Ordinals and Knuth-Bendix Orders

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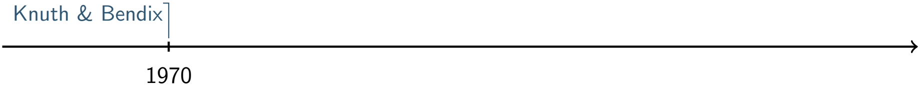
Glimpses into the History of the Knuth-Bendix Order



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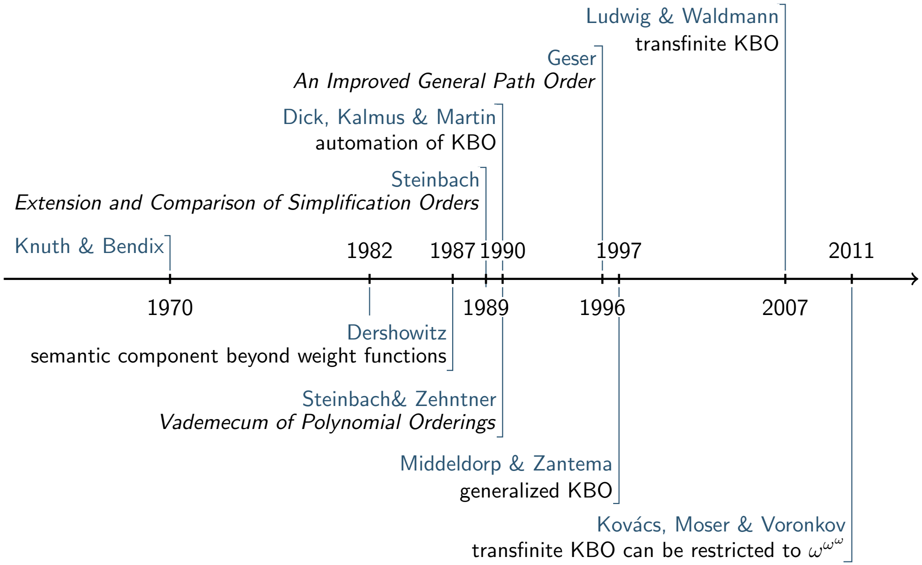

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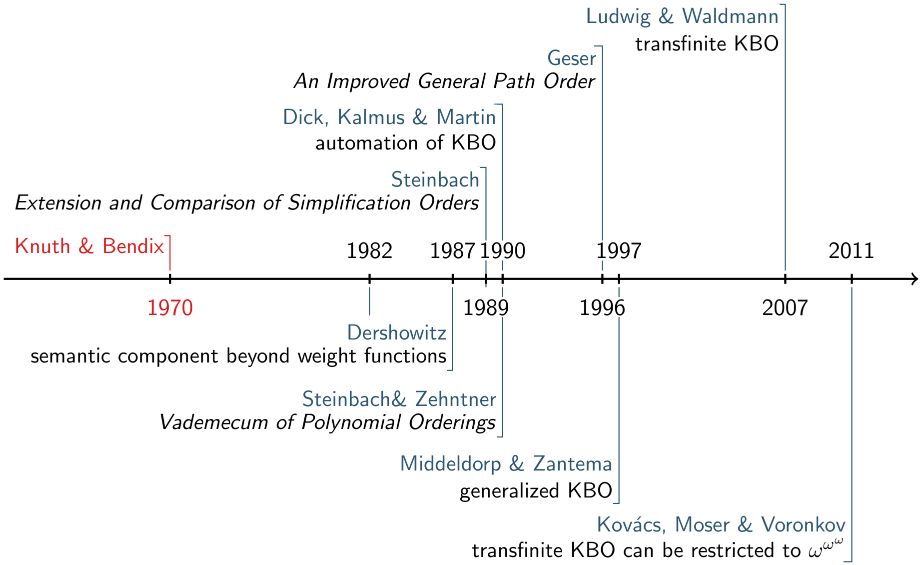

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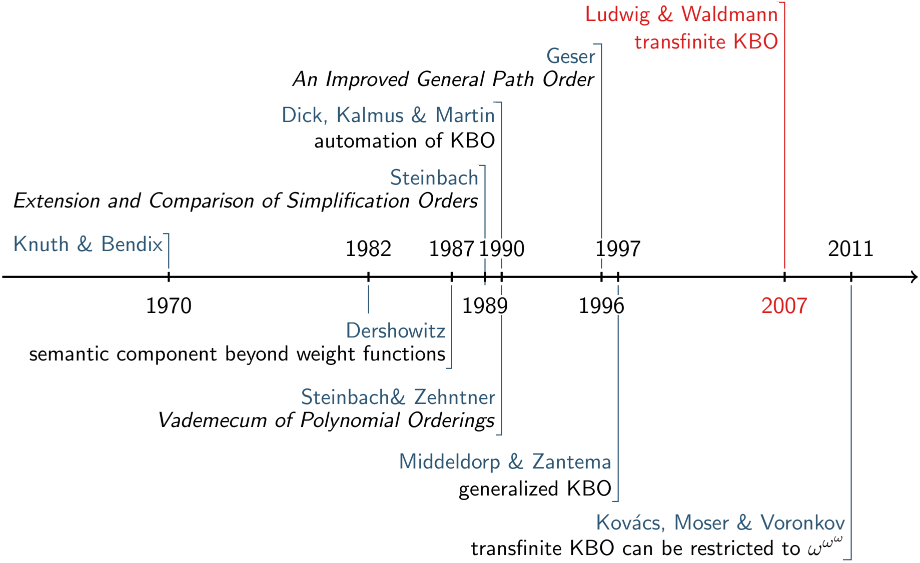

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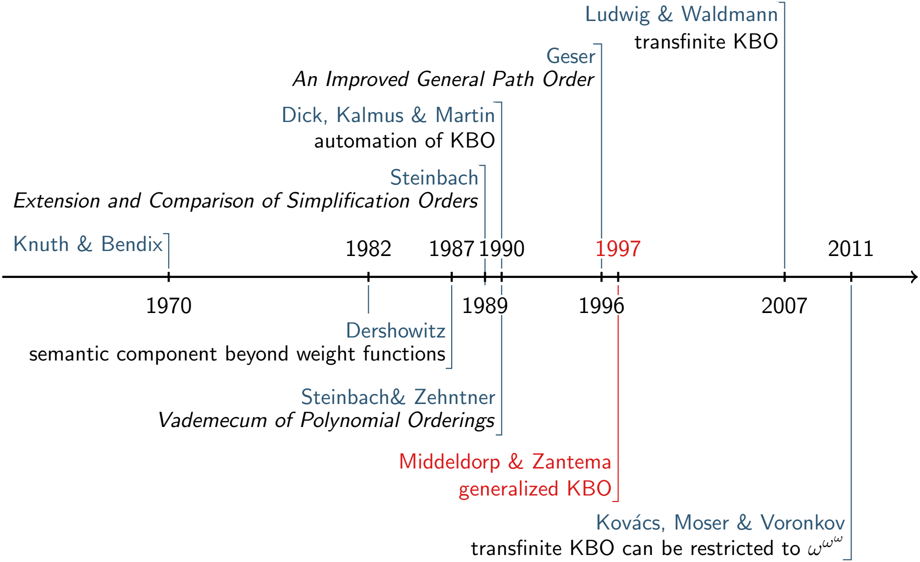

Overview

- Knuth-Bendix Order
- Transfinite Knuth-Bendix Order
- Generalized Knuth-Bendix Order
- Implementation
- Conclusion

Knuth-Bendix Order

Definition (precedence)

precedence $\succ$ is strict order on signature $\mathcal{F}$

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$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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(w, w_0) is admissible for $\succ$ if f unary with $w(f) = 0$ implies $f \succ g$ for $g \in \mathcal{F} \setminus \{f\}$

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in sequel assume (w, w_0) admissible for $\succ$

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Definition (KBO)

$s >_{\mathrm{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and

1. $w(s) > w(t)$, or
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   1. $s = f^k(t)$ and $t \in \mathcal{V}$ for some $k > 0$, or
   2. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$ and $f \succ g$, or
   3. $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $s_1 = t_1, \ldots, s_{k-1} = t_{k-1}$ and $s_k >_{\mathrm{kbo}} t_k$ with $1 \leqslant t \leqslant n$

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Theorem (Knuth, Bendix 1970)

TRS $\mathcal{R}$ terminating if $\ell >_{\mathsf{kbo}} r$ for each $\ell \to r \in \mathcal{R}$

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$w_5(\mathsf{h}) = 5$	$w_5(\mathsf{f}) = 10$	$(w_0)_5 = 5$	$\mathsf{h} \succ \mathsf{f}$	✓

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Effects

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- (tiny) improvement in execution time

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Ordinals are transitive sets well-ordered by $\in$

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$0, 1, 2, 3, 4, 5, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots$

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$\omega \cdot 42, \dots, \omega^2, \omega^2 + 1, \dots, \omega^3, \dots, \omega^4 + \omega \cdot 23, \dots, \omega^{1000} + \omega^2 + 54, \dots$

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Theorem (Cantor Normal Form (CNF))

every ordinal $\alpha \in \mathcal{O}$

set of ordinals below ϵ_0

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every ordinal $\alpha \in \mathcal{O}$ *can be uniquely written as*

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such that $\alpha > \alpha_1 > \cdots > \alpha_n$ *are in CNF and* $a_1, \ldots, a_n \in \mathbb{N}$ *are positive*

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Natural Addition $\oplus$ and Multiplication $\odot$ on Ordinals

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- distributive $\quad \alpha \odot (\beta \oplus \gamma) = \alpha \odot \beta \oplus \alpha \odot \gamma$
- monotone $\quad \alpha \oplus \gamma < \beta \oplus \gamma \quad \alpha \odot \delta < \beta \odot \delta$

  for $\alpha < \beta$ and $\delta > 0$

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and subterm coefficient function

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Example

$$w(\mathsf{f}) = \omega \quad s(\mathsf{f}, 1) = 2 \quad w(\mathsf{h}) = 1000 \quad s(\mathsf{h}, 1) = 1 \quad s(\mathsf{h}, 2) = 1 \quad w_0 = 1$$

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- $w(\mathsf{f}(x)) = \omega \oplus w_0 \odot 2 = \omega + 2$
- $w(\mathsf{h}(x, x)) = 1000 \oplus w_0 \odot 1 \oplus w_0 \odot 1 = 1000 \oplus w_0 \odot 2 = 1002$

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sum of coefficients
above all occurrences of x in s

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Example

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Theorem (Kovács, Moser and Voronkov, CADE 2011)

Let $\mathcal{R}$ be TRS over finite signature.
$\mathcal{R}$ *TKBO terminating* $\Longleftrightarrow$ $\mathcal{R}$ *TKBO terminating with weights* $< \omega^{\omega^{\omega}}$

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- $k = \max\{M(\alpha) + 1 \mid \alpha = w(t) \text{ for } t \text{ occurring in } \mathcal{R}\}$

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$$w_0 = 1 \quad w(\mathsf{g}) = 1 \quad w(\mathsf{f}) = 5 \quad w(\mathsf{h}) = \omega \quad w(\mathsf{k}) = \omega^2 + 1 \qquad \checkmark$$

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Generalized Knuth-Bendix Order

Standard Addition $+$ and Multiplication $\cdot$ on Ordinals

- not commutative $1 + \omega \neq \omega + 1$ $2 \cdot \omega \neq \omega \cdot 2$
- not right-distributive $(1 + 1) \cdot \omega \neq 1 \cdot \omega + 1 \cdot \omega$
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GKBO: $\mathsf{a}_{\mathcal{O}}(x) = x + \omega + 1$ $\qquad$ $\mathsf{b}_{\mathcal{O}}(x) = x + \omega$ $\qquad$ $\mathsf{c}_{\mathcal{O}}(x) = x + 2$ $\qquad$ ✓

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GKBO: $\mathsf{a}_{\mathcal{N}}(x) = 3x + 1 \quad \mathsf{b}_{\mathcal{N}}(x) = \mathsf{c}_{\mathcal{N}}(x) = x + 1 \quad \mathsf{b} \succ \mathsf{c} \quad \checkmark$

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GKBO with linear polynomials subsumes linear polynomials

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GKBO with polynomials does not subsume polynomials

Implementing Ordinal Interpretations (for SRSs)

Canonical Shape

$$f_{\mathcal{O}}(x) = x \cdot f' + \omega^d \cdot f_d + \cdots + \omega^1 \cdot f_1 + f_0$$

where $f', f_0, \ldots, f_d \in \mathbb{N}$ and $d \in \mathbb{N}$ is fixed dimension

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Example

encoding $\mathrm{a}(\mathrm{b}(x))$ with $d = 1$ yields

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Constraints

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encoded in SMT problem with non-linear integer arithmetic

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Evaluation

- termination experiments with $\mathsf{T_TT_2}$

Termination

method	1416 TRSs* yes	1416 TRSs* time	720 SRSs* yes	720 SRSs* time
KBO	107	0.5	33	0.5
TKBO	192	0.9	43	1.7
POLY	149	0.9	22	1.6
LPO	159	0.5	5	0.5
$\sum$	262	-	45	-

* taken from TPDB 7.0.2

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- termination experiments with $\mathsf{T_TT_2}$

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method	1416 TRSs* yes	1416 TRSs* time	720 SRSs* yes	720 SRSs* time
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$\sum$	262	-	45	-

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Evaluation

- termination experiments with $\mathsf{T_TT_2}$
- ordered completion experiments with OMKBTT

	Termination				Ordered Completion		
	1416 TRSs*		720 SRSs*		42 TPTP theories†		
method	yes	time	yes	time	yes	time	tc
KBO/KBO$_\omega$	107/-	0.5	33/34	0.5/0.7	31/-	16	19%
TKBO/TKBO$_\omega$	192/-	0.9	43/44	1.7/2.9	34/-	56	35%
POLY	149	0.9	22	1.6	15	4	70%
LPO	159	0.5	5	0.5	26	37	7%
$\sum$	262	-	45	-	34		

* taken from TPDB 7.0.2

† taken from UEQ division in TPTP 3.6.0

Conclusion

Summary

- w_0 in KBO can be fixed to arbitrary positive value
- finite weights suffice for finite TRSs in TKBO
- ordinals add power in GKBO
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Future Work

- implementation of ordinal interpretations for TRSs
- long-term goal: refutation of

Conjecture

If $\mathcal{R}$ admits a sound termination proof by $\mathsf{T_TT_2}$/AProVE then $\mathrm{dc}(\mathcal{R})$ is bounded by a multiply recursive function.