordered Completion)

$\mathcal{E}:$

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$
$$\mathsf{true} \approx \mathsf{true}$$
$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

$\mathcal{R}:$

$$x - 0 \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$
$$0 - y \to 0$$
$$\mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$
$$x \div y \to \langle 0, y\rangle$$
$$\mathsf{s}(x) \succ 0 \to \mathsf{true}$$
$$\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y$$
$$0 \preccurlyeq x \to \mathsf{true}$$

- fix some KBO (...)
- delete $\mathsf{true} \approx \mathsf{true}$

Example (Ordered Completion)

$\mathcal{E}$:

$$\langle 0, x\rangle \approx \langle 0, y\rangle$$
$$\langle \mathsf{s}(x), y\rangle \approx \langle \mathsf{s}(q), r\rangle$$
$$\langle 0, y\rangle \approx \langle \mathsf{s}(q), r\rangle$$

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Example (Ordered Completion)

$\mathcal{E}:$

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x \div y &\to \langle 0, y\rangle \\
\mathsf{s}(x) \succ 0 &\to \mathsf{true} \\
\mathsf{s}(x) \preccurlyeq \mathsf{s}(y) &\to x \preccurlyeq y \\
0 \preccurlyeq x &\to \mathsf{true}
\end{aligned}$$

- fix some KBO (...)
- oKB run produced ground complete system

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Formalization in IsaFoR

Lemma

If $(\mathcal{E}, \mathcal{R}) \vdash^*_{\mathsf{oKB}} (\mathcal{E}', \mathcal{R}')$ *and* $\mathcal{R} \subseteq >$ *then* $\mathcal{R}' \subseteq >$.

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If $(\mathcal{E}, \mathcal{R}) \vdash^*_{\mathsf{oKB}} (\mathcal{E}', \mathcal{R}')$ *then* $\leftrightarrow^*_{\mathcal{E} \cup \mathcal{R}} = \leftrightarrow^*_{\mathcal{E}' \cup \mathcal{R}'}$.

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Lemma

If $\forall\ s \approx t \in \mathsf{CP}_>(\mathcal{E})$ *have* $s \downarrow^g_{\mathcal{E}_>} t$ *then* $\mathcal{E}^>$ *is ground complete.*

. . . and check ground-joinability of critical pairs, see?

Formalization in IsaFoR

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If $\forall\ s \approx t \in \mathsf{CP}_{>}(\mathcal{E})$ *have* $s \downarrow^{g}_{\mathcal{E}_{>}} t$ *then* $\mathcal{E}^{>}$ *is ground complete.*

Lemma

If $>$ *is total precedence then* $>_{\mathsf{lpo}}$ *and* $>_{\mathsf{kbo}}$ *are total on ground terms.*

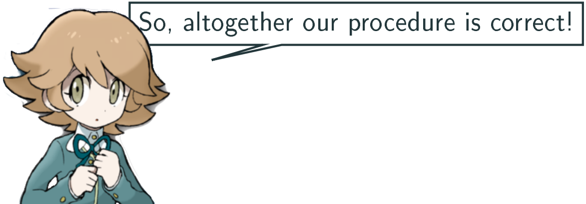
Formalization in IsaFoR

Theorem (Correctness)

Suppose $(\mathcal{E}_0, \varnothing) \vdash^*_{\mathsf{oKB}} (\mathcal{E}, \mathcal{R})$

- *using LPO or KBO as ground-total reduction order* $>$, *and*
- $\forall\ s \approx t \in \mathsf{CP}_{>}(\mathcal{E} \cup \mathcal{R})$ *have* $s \downarrow^{g}_{\mathcal{R} \cup \mathcal{E}^{>}} t$

Then $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R} \cup \mathcal{E}}$ *and* $\mathcal{R} \cup \mathcal{E}^{>}$ *is ground complete.*



CERTIFIED

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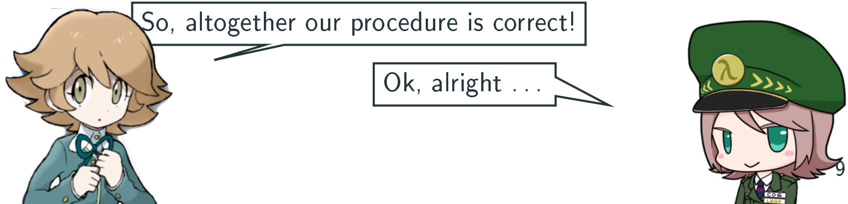

CERTIFIED

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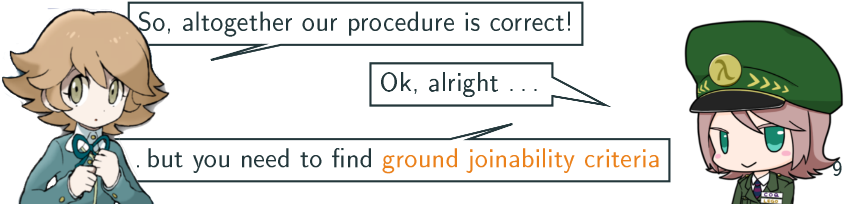

Outline

Preliminaries

Ordered Completion

Ground Joinability Criteria

Proof Checking

Conclusion

Lemma (Criterion 1)

relationship $s \downarrow^{g}_{\mathcal{E}_>} t$ *holds if*

- $s \downarrow_{\mathcal{E}_>} t$, *or*
- $s \approx t$ *is instance of equation in* $\mathcal{E}^{\pm}$

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Example

for $\mathcal{R}$ and $\mathcal{E}$ derived by ConCon from Cops 361:

$$-x \cdot 0 \to x \qquad -0 \cdot x \to 0 \qquad -\mathsf{s}(x) \cdot \mathsf{s}(y) \to -x \cdot y$$

$$0 \preccurlyeq x \to \mathsf{true} \qquad \mathsf{s}(x) \preccurlyeq \mathsf{s}(y) \to x \preccurlyeq y \qquad x \div y \to \langle 0, y \rangle$$

$$\mathsf{s}(x) \succ 0 \to \mathsf{true} \qquad \mathsf{s}(x) \succ \mathsf{s}(y) \to x \succ y$$

$$\langle \mathsf{s}(x), y \rangle \approx \langle \mathsf{s}(q), r \rangle \qquad \langle 0, y \rangle \approx \langle \mathsf{s}(q), r \rangle \qquad \langle 0, x \rangle \approx \langle 0, y \rangle$$

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ground confluence can be established by Criterion 1:

- critical overlap between first two equations:
  $\langle 0, y \rangle \leftarrow \langle \mathsf{s}(q), r \rangle \to \langle \mathsf{s}(x), y \rangle$
- critical overlap between first two rules:
  $0 \leftarrow -0 \cdot 0 \to 0$

Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x \qquad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x \qquad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

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$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

- Criterion 1 fails to show $s \downarrow^{\mathrm{g}}_{\mathcal{E}_>} t$

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Observation

for any grounding substitution σ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered

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set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x \qquad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

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$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

- Criterion 1 fails to show $s \downarrow^{\mathrm{g}}_{\mathcal{E}_>} t$
- if $>$ is "extended" by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

$$z\sigma \cdot (x\sigma \cdot y\sigma) \rightarrow z\sigma \cdot (y\sigma \cdot x\sigma) \rightarrow y\sigma \cdot (z\sigma \cdot x\sigma) \leftarrow y\sigma \cdot (x\sigma \cdot z\sigma) \leftarrow x\sigma \cdot (y\sigma \cdot z\sigma)$$

Observation

for any grounding substitution σ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered

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set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x \qquad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

- Criterion 1 fails to show $s \downarrow^{\mathrm{g}}_{\mathcal{E}_>} t$
- if $>$ is "extended" by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

$$z\sigma \cdot (\underline{x\sigma \cdot y\sigma}) \rightarrow \underline{z\sigma \cdot (y\sigma \cdot x\sigma)} \rightarrow y\sigma \cdot (z\sigma \cdot x\sigma) \leftarrow \underline{y\sigma \cdot (x\sigma \cdot z\sigma)} \leftarrow x\sigma \cdot (\underline{y\sigma \cdot z\sigma})$$

- can verify $s\sigma \downarrow_{\mathcal{E}_>} t\sigma$ for all 13 orderings of $x\sigma$, $y\sigma$, $z\sigma$

Observation

for any grounding substitution σ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered

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- can verify $s\sigma \downarrow_{\mathcal{E}_>} t\sigma$ for all 13 orderings of $x\sigma$, $y\sigma$, $z\sigma$
- repeat this check for all CPs: $\mathcal{E}_>$ is ground complete

Observation

for any grounding substitution σ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered

Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x \qquad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

- Criterion 1 fails to show $s \downarrow^{\mathrm{g}}_{\mathcal{E}_>} t$
- if $>$ is "extended" by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

$z\sigma \cdot (x\sigma \cdot y\sigma)$ $\qquad$ $x\sigma \cdot (y\sigma \cdot z\sigma)$

$\rightarrow z\sigma \cdot (y\sigma \cdot x\sigma)$

$\rightarrow y\sigma \cdot (z\sigma$

U. Martin and T. Nipkow.
Ordered Rewriting and Confluence.
Proc. 10th CADE, 1990.

- can verify $s\sigma \downarrow_{\mathcal{E}_>} t\sigma$ for all 13 o
- repeat this check for all CPs: $\mathcal{E}_>$ is ground complete

Observation

for any grounding substitution σ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered

Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x \qquad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

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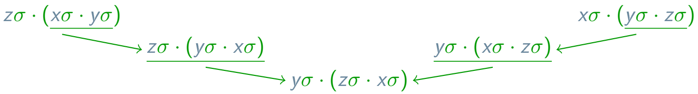

- can verify $s\sigma \downarrow_{\mathcal{E}_>} t\sigma$ for all 13 orderings of $x\sigma$, $y\sigma$, $z\sigma$
- repeat this check for all CPs: $\mathcal{E}_>$ is ground complete

Definition (Joinable wrt Closure)

write $s \downarrow^{\mathcal{C}}_{\mathcal{E}} t$ if $\forall$ equivalence relations $\equiv$ on $\mathcal{V}ar(s,t)$ $\forall$ order $\succ$ on $\equiv$

$$\hat{\equiv}(s) \xrightarrow[\mathcal{E}_{\mathcal{C}(\succ)}]{*} \cdot \xleftarrow[\mathcal{E}_{\mathcal{C}(\succ)}]{*} \hat{\equiv}(t)$$

Definition

inductively defined ground joinability predicate $\mathrm{gj}(\cdot,\cdot)$

delete			$\mathrm{gj}(t,t)$
closure	$s \downarrow_{\mathcal{E}}^{\mathcal{C}} t$	$\Longrightarrow$	$\mathrm{gj}(s,t)$
step	$s \underset{\mathcal{E}}{\longleftrightarrow} t$	$\Longrightarrow$	$\mathrm{gj}(s,t)$
rewrite left	$s \underset{\mathcal{E}_>}{\longrightarrow} u$ and $\mathrm{gj}(u,t)$	$\Longrightarrow$	$\mathrm{gj}(s,t)$
rewrite right	$t \underset{\mathcal{E}_>}{\longrightarrow} u$ and $\mathrm{gj}(s,u)$	$\Longrightarrow$	$\mathrm{gj}(s,t)$
congruence	$\mathrm{gj}(s_i,t_i)$ for all $1 \leqslant i \leqslant n$	$\Longrightarrow$	$\mathrm{gj}(f(\overline{s}),f(\overline{t}))$

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$\mathrm{gj}(s,t)$ *implies* $s \downarrow_{\mathcal{E}_>}^{g} t$

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gain flexibility/efficiency
over Martin & Nipkow criterion

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Example

for set of equations $\mathcal{E}$:

$$\mathsf{f}(x) \approx \mathsf{f}(y) \qquad\qquad \mathsf{g}(x,y) \approx \mathsf{f}(x)$$

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$$
\begin{array}{rrcl}
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can show $\mathsf{g}(x,y) \downarrow_{\mathcal{R}}^{\mathsf{g}} \mathsf{g}(z,w)$:

$$\xRightarrow[\text{step}]{\mathsf{f}(x) \leftrightarrow_{\mathcal{E}} \mathsf{f}(z)} \mathrm{gj}(\mathsf{f}(x),\mathsf{f}(z)) \xRightarrow[\text{rewrite left}]{\mathsf{g}(x,y) \to_{\mathcal{E}_>} \mathsf{f}(x)} \mathrm{gj}(\mathsf{g}(x,y),\mathsf{f}(z)) \xRightarrow[\text{rewrite right}]{\mathsf{g}(z,w) \to_{\mathcal{E}_>} \mathsf{f}(z)} \mathrm{gj}(\mathsf{g}(x,y),\mathsf{g}(z,w))$$

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for set of equations $\mathcal{E}$

MN90 approach would need to check 81 relations

$\mathsf{f}(x)$ ... $\mathsf{f}(x)$

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Outline

Preliminaries

Ordered Completion

Ground Joinability Criteria

Proof Checking

Conclusion

Ordered Completion

Certificate Components

- initial equations $\mathcal{E}_0$
- reduction order $>$
- resulting system $(\mathcal{E}, \mathcal{R})$
- oKB steps from $\mathcal{E}_0$ to $(\mathcal{E}, \mathcal{R})$

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Checks Done in CeTA

1. valid run $(\mathcal{E}_0\pi, \varnothing) \vdash^*_{\mathsf{oKB}} (\mathcal{E}, \mathcal{R})$, termination of $\mathcal{R}$, $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R} \cup \mathcal{E}}$
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Certified Ordered Completion Proofs

- 94% of MædMax oKB proofs with KBO (58% of all oKB proofs)
- missing: LPO + trick to ignore CPs with ground joinable equations

Equational Satisfiability

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- initial equations $\mathcal{E}_0$
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Certified Satisfiability Proofs

- 100% of MædMax proofs with KBO (79% of all proofs)
- missing: LPO

Infeasibility

Certificate Components

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- ...

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Certified Conditional Confluence Proofs

- previously: 112 ConCon proofs, 109 certified
- with infeasibility checks using MædMax: 114 proofs, 111 certified

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Summary

- formalized finite ordered completion:
  - allowing variants: greatly simplifies output for tools
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Future Work

- support other orders
- support equational disproofs with narrowing
- certify more TPTP proofs (Instgen-Eq?)