Are ground-complete systems unique?

Sarah Winkler

Institute for Computer Science
University of Innsbruck

Workshop Paris - Innsbruck - Tbilisi

May 20, 2010

Outline

- Preliminaries
- Are ground-complete systems unique?
- Conclusion

Definition

TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$

Definition

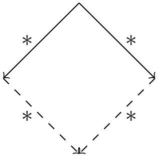
TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$
- confluent if $\forall\ s\ {}^*\!\!\leftarrow u \to^* t$
  there is some v such that $s \to^* v\ {}^*\!\!\leftarrow t$

Definition

TRS R is

- terminating if $\nexists\ t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow \cdots$
- confluent if $\forall\ s\ {}^*\!\!\longleftarrow u \longrightarrow^* t$
  there is some v such that $s \longrightarrow^* v\ {}^*\!\!\longleftarrow t$



Definition

TRS R is

- terminating if $\nexists\ t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow \cdots$
- confluent if $\forall\ s\ {}^*\!\!\longleftarrow u \longrightarrow^* t$
  there is some v such that $s \longrightarrow^* v\ {}^*\!\!\longleftarrow t$
- complete if terminating and confluent

Definition

TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$
- confluent if $\forall\ s \ {}^*\!\!\leftarrow u \to^* t$
  there is some v such that $s \to^* v \ {}^*\!\!\leftarrow t$
- complete if terminating and confluent
- reduced if $\forall\ l \to r$ in R
  $r \in \mathsf{NF}(R)$ and $l \in \mathsf{NF}(R \setminus \{l \to r\})$

normal forms of R

Definition

TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$
- confluent if $\forall\ s\ {}^*\!\leftarrow u \to^* t$
  there is some v such that $s \to^* v\ {}^*\!\leftarrow t$
- complete if terminating and confluent
- reduced if $\forall\ l \to r$ in R
  $r \in \mathrm{NF}(R)$ and $l \in \mathrm{NF}(R \setminus \{l \to r\})$

Example

$$
\begin{array}{rl}
g(f(x)) \to a & f(x) \to g(x) \\
g(g(x)) \to a & f(x) \to a \\
g(a) \to a &
\end{array}
$$

Definition

TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$
- confluent if $\forall\ s\ {}^*\!\!\leftarrow u \to^* t$
  there is some v such that $s \to^* v\ {}^*\!\!\leftarrow t$
- complete if terminating and confluent
- reduced if $\forall\ l \to r$ in R
  $r \in \mathrm{NF}(R)$ and $l \in \mathrm{NF}(R \setminus \{l \to r\})$

Example

$$g(f(x)) \to a \quad f(x) \to g(x)$$
$$g(g(x)) \to a \quad f(x) \to a$$
$$g(a) \to a$$

terminating ✓

confluent

Definition

TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$
- confluent if $\forall\ s\ {}^*\!\!\leftarrow u \to^* t$
  there is some v such that $s \to^* v\ {}^*\!\!\leftarrow t$
- complete if terminating and confluent
- reduced if $\forall\ l \to r$ in R
  $r \in \mathrm{NF}(R)$ and $l \in \mathrm{NF}(R \setminus \{l \to r\})$

Example

$$g(f(x)) \to a \quad f(x) \to g(x)$$
$$g(g(x)) \to a \quad f(x) \to a$$
$$g(a) \to a$$

terminating ✓
confluent ϟ
complete ϟ

$g(x) \leftarrow f(x) \to a$
not joinable

Definition

TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$
- confluent if $\forall\ s \ {}^*\!\!\leftarrow u \to^* t$
  there is some v such that $s \to^* v \ {}^*\!\!\leftarrow t$
- complete if terminating and confluent
- reduced if $\forall\ l \to r$ in R
  $r \in \mathrm{NF}(R)$ and $l \in \mathrm{NF}(R \setminus \{l \to r\})$

Example

$$
\begin{array}{rl}
g(f(x)) \to a & f(x) \to g(x) \\
g(g(x)) \to a & f(x) \to a \\
g(a) \to a &
\end{array}
$$

terminating	✓
confluent	↯
complete	↯
reduced	

Definition

TRS R is

- terminating if $\nexists\ t_1 \to t_2 \to t_3 \to \cdots$
- confluent if $\forall\ s\ {}^*\!\!\leftarrow u \to^* t$
  there is some v such that $s \to^* v\ {}^*\!\!\leftarrow t$
- complete if terminating and confluent
- reduced if $\forall\ l \to r$ in R
  $r \in \mathsf{NF}(R)$ and $l \in \mathsf{NF}(R \setminus \{l \to r\})$

Example

$$g(f(x)) \to a \quad f(x) \to g(x)$$
$$g(g(x)) \to a \quad f(x) \to a$$
$$g(a) \to a$$

terminating	✓
confluent	↯
complete	↯
reduced	↯

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Example

for $E = \{x + y \approx y + x\}$ and $>$ being LPO with precedence $f > a > b$

$$E_> = \{a + b \to b + a,$$

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Example

for $E = \{x + y \approx y + x\}$ and $>$ being LPO with precedence $f > a > b$

$$E_> = \{a + b \to b + a,\ f(a) + a \to a + f(a),$$

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Example

for $E = \{x + y \approx y + x\}$ and $>$ being LPO with precedence $f > a > b$

$$E_> = \{a + b \to b + a,\ f(a) + a \to a + f(a),\ f(b) + a \to a + f(b),$$

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Example

for $E = \{x + y \approx y + x\}$ and $>$ being LPO with precedence $f > a > b$

$$\begin{aligned} E_> = \{&a + b \to b + a,\ f(a) + a \to a + f(a),\ f(b) + a \to a + f(b), \\ &f(a) + f(b) \to f(b) + f(a),\ f(b) + b \to b + f(b), \\ &f(f(a)) + a \to a + f(f(a)),\ \ldots\} \end{aligned}$$

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Definition

system (E, R) is ground-complete wrt $>$

$\iff$ $\forall$ ground terms s, t with $s \leftrightarrow^* t$ using rules in $E_> \cup R$
$\exists$ v such that $s \to^* v \; {}^*\!\!\leftarrow t$ in $E_> \cup R$

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Definition

system (E, R) is ground-complete wrt $>$

$\iff$ $\forall$ ground terms s, t with $s \leftrightarrow^* t$ using rules in $E_> \cup R$
$\exists$ v such that $s \to^* v \; {}^*\!\leftarrow t$ in $E_> \cup R$

Example (1)

$$(E, R) = \begin{cases} g(f(x)) \to a & f(x) \to g(x) \\ g(g(x)) \to a & f(x) \to a \\ g(a) \to a & \end{cases}$$

ground complete (for $>$ being LPO with precedence $f > g$)

Consider reduction orders $\succ \subseteq >$ where $>$ is total on $\mathcal{T}(\mathcal{F})$, equations E and TRS R.

Definition

orientable instances of set of equations E wrt total reduction order $>$ is TRS

$$E_> = \{\hat{s} \to \hat{t} \mid \hat{s} \approx \hat{t} \text{ is instance of } s \approx t \in E \text{ and } \hat{s} > \hat{t}\}$$

Definition

system (E, R) is ground-complete wrt $>$

$\iff$ $\forall$ ground terms s, t with $s \leftrightarrow^* t$ using rules in $E_> \cup R$
$\exists$ v such that $s \to^* v \; {}^*\!\leftarrow t$ in $E_> \cup R$

Example (2)

$$(E, R) = \begin{cases} x \cdot i(x) \approx i(y) \cdot y \\ x \cdot i(x) \approx y \cdot i(y) \\ i(x) \cdot x \approx i(y) \cdot y \\ \mathsf{f}(x \cdot i(x)) \to 0 \\ \mathsf{f}(i(x) \cdot x) \to 0 \end{cases}$$

ground complete (for $>$ being LPO with precedence $f > g$)

Definition (Extended critical pair)

if $t \xleftarrow{r_1\sigma \leftarrow l_1\sigma} u \xrightarrow{l_2\sigma \rightarrow r_2\sigma} s$ such that $l_i \approx r_i \in E \cup R$ and $r_i\sigma \not\succ l_i\sigma$

Definition (Extended critical pair)

if $t \xleftarrow{r_1\sigma \leftarrow l_1\sigma} u \xrightarrow{l_2\sigma \rightarrow r_2\sigma} s$ such that $l_i \approx r_i \in E \cup R$ and $r_i\sigma \not\succ l_i\sigma$ then $s \approx t$ is in $CP_{\succ}(E \cup R)$

Definition (Extended critical pair)

if $t \xleftarrow{r_1\sigma \leftarrow l_1\sigma} u \xrightarrow{l_2\sigma \rightarrow r_2\sigma} s$ such that $l_i \approx r_i \in E \cup R$ and $r_i\sigma \not\succ l_i\sigma$
then $s \approx t$ is in $CP_{\succ}(E \cup R)$

Example

$\xleftarrow{x \cdot i(x) \approx i(y) \cdot y}$ $\xrightarrow{f(x \cdot i(x)) \rightarrow 0}$

Definition (Extended critical pair)

if $t \xleftarrow{r_1\sigma \leftarrow l_1\sigma} u \xrightarrow{l_2\sigma \to r_2\sigma} s$ such that $l_i \approx r_i \in E \cup R$ and $r_i\sigma \not\succ l_i\sigma$ then $s \approx t$ is in $CP_{\succ}(E \cup R)$

Example

$$\xleftarrow{x \cdot i(x) \approx i(y) \cdot y} f(x \cdot i(x)) \xrightarrow{f(x \cdot i(x)) \to 0}$$

Definition (Extended critical pair)

if $t \xleftarrow{r_1\sigma \leftarrow l_1\sigma} u \xrightarrow{l_2\sigma \rightarrow r_2\sigma} s$ such that $l_i \approx r_i \in E \cup R$ and $r_i\sigma \not\succ l_i\sigma$ then $s \approx t$ is in $CP_{\succ}(E \cup R)$

Example

$$f(i(y) \cdot y) \xleftarrow{x \cdot i(x) \approx i(y) \cdot y} f(x \cdot i(x)) \xrightarrow{f(x \cdot i(x)) \rightarrow 0} 0$$

Definition (Extended critical pair)

if $t \xleftarrow{r_1\sigma \leftarrow l_1\sigma} u \xrightarrow{l_2\sigma \rightarrow r_2\sigma} s$ such that $l_i \approx r_i \in E \cup R$ and $r_i\sigma \not\succ l_i\sigma$ then $s \approx t$ is in $CP_{\succ}(E \cup R)$

Example

$$f(i(y) \cdot y) \xleftarrow{x \cdot i(x) \approx i(y) \cdot y} f(x \cdot i(x)) \xrightarrow{f(x \cdot i(x)) \rightarrow 0} 0$$

yields $CP_{\succ}$ $f(i(y) \cdot y) \approx 0$

Definition (Extended critical pair)

if $t \xleftarrow{r_1\sigma \leftarrow l_1\sigma} u \xrightarrow{l_2\sigma \rightarrow r_2\sigma} s$ such that $l_i \approx r_i \in E \cup R$ and $r_i\sigma \not\succ l_i\sigma$ then $s \approx t$ is in $CP_{\succ}(E \cup R)$

Example

$$f(i(y) \cdot y) \xleftarrow{x \cdot i(x) \approx i(y) \cdot y} f(x \cdot i(x)) \xrightarrow{f(x \cdot i(x)) \rightarrow 0} 0$$

yields $CP_{\succ}$ $f(i(y) \cdot y) \approx 0$

Lemma (Extended critical pair lemma)

for ground peak $u \leftarrow \cdot \rightarrow v$ in $E_{>} \cup R$

- *either $\exists w$ such that $u \rightarrow^* w \; {}^*\!\!\leftarrow v$*
- *or $u \approx v$ is $C[s\sigma] \approx C[t\sigma]$ where $s \approx t \in CP_{>}(E \cup R)$*

Equational Proofs

Definition

proof of $s_0 \leftrightarrow^* s_n$ in (E, R) is sequence $s_0, s_1, \ldots, s_n$ of terms

Equational Proofs

Definition

proof of $s_0 \leftrightarrow^* s_n$ in (E, R) is sequence $s_0, s_1, \ldots, s_n$ of terms such that for all $0 \leq i < n$

$$s_i \leftrightarrow_E s_{i+1}$$

Equational Proofs

Definition

proof of $s_0 \leftrightarrow^* s_n$ in (E, R) is sequence $s_0, s_1, \ldots, s_n$ of terms such that for all $0 \leq i < n$

$$s_i \leftrightarrow_E s_{i+1} \qquad \text{or} \qquad s_i \rightarrow_R s_{i+1} \qquad \text{or} \qquad s_{i+1} \rightarrow_R s_i$$

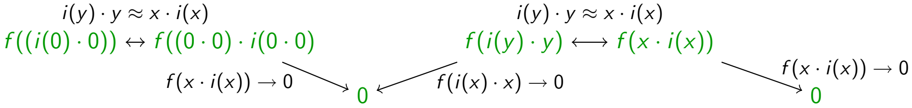
Equational Proofs

Definition

proof of $s_0 \leftrightarrow^* s_n$ in (E, R) is sequence $s_0, s_1, \ldots, s_n$ of terms such that for all $0 \leq i < n$

$$s_i \leftrightarrow_E s_{i+1} \quad \text{or} \quad s_i \to_R s_{i+1} \quad \text{or} \quad s_{i+1} \to_R s_i$$

Example



Equational Proofs

Definition

proof of $s_0 \leftrightarrow^* s_n$ in (E, R) is sequence $s_0, s_1, \ldots, s_n$ of terms such that for all $0 \leq i < n$

$$s_i \leftrightarrow_E s_{i+1} \quad \text{or} \quad s_i \rightarrow_R s_{i+1} \quad \text{or} \quad s_{i+1} \rightarrow_R s_i$$

Definition

Given proof $P = s_0, s_1, \ldots, s_n$,

- for substitution σ, $P\sigma = s_0\sigma, s_1\sigma, \ldots, s_n\sigma$

Equational Proofs

Definition

proof of $s_0 \leftrightarrow^* s_n$ in (E, R) is sequence $s_0, s_1, \ldots, s_n$ of terms such that for all $0 \leq i < n$

$$s_i \leftrightarrow_E s_{i+1} \qquad \text{or} \qquad s_i \rightarrow_R s_{i+1} \qquad \text{or} \qquad s_{i+1} \rightarrow_R s_i$$

Definition

Given proof $P = s_0, s_1, \ldots, s_n$,

- for substitution σ, $P\sigma = s_0\sigma, s_1\sigma, \ldots, s_n\sigma$
- for context C, $C[P] = C[s_0], C[s_1], \ldots, C[s_n]$

Equational Proofs

Definition

proof of $s_0 \leftrightarrow^* s_n$ in (E, R) is sequence $s_0, s_1, \ldots, s_n$ of terms such that for all $0 \leq i < n$

$$s_i \leftrightarrow_E s_{i+1} \qquad \text{or} \qquad s_i \rightarrow_R s_{i+1} \qquad \text{or} \qquad s_{i+1} \rightarrow_R s_i$$

Definition

Given proof $P = s_0, s_1, \ldots, s_n$,

- for substitution σ, $P\sigma = s_0\sigma, s_1\sigma, \ldots, s_n\sigma$
- for context C, $C[P] = C[s_0], C[s_1], \ldots, C[s_n]$
- write $Q[P]$ if Q contains P as a subproof

Uniqueness of complete systems

Theorem *Métivier 83*

Let R_1 and R_2 be

- *reduced and*
- *complete such that*
- $R_1 \subseteq \succ$ *and* $R_2 \subseteq \succ$
- *and* $\leftrightarrow^*_{R_1} = \leftrightarrow^*_{R_2}$.

Then R_1 and R_2 are the same (up to renaming variables).

Uniqueness of complete systems

Theorem

Métivier 83

Let R_1 and R_2 be

- *reduced and*
- *complete such that*
- $R_1 \subseteq \succ$ *and* $R_2 \subseteq \succ$
- *and* $\leftrightarrow^*_{R_1} = \leftrightarrow^*_{R_2}$.

Then R_1 and R_2 are the same (up to renaming variables).

Question

How about ground-complete systems for same theory and reduction order?

Are ground-complete systems unique?

Definition

(E, R) is compatible with $\succ$

$\iff \quad R \subseteq \succ$ and $\succ \cap E = \varnothing$

Are ground-complete systems unique?

Definition

(E, R) is compatible with $\succ$
$\iff \quad R \subseteq \succ$ and $\succ \cap E = \varnothing$

Corollary

Assume (E_1, R_1) and (E_2, R_2) are compatible with $\succ$ and ground-complete wrt $> \supseteq \succ$ such that $\leftrightarrow^*_{E_1 \cup R_1}$ and $\leftrightarrow^*_{E_2 \cup R_2}$ coincide on ground terms.

Are ground-complete systems unique?

Definition

(E, R) is compatible with $\succ$

$\iff \quad R \subseteq \succ$ and $\succ \cap E = \varnothing$

Corollary

Assume (E_1, R_1) and (E_2, R_2) are compatible with $\succ$ and ground-complete wrt $> \supseteq \succ$ such that $\leftrightarrow^*_{E_1 \cup R_1}$ and $\leftrightarrow^*_{E_2 \cup R_2}$ coincide on ground terms.

Then reduced forms of TRSs containing all ground rules in $(E_1 \cup R_1)_>$ and $(E_2 \cup R_2)_>$ are the same (up to renaming variables).

Are ground-complete systems unique?

Definition

(E, R) is compatible with $\succ$
$\iff \quad R \subseteq \succ$ and $\succ \cap E = \varnothing$

Corollary

Assume (E_1, R_1) and (E_2, R_2) are compatible with $\succ$ and ground-complete wrt $> \supseteq \succ$ such that $\leftrightarrow^*_{E_1 \cup R_1}$ and $\leftrightarrow^*_{E_2 \cup R_2}$ coincide on ground terms.
Then reduced forms of TRSs containing all ground rules in $(E_1 \cup R_1)_>$ and $(E_2 \cup R_2)_>$ are the same (up to renaming variables).

Question

Are also (E_1, R_1) and (E_2, R_2) the same?

Example (1)

ground-complete systems for same theory

$$(E_1, R_1) = \left\{ \begin{array}{c} x + y \approx y + x \\ \mathsf{g}(x + y) \approx \mathsf{g}(y + x) \\ \mathsf{f}(x, x) \to \mathsf{g}(x) \end{array} \right. \quad (E_2, R_2) = \left\{ \begin{array}{r} x + y \approx y + x \\ \mathsf{f}(x, x) \to \mathsf{g}(x) \\ \mathsf{f}(x + y, y + x) \to \mathsf{g}(x + y) \end{array} \right.$$

are compatible with $\succ$ being LPO with precedence $f > g$

Example (1)

ground-complete systems for same theory

$$(E_1, R_1) = \left\{ \begin{array}{c} x + y \approx y + x \\ \mathsf{g}(x + y) \approx \mathsf{g}(y + x) \\ \mathsf{f}(x, x) \to \mathsf{g}(x) \end{array} \right. \quad (E_2, R_2) = \left\{ \begin{array}{rl} x + y & \approx\ y + x \\ \mathsf{f}(x, x) & \to \mathsf{g}(x) \\ \mathsf{f}(x + y, y + x) & \to \mathsf{g}(x + y) \end{array} \right.$$

are compatible with $\succ$ being LPO with precedence $f > g$

Problem

superfluous equations

Example (1)

ground-complete systems for same theory

$$(E_1, R_1) = \left\{ \begin{array}{rcl} x + y & \approx & y + x \\ \mathsf{g}(x + y) & \approx & \mathsf{g}(y + x) \\ \mathsf{f}(x, x) & \rightarrow & \mathsf{g}(x) \end{array} \right. \quad (E_2, R_2) = \left\{ \begin{array}{rcl} x + y & \approx & y + x \\ \mathsf{f}(x, x) & \rightarrow & \mathsf{g}(x) \\ \mathsf{f}(x + y, y + x) & \rightarrow & \mathsf{g}(x + y) \end{array} \right.$$

are compatible with $\succ$ being LPO with precedence $f > g$

Problem

superfluous equations and rules prevent uniqueness

Example (1)

ground-complete systems for same theory

$$(E_1, R_1) = \left\{ \begin{array}{rl} x + y & \approx y + x \\ \mathsf{g}(x + y) & \approx \mathsf{g}(y + x) \\ \mathsf{f}(x, x) & \to \mathsf{g}(x) \end{array} \right. \quad (E_2, R_2) = \left\{ \begin{array}{rl} x + y & \approx y + x \\ \mathsf{f}(x, x) & \to \mathsf{g}(x) \\ \mathsf{f}(x + y, y + x) & \to \mathsf{g}(x + y) \end{array} \right.$$

are compatible with $\succ$ being LPO with precedence $f > g$

Problem

superfluous equations and rules prevent uniqueness

Solution

restrict to reduced systems

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \to^{p}_{l \to r} t \text{ for some } l \to r \text{ in } R \\ & \\ & \end{cases}$$

When is (E, R) reduced?

Definition

- cost function for proof step

$\top > t$ for all terms t

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \rightarrow^{p}_{l \rightarrow r} t \text{ for some } l \rightarrow r \text{ in } R \\ (\{s\}, s|_p, l, \{t, \top\}) & \text{if } s \rightarrow^{p}_{l \rightarrow r} t \text{ for some } l \rightarrow r \text{ in } E_{>} \\ \end{cases}$$

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \to^p_{l \to r} t \text{ for some } l \to r \text{ in } R \\ (\{s\}, s|_p, l, \{t, \top\}) & \text{if } s \to^p_{l \to r} t \text{ for some } l \to r \text{ in } E_> \\ (\{s, t\}, -, -, -) & \text{if } s \approx_E t \end{cases}$$

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \to^{p}_{l \to r} t \text{ for some } l \to r \text{ in } R \\ (\{s\}, s|_p, l, \{t, \top\}) & \text{if } s \to^{p}_{l \to r} t \text{ for some } l \to r \text{ in } E_{>} \\ (\{s, t\}, -, -, -) & \text{if } s \approx_E t \end{cases}$$

- order $>^c$ on costs is lexicographic combination of $>_{\text{mul}}$, $\rhd$, $\mathrel{\cdot\!\!\!\rhd}$ and $>_{\text{mul}}$

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \to^{p}_{l\to r} t \text{ for some } l \to r \text{ in } R \\ (\{s\}, s|_p, l, \{t, \top\}) & \text{if } s \to^{p}_{l\to r} t \text{ for some } l \to r \text{ in } E_{>} \\ (\{s, t\}, -, -, -) & \text{if } s \approx_E t \end{cases}$$

- order $>^c$ on costs is lexicographic combination of $>_{\mathrm{mul}}$, $\triangleright$, $\dot{\triangleright}$ and $>_{\mathrm{mul}}$
- order $>_{\mathcal{U}}$ on ground proofs: $(s_0, \ldots, s_n) >_{\mathcal{U}} (t_0, \ldots, t_m)$ iff

$$\{c(s_0, s_1), \ldots, c(s_{n-1}, s_n)\} >^{c}_{\mathrm{mul}} \{c(t_0, t_1), \ldots, c(t_{m-1}, t_m)\}$$

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \to^p_{l\to r} t \text{ for some } l \to r \text{ in } R \\ (\{s\}, s|_p, l, \{t, \top\}) & \text{if } s \to^p_{l\to r} t \text{ for some } l \to r \text{ in } E_> \\ (\{s,t\}, -, -, -) & \text{if } s \approx_E t \end{cases}$$

- order $>^c$ on costs is lexicographic combination of $>_{\mathrm{mul}}$, $\rhd$, $\mathrel{\ooalign{$\rhd$\cr$\cdot$}}$ and $>_{\mathrm{mul}}$
- order $>_{\mathcal{U}}$ on ground proofs: $(s_0, \dots, s_n) >_{\mathcal{U}} (t_0, \dots, t_m)$ iff

$$\{c(s_0, s_1), \dots, c(s_{n-1}, s_n)\} >^c_{\mathrm{mul}} \{c(t_0, t_1), \dots, c(t_{m-1}, t_m)\}$$

Lemma

$>_{\mathcal{U}}$ is well-founded ordering on proofs and for all proofs P, P' with $P >_{\mathcal{U}} P'$

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \to^p_{l \to r} t \text{ for some } l \to r \text{ in } R \\ (\{s\}, s|_p, l, \{t, \top\}) & \text{if } s \to^p_{l \to r} t \text{ for some } l \to r \text{ in } E_> \\ (\{s, t\}, -, -, -) & \text{if } s \approx_E t \end{cases}$$

- order $>^c$ on costs is lexicographic combination of $>_{\text{mul}}$, $\rhd$, $\unrhd$ and $>_{\text{mul}}$
- order $>_{\mathcal{U}}$ on ground proofs: $(s_0, \ldots, s_n) >_{\mathcal{U}} (t_0, \ldots, t_m)$ iff

$$\{c(s_0, s_1), \ldots, c(s_{n-1}, s_n)\} >^c_{\text{mul}} \{c(t_0, t_1), \ldots, c(t_{m-1}, t_m)\}$$

Lemma

$>_{\mathcal{U}}$ is well-founded ordering on proofs and for all proofs P, P' with $P >_{\mathcal{U}} P'$

- $C[P\sigma] >_{\mathcal{U}} C[P'\sigma]$ *for all contexts C and substitutions σ*

When is (E, R) reduced?

Definition

- cost function for proof step (s, t) in (E, R)

$$c(s,t) = \begin{cases} (\{s\}, s|_p, l, \{t\}) & \text{if } s \to^p_{l \to r} t \text{ for some } l \to r \text{ in } R \\ (\{s\}, s|_p, l, \{t, \top\}) & \text{if } s \to^p_{l \to r} t \text{ for some } l \to r \text{ in } E_> \\ (\{s, t\}, -, -, -) & \text{if } s \approx_E t \end{cases}$$

- order $>^c$ on costs is lexicographic combination of $>_{\text{mul}}$, $\rhd$, $\blacktriangleright$ and $>_{\text{mul}}$
- order $>_{\mathcal{U}}$ on ground proofs: $(s_0, \dots, s_n) >_{\mathcal{U}} (t_0, \dots, t_m)$ iff

$$\{c(s_0, s_1), \dots, c(s_{n-1}, s_n)\} >^c_{\text{mul}} \{c(t_0, t_1), \dots, c(t_{m-1}, t_m)\}$$

Lemma

$>_{\mathcal{U}}$ is well-founded ordering on proofs and for all proofs P, P' with $P >_{\mathcal{U}} P'$

- *$C[P\sigma] >_{\mathcal{U}} C[P'\sigma]$ for all contexts C and substitutions σ*
- *$Q[P] >_{\mathcal{U}} Q[P']$ for all proofs Q*

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff \quad \exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ $\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$
- non-ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ all its ground instances are redundant

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ $\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$
- non-ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ all its ground instances are redundant
- (E, R) is reduced with respect to $>$
  $\iff$ no equation or rule in (E, R) is redundant with respect to $>$

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ $\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$
- non-ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ all its ground instances are redundant
- (E, R) is reduced with respect to $>$
  $\iff$ no equation or rule in (E, R) is redundant with respect to $>$

Example

$$(E_2, R_2) = \left\{ \begin{array}{cl} x + y \approx y + x & (1) \\ \mathsf{g}(x + y) \approx \mathsf{g}(y + x) & (2) \\ \mathsf{f}(x, x) \to \mathsf{g}(x) & (3) \end{array} \right.$$

For example $g(a + b) \xrightarrow{(2)} g(b + a)$ $\quad$ $g(a + b) \xrightarrow{(1)} g(b + a)$

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ $\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$
- non-ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ all its ground instances are redundant
- (E, R) is reduced with respect to $>$
  $\iff$ no equation or rule in (E, R) is redundant with respect to $>$

Example

$$(E_2, R_2) = \begin{cases} x + y \approx y + x & (1) \\ \mathsf{g}(x + y) \approx \mathsf{g}(y + x) & (2) \\ \mathsf{f}(x, x) \to \mathsf{g}(x) & (3) \end{cases}$$

For example $g(a + b) \xrightarrow{(2)} g(b + a) >_{\mathcal{U}} g(a + b) \xrightarrow{(1)} g(b + a)$

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ $\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$
- non-ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ all its ground instances are redundant
- (E, R) is reduced with respect to $>$
  $\iff$ no equation or rule in (E, R) is redundant with respect to $>$

Example

$$(E_2, R_2) = \begin{cases} x + y \approx y + x & (1) \\ \mathsf{g}(x + y) \approx \mathsf{g}(y + x) & (2) \\ \mathsf{f}(x, x) \to \mathsf{g}(x) & (3) \end{cases}$$

For example $g(a + b) \xrightarrow{(2)} g(b + a) >_{\mathcal{U}} g(a + b) \xrightarrow{(1)} g(b + a)$ because
$(\{g(a + b)\}, g(a + b), \ldots) >_c (\{g(a + b)\}, a + b, \ldots\})$

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ $\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$
- non-ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ all its ground instances are redundant
- (E, R) is reduced with respect to $>$
  $\iff$ no equation or rule in (E, R) is redundant with respect to $>$

Example

$$(E_2, R_2) = \begin{cases} x + y \approx y + x & (1) \\ \mathsf{g}(x + y) \approx \mathsf{g}(y + x) & (2) \text{ redundant} \\ \mathsf{f}(x, x) \to \mathsf{g}(x) & (3) \end{cases}$$

For example $g(a+b) \xrightarrow{(2)} g(b+a) >_{\mathcal{U}} g(a+b) \xrightarrow{(1)} g(b+a)$ because
$(\{g(a+b)\}, g(a+b), \ldots) >_c (\{g(a+b)\}, a+b, \ldots\})$

Definition

Löchner 04

- ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ $\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $s \leftrightarrow t >_{\mathcal{U}} P$
- non-ground $s \leftrightarrow t$ is redundant in (E, R) with respect to $>$
  $\iff$ all its ground instances are redundant
- (E, R) is reduced with respect to $>$
  $\iff$ no equation or rule in (E, R) is redundant with respect to $>$

Example

$$(E_2, R_2) = \begin{cases} x + y \approx y + x & (1) \\ \mathsf{g}(x + y) \approx \mathsf{g}(y + x) & (2)\ \text{redundant} \\ \mathsf{f}(x, x) \rightarrow \mathsf{g}(x) & (3) \end{cases}$$

For example $g(a + b) \xrightarrow{(2)} g(b + a) >_{\mathcal{U}} g(a + b) \xrightarrow{(1)} g(b + a)$ because $(\{g(a + b)\}, g(a + b), \ldots) >_c (\{g(a + b)\}, a + b, \ldots\})$

▶ not reduced

Example (2)

ground-complete systems for same theory

$$(E_1, R_1) = \begin{cases} x + y \approx y + x \\ \mathrm{f}(x, y) \to \mathrm{g}(x + y) \end{cases} \qquad (E_2, R_2) = \begin{cases} x + y \approx y + x \\ \mathrm{f}(x, y) \to \mathrm{g}(y + x) \end{cases}$$

compatible with $\succ$ being LPO with precedence $f > g$

Example (2)

ground-complete systems for same theory

$$(E_1, R_1) = \left\{ \begin{array}{c} x + y \approx y + x \\ \mathsf{f}(x, y) \to \mathsf{g}(x + y) \end{array} \right. \qquad (E_2, R_2) = \left\{ \begin{array}{c} x + y \approx y + x \\ \mathsf{f}(x, y) \to \mathsf{g}(y + x) \end{array} \right.$$

compatible with $\succ$ being LPO with precedence $f > g$

Problem

different right-hand sides of rewrite rules

Example (3)

ground-complete systems for same theory

$$(E_1, R_1) = \left\{ \begin{array}{rl} g(x) & \to a \\ f(x) & \to g(x) \\ f(x) & \to a \end{array} \right. \qquad (E_2, R_2) = \left\{ \begin{array}{rl} g(f(x)) & \to a \\ g(g(x)) & \to a \\ g(a) & \to a \\ f(x) & \to g(x) \\ f(x) & \to a \end{array} \right.$$

compatible with $\succ$ being LPO where $f > g$

Example (3)

ground-complete systems for same theory

$$(E_1, R_1) = \begin{cases} g(x) \to a \\ f(x) \to g(x) \\ f(x) \to a \end{cases} \qquad (E_2, R_2) = \begin{cases} g(f(x)) \to a \\ g(g(x)) \to a \\ g(a) \to a \\ f(x) \to g(x) \\ f(x) \to a \end{cases}$$

compatible with $\succ$ being LPO where $f > g$

Problem

one rule in R_1 plays role of three rules in R_2

Example (3)

ground-complete systems for same theory

$$(E_1, R_1) = \begin{cases} g(x) \to a \\ f(x) \to g(x) \\ f(x) \to a \end{cases} \qquad (E_2, R_2) = \begin{cases} g(f(x)) \to a \\ g(g(x)) \to a \\ g(a) \to a \\ f(x) \to g(x) \\ f(x) \to a \end{cases}$$

compatible with $\succ$ being LPO where $f > g$

Problem

one rule in R_1 plays role of three rules in R_2

Definition

(E, R) compatible with reduction order $\succ$ is fairly constructed

$\iff$ for every $s \leftarrow u \rightarrow t$ in $CP_{\succ}(E \cup R)$

$\exists$ proof P of $s \leftrightarrow^* t$ in (E, R) such that $(s, u, t) \succ_{\mathcal{U}} P$

A (non-)result

Assume all $u \approx v$ in $E_1 \cup E_2$ satisfy $\mathrm{Var}(u) = \mathrm{Var}(v)$.

A (non-)result

Assume all $u \approx v$ in $E_1 \cup E_2$ satisfy $\mathrm{Var}(u) = \mathrm{Var}(v)$.

Claim

Let (E_1, R_1) and (E_2, R_2) be two systems

- compatible with reduction order $\succ$,
- ground-complete and reduced for total reduction order $>\supseteq\succ$,and
- fairly constructed
- such that $\leftrightarrow^*_{E_1 \cup R_1} = \leftrightarrow^*_{E_2 \cup R_2}$ on ground terms.

A (non-)result

Assume all $u \approx v$ in $E_1 \cup E_2$ satisfy $\mathrm{Var}(u) = \mathrm{Var}(v)$.

Claim

Let (E_1, R_1) and (E_2, R_2) be two systems

- compatible with reduction order $\succ$,
- ground-complete and reduced for total reduction order $>\supseteq\succ$,and
- fairly constructed
- such that $\leftrightarrow^*_{E_1 \cup R_1} = \leftrightarrow^*_{E_2 \cup R_2}$ on ground terms.

Then

- for ground instance $\hat{u} \approx \hat{v}$ of $u \approx v$ in E_i
  $\exists\ u' \approx v'$ in E_j such that $\hat{u} \approx \hat{v}$ is instance of $u' \approx v'$
- reducible ground terms in R_1 and R_2 coincide

up to renaming variables.

Proof attempt (1)

- assume there is some
  - equation that is instance of $u \approx v$ in E_i but not of any $u' \approx v'$ in E_j
  - term reducible by $u \to v$ in R_i but not in R_j

then $\exists$ ground instance $\hat{u} \leftrightarrow \hat{v}$ having no smaller proof in E_i, R_i

Proof attempt (1)

- assume there is some
  - equation that is instance of $u \approx v$ in E_i but not of any $u' \approx v'$ in E_j
  - term reducible by $u \rightarrow v$ in R_i but not in R_j

  then $\exists$ ground instance $\hat{u} \leftrightarrow \hat{v}$ having no smaller proof in E_i, R_i
- choose such $(\hat{u}, \hat{v})$ minimal wrt to $>_{\mathcal{U}}$ (wlog, $u \leftrightarrow v$ in (E_1, R_1))

Proof attempt (1)

- assume there is some
  - equation that is instance of $u \approx v$ in E_i but not of any $u' \approx v'$ in E_j
  - term reducible by $u \to v$ in R_i but not in R_j

  then $\exists$ ground instance $\hat{u} \leftrightarrow \hat{v}$ having no smaller proof in E_i, R_i
- choose such $(\hat{u}, \hat{v})$ minimal wrt to $>_{\mathcal{U}}$ (wlog, $u \leftrightarrow v$ in (E_1, R_1))

note that $\forall\ (\hat{s}, \hat{t}) <_{\mathcal{U}} (\hat{u}, \hat{v})$

- if $\hat{s} \approx \hat{t}$ instance of $s \approx t$ in E_2
  either $\exists\ s' \approx t'$ in E_1, or $\exists$ proof Q of $\hat{s} \leftrightarrow^* \hat{t}$ in (E_1, R_1) with $(\hat{s}, \hat{t}) >_{\mathcal{U}} Q$
- if $\hat{s} \to \hat{t}$ instance of $s \to t$ in R_2
  either $\hat{s}$ reducible in R_1, or $\exists$ proof Q of $\hat{s} \leftrightarrow^* \hat{t}$ in (E_1, R_1) with $(\hat{s}, \hat{t}) >_{\mathcal{U}} Q$

Proof attempt (1)

- assume there is some
  - equation that is instance of $u \approx v$ in E_i but not of any $u' \approx v'$ in E_j
  - term reducible by $u \to v$ in R_i but not in R_j

  then $\exists$ ground instance $\hat{u} \leftrightarrow \hat{v}$ having no smaller proof in E_i, R_i
- choose such $(\hat{u}, \hat{v})$ minimal wrt to $>_{\mathcal{U}}$ (wlog, $u \leftrightarrow v$ in (E_1, R_1))

note that $\forall\ (\hat{s}, \hat{t}) <_{\mathcal{U}} (\hat{u}, \hat{v})$

- if $\hat{s} \approx \hat{t}$ instance of $s \approx t$ in E_2
  either $\exists\ s' \approx t'$ in E_1, or $\exists$ proof Q of $\hat{s} \leftrightarrow^* \hat{t}$ in (E_1, R_1) with $(\hat{s}, \hat{t}) >_{\mathcal{U}} Q$
- if $\hat{s} \to \hat{t}$ instance of $s \to t$ in R_2
  either $\hat{s}$ reducible in R_1, or $\exists$ proof Q of $\hat{s} \leftrightarrow^* \hat{t}$ in (E_1, R_1) with $(\hat{s}, \hat{t}) >_{\mathcal{U}} Q$

- ground-complete system (E_2, R_2) allows for proof P

$$\hat{u} \to t_1 \to t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

  which is minimal wrt $>_{\mathcal{U}}$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \ldots \rightarrow t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ldots \to t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \ldots \ \to t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$

  - $u_1 \approx v_1$ and $u_2 \leftrightarrow v_2$ form extended critical pair in $CP_{\succ}(E_2, R_2)$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$ ↯

  - $u_1 \approx v_1$ and $u_2 \leftrightarrow v_2$ form extended critical pair in $CP_{\succ}(E_2, R_2)$
  - P not minimal as (E_2, R_2) fairly constructed

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \dots \rightarrow t_k \leftarrow \dots \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{Pos}_{\mathcal{F}}(v_1)$ ↯
  case $p_2 = q_0 q_1$ for $q_0 \in \mathcal{Pos}_{\mathcal{V}}(v_1)$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \ \ldots \ \rightarrow t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$ ↯
  case $p_2 = q_0 q_1$ for $q_0 \in \mathcal{P}os_{\mathcal{V}}(v_1)$

  - $\exists\, \hat{u}'$ such that $\hat{u} \xrightarrow{u_2 \leftrightarrow v_2} \hat{u}'$
    (as u_1, v_1 have same variables)

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$ ↯

  case $p_2 = q_0 q_1$ for $q_0 \in \mathcal{P}os_{\mathcal{V}}(v_1)$

  - $\exists\, \hat{u}'$ such that $\hat{u} \xrightarrow{u_2 \leftrightarrow v_2} \hat{u}'$
    (as u_1, v_1 have same variables)
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$
    so some proof of $u_2\sigma_2 \leftrightarrow^* v_2\sigma_2$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$ ($\bigstar$)

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \ldots \ \to t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$ ↯

  case $p_2 = q_0 q_1$ for $q_0 \in \mathcal{P}os_{\mathcal{V}}(v_1)$

  - $\exists\, \hat{u}'$ such that $\hat{u} \xrightarrow{u_2 \leftrightarrow v_2} \hat{u}'$
    (as u_1, v_1 have same variables)
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$
    so some proof of $u_2\sigma_2 \leftrightarrow^* v_2\sigma_2$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$ (★)
  - ground-complete (E_1, R_1) has valley proof for $\hat{u}' \leftrightarrow^* \hat{v}$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$ ↯

  case $p_2 = q_0 q_1$ for $q_0 \in \mathcal{P}os_{\mathcal{V}}(v_1)$ ↯

  - $\exists\, \hat{u}'$ such that $\hat{u} \xrightarrow{u_2 \leftrightarrow v_2} \hat{u}'$
    (as u_1, v_1 have same variables)
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$
    so some proof of $u_2\sigma_2 \leftrightarrow^* v_2\sigma_2$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$ (★)
  - ground-complete (E_1, R_1) has valley proof for $\hat{u}' \leftrightarrow^* \hat{v}$
  - $\hat{u} \leftrightarrow^* \hat{u}' \leftrightarrow^* \hat{v}$ is smaller proof of $(\hat{u}, \hat{v})$ in (E_1, R_1),
    contradicting choice of $(\hat{u}, \hat{v})$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \dots \to t_k \leftarrow \dots \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ $\lightning$

  case $p_2 \in \mathcal{P}os_{\mathcal{F}}(v_1)$ $\lightning$

  case $p_2 = q_0 q_1$ for $q_0 \in \mathcal{P}os_{\mathcal{V}}(v_1)$ $\lightning$
  - $\exists\, \hat{u}'$ such that $\hat{u} \xrightarrow{u_2 \leftrightarrow v_2} \hat{u}'$
    (as u_1, v_1 have same variables)
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$
    so some proof of $u_2\sigma_2 \leftrightarrow^* v_2\sigma_2$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_2\sigma_2, v_2\sigma_2)$ $(\bigstar)$
  - ground-complete (E_1, R_1) has valley proof for $\hat{u}' \leftrightarrow^* \hat{v}$
  - $\hat{u} \leftrightarrow^* \hat{u}' \leftrightarrow^* \hat{v}$ is smaller proof of $(\hat{u}, \hat{v})$ in (E_1, R_1),
    contradicting choice of $(\hat{u}, \hat{v})$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \;\ldots\; \rightarrow t_k \leftarrow \;\ldots\; \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ ↯
  case $p > \epsilon$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \ \ldots \ \rightarrow t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ ↯
  case $p > \epsilon$

    - $u \rhd u_1$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \ldots \rightarrow t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ $\lightning$
  case $p > \epsilon$

  - $u \rhd u_1$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, u_1, \ldots)$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ ↯
  case $p > \epsilon$

  - $u \rhd u_1$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, u_1, \dots)$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
    so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \ \ldots \ \rightarrow t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ $\lightning$
  case $p > \epsilon$

  - $u \rhd u_1$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, u_1, \ldots)$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
    so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)
  - ground-complete (E_1, R_1) has valley proof for $t_1 \leftrightarrow^* \hat{v}$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \dots \to t_k \leftarrow \dots \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ ↯
  case $p > \epsilon$

  - $u \rhd u_1$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, u_1, \dots)$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
    so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ (★)
  - ground-complete (E_1, R_1) has valley proof for $t_1 \leftrightarrow^* \hat{v}$
  - $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step ↯
  by compatibility, P has more than one step

  case $p_1 = \epsilon$ ↯
  case $p > \epsilon$ ↯

  - $u \rhd u_1$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, u_1, \dots)$
  - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
    so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ $(\bigstar)$
  - ground-complete (E_1, R_1) has valley proof for $t_1 \leftrightarrow^* \hat{v}$
  - $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$
  - contradicts choice of $(\hat{u}, \hat{v})$

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step ↯

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \rightarrow \ \ldots \ \rightarrow t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \rightarrow v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step $\lightning$
- $u_1 \rightarrow v_1$ must be rewrite step

Proof attempt (2)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_2} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \to v$ is rule in R_1

- assume $u_1 \approx v_1$ is equation step ↯
- $u_1 \to v_1$ must be rewrite step
- reducible ground terms in R_1, R_2 coincide

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \ldots \to t_k \leftarrow \ \ldots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \dots \to t_k \leftarrow \dots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \rightarrow \ \ldots \ \rightarrow t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \ldots \ \to t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ldots \to t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as
  $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
  so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \rightarrow \ldots \rightarrow t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)

case $u_1 \approx v_1$ is equation

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, \dots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)

case $u_1 \approx v_1$ is equation

- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \ldots \ \to t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)

case $u_1 \approx v_1$ is equation

- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$
- ground-complete (E_1, R_1) has valley proof for $t_1 \leftrightarrow^* \hat{v}$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, \dots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ (★)

case $u_1 \approx v_1$ is equation

- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$
- ground-complete (E_1, R_1) has valley proof for $t_1 \leftrightarrow^* \hat{v}$
- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ldots \to t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ (★)

case $u_1 \approx v_1$ is equation ↯

- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$
- ground-complete (E_1, R_1) has valley proof for $t_1 \leftrightarrow^* \hat{v}$
- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$
- contradicts choice of $(\hat{u}, \hat{v})$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, \dots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)
  case $u_1 \approx v_1$ is equation ↯
  case $u_1 \to v_1$ is rule

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \ldots \ \to t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as
  $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
  so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ $(\bigstar)$
  case $u_1 \approx v_1$ is equation $\lightning$
  case $u_1 \to v_1$ is rule
  - $\hat{u}$ reduces to t_1' in (E_1, R_1)

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \rightarrow \ \ldots \ \rightarrow t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ (★)

case $u_1 \approx v_1$ is equation ϟ

case $u_1 \rightarrow v_1$ is rule

- $\hat{u}$ reduces to t_1' in (E_1, R_1)
- ground-complete (E_1, R_1) has valley proof for $t_1' \leftrightarrow^* \hat{v}$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \dots \to t_k \leftarrow \dots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, \dots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)

case $u_1 \approx v_1$ is equation ↯

case $u_1 \to v_1$ is rule

- $\hat{u}$ reduces to t_1' in (E_1, R_1)
- ground-complete (E_1, R_1) has valley proof for $t_1' \leftrightarrow^* \hat{v}$
- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \rightarrow \ \ldots \ \rightarrow t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, \ldots) >_c (\{\hat{u}\}, u_1\sigma_1, \ldots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)

case $u_1 \approx v_1$ is equation ↯

case $u_1 \rightarrow v_1$ is rule ↯

- $\hat{u}$ reduces to t_1' in (E_1, R_1)
- ground-complete (E_1, R_1) has valley proof for $t_1' \leftrightarrow^* \hat{v}$
- $\hat{u} \leftrightarrow^* t_1 \leftrightarrow^* \hat{v}$ yields proof Q in (E_1, R_1) such that $Q <_{\mathcal{U}} (\hat{u}, \hat{v})$
- contradicts choice of $(\hat{u}, \hat{v})$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$ $\lightning$

- have $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as
  $(\{\hat{u}\}, \hat{u}, \dots) >_c (\{\hat{u}\}, u_1\sigma_1, \dots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
  so some proof of $u_1\sigma_1 \leftrightarrow^* v_1\sigma_1$ in (E_1, R_1) is $\leq_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$ ($\bigstar$)
  case $u_1 \approx v_1$ is equation $\lightning$
  case $u_1 \to v_1$ is rule $\lightning$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \rightarrow \ \dots \ \rightarrow t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$ ↯

case $p_1 = \epsilon$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \ldots \ \to t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ $\lightning$
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation $\lightning$ argument as before

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ $\lightning$
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation $\lightning$ argument as before
  - case $u_1 \to v_1$ is rule

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ ↯
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation ↯ argument as before
  - case $u_1 \to v_1$ is rule
    - case $u \triangleright u_1$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ldots \to t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ ↯
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation ↯ argument as before
  - case $u_1 \to v_1$ is rule
    - case $u \rhd u_1$
      - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \ldots) >_c (\{\hat{u}\}, \hat{u}, u_1, \ldots)$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \dots \ \to t_k \leftarrow \ \dots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ ↯
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation ↯ argument as before
  - case $u_1 \to v_1$ is rule
    - case $u \rhd u_1$
      - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \dots) >_c (\{\hat{u}\}, \hat{u}, u_1, \dots)$
      - hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \rightarrow \ldots \rightarrow t_k \leftarrow \ldots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ ↯
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation ↯ argument as before
  - case $u_1 \rightarrow v_1$ is rule
    - case $u \trianglerighteq u_1$
      - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \ldots) >_c (\{\hat{u}\}, \hat{u}, u_1, \ldots)$
      - hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
      - so $\exists t_1'$ such that $\hat{u} \leftrightarrow^* t_1'$ in (E_1, R_1) and $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1')i$ (★)

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \dots \to t_k \leftarrow \dots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

case $p_1 > \epsilon$ ↯

case $p_1 = \epsilon$

case $u_1 \approx v_1$ is equation ↯ argument as before

case $u_1 \to v_1$ is rule

case $u \rhd u_1$

- $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \dots) >_c (\{\hat{u}\}, \hat{u}, u_1, \dots)$
- hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
- so $\exists t_1'$ such that $\hat{u} \leftrightarrow^* t_1'$ in (E_1, R_1) and $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1')i$ (★)
- (E_1, R_1) allows for proof $(\hat{u} \leftrightarrow^* t_1' \leftrightarrow^* \hat{v}) <_{\mathcal{U}} (\hat{u}, \hat{v})$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \dots \to t_k \leftarrow \dots \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ ↯
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation ↯ argument as before
  - case $u_1 \to v_1$ is rule
    - case $u \triangleright u_1$ ↯
      - $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1)$ as $(\{\hat{u}\}, \hat{u}, u, \dots) >_c (\{\hat{u}\}, \hat{u}, u_1, \dots)$
      - hence also $(\hat{u}, \hat{v}) >_{\mathcal{U}} (u_1\sigma_1, v_1\sigma_1)$
      - so $\exists t_1'$ such that $\hat{u} \leftrightarrow^* t_1'$ in (E_1, R_1) and $(\hat{u}, \hat{v}) >_{\mathcal{U}} (\hat{u}, t_1')i$ (★)
      - (E_1, R_1) allows for proof $(\hat{u} \leftrightarrow^* t_1' \leftrightarrow^* \hat{v}) <_{\mathcal{U}} (\hat{u}, \hat{v})$
      - contradicts choice of $(\hat{u}, \hat{v})$

Proof attempt (3)

(E_2, R_2) allows for minimal proof P

$$\hat{u} \xrightarrow[\sigma_1]{u_1 \leftrightarrow v_1}_{p_1} t_1 \xrightarrow[\sigma_2]{u_2 \leftrightarrow v_2}_{p_1} t_2 \to \ \ldots \ \to t_k \leftarrow \ \ldots \ \leftarrow \hat{v}$$

case $u \approx v$ is equation in E_1 (wlog, $\hat{u} > \hat{v}$)

case P consists of more than one step

- case $p_1 > \epsilon$ ↯
- case $p_1 = \epsilon$
  - case $u_1 \approx v_1$ is equation ↯ argument as before
  - case $u_1 \to v_1$ is rule
    - case $u \triangleright u_1$ ↯
    - case $u_1 \triangleright u$

Example

yet another pair of ground-complete systems for same theory

$$(E_1, R_1) = \begin{cases} 0' + y \approx y + 0 \\ 0 + 0 \to 0 \end{cases} \qquad (E_2, R_2) = \begin{cases} 0' + (x + y) \approx (x + y) + 0 \\ 0 + 0 \to 0 \\ 0' + 0 \to 0 \end{cases}$$

compatible with simplification order

Conclusion

- ground-complete systems are “less unique” than complete ones
- reducedness becomes undecidable property

Conclusion

- ground-complete systems are "less unique" than complete ones
- reducedness becomes undecidable property

Further work

- fix proof