Normalized Completion Revisited

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Example (Abelian Group)

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x$$

$$x \cdot \mathsf{e} \approx x \qquad x \cdot x^{-1} \approx \mathsf{e}$$

Example (Abelian Group + Endomorphisms)

$$
\begin{aligned}
(x \cdot y) \cdot z &\approx x \cdot (y \cdot z) & x \cdot y &\approx y \cdot x \\
x \cdot \mathrm{e} &\approx x & x \cdot x^{-1} &\approx \mathrm{e} \\
\mathrm{f}(x \cdot y) &\approx \mathrm{f}(x) \cdot \mathrm{f}(y) & \mathrm{f}(\mathrm{e}) &\approx \mathrm{e} \\
\mathrm{g}(x \cdot y) &\approx \mathrm{g}(x) \cdot \mathrm{g}(y) & \mathrm{g}(\mathrm{e}) &\approx \mathrm{e}
\end{aligned}
$$

Example (Abelian Group + Endomorphisms + Group Action)

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\begin{aligned}
(x \cdot y) \cdot z &\approx x \cdot (y \cdot z) & x \cdot y &\approx y \cdot x \\
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\mathsf{f}(x \cdot y) &\approx \mathsf{f}(x) \cdot \mathsf{f}(y) & \mathsf{f}(\mathsf{e}) &\approx \mathsf{e} \\
\mathsf{g}(x \cdot y) &\approx \mathsf{g}(x) \cdot \mathsf{g}(y) & \mathsf{g}(\mathsf{e}) &\approx \mathsf{e} \\
\phi(x, \phi(y, z)) &\approx \phi(x \cdot y, z) & \phi(\mathsf{e}, x) &\approx x \\
\phi(\mathsf{f}(x), \mathsf{g}(y)) &\approx \phi(\mathsf{g}(y), \mathsf{f}(x))
\end{aligned}
$$

Example (Abelian Group + Endomorphisms + Group Action)

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\phi(x, \phi(y, z)) &\approx \phi(x \cdot y, z) & \phi(\mathsf{e}, x) &\approx x \\
\phi(\mathsf{f}(x), \mathsf{g}(y)) &\approx \phi(\mathsf{g}(y), \mathsf{f}(x)) & &
\end{aligned}
$$

How to decide theory? ... e.g., check that $\phi(\mathsf{g}(x), y) \not\approx \phi(x, \mathsf{g}(y))$?

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$$\phi(\mathrm{f}(x), \mathrm{g}(y)) \approx \phi(\mathrm{g}(y), \mathrm{f}(x))$$

How to decide theory? . . . e.g., check that $\phi(\mathrm{g}(x), y) \not\approx \phi(x, \mathrm{g}(y))$?

Solve with term rewriting?

- Knuth-Bendix completion

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How to decide theory? ... e.g., check that $\phi(\mathsf{g}(x), y) \not\approx \phi(x, \mathsf{g}(y))$?

Solve with term rewriting?

- Knuth-Bendix completion ☹ unorientable equation

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\phi(\mathsf{f}(x), \mathsf{g}(y)) &\approx \phi(\mathsf{g}(y), \mathsf{f}(x)) & &
\end{aligned}
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How to decide theory? ... e.g., check that $\phi(\mathsf{g}(x), y) \not\approx \phi(x, \mathsf{g}(y))$?

Solve with term rewriting?

- Knuth-Bendix completion ☹ unorientable equation
- ordered completion

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$$\phi(x, \phi(y, z)) \approx \phi(x \cdot y, z) \qquad \phi(\mathsf{e}, x) \approx x$$
$$\phi(\mathsf{f}(x), \mathsf{g}(y)) \approx \phi(\mathsf{g}(y), \mathsf{f}(x))$$

How to decide theory? ... e.g., check that $\phi(\mathsf{g}(x), y) \not\approx \phi(x, \mathsf{g}(y))$?

Solve with term rewriting?

- Knuth-Bendix completion ☹ unorientable equation
- ordered completion ☹ inefficient in presence of AC

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How to decide theory? ... e.g., check that $\phi(\mathsf{g}(x), y) \not\approx \phi(x, \mathsf{g}(y))$?

Solve with term rewriting?

- Knuth-Bendix completion ☹ unorientable equation
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- AC completion

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- Knuth-Bendix completion ☹ unorientable equation
- ordered completion ☹ inefficient in presence of AC
- AC completion ☹ many CPs

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$$\mathsf{g}(x \cdot y) \approx \mathsf{g}(x) \cdot \mathsf{g}(\ldots$$
$$\phi(x, \phi(y, z)) \approx \phi(x \cdot y, \ldots$$
$$\phi(\mathsf{f}(x), \mathsf{g}(y)) \approx \phi(\mathsf{g}(y), \ldots$$

AC-convergent TRS G:

$$x \cdot \mathsf{e} \to x \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$
$$x \cdot x^{-1} \to \mathsf{e} \qquad \mathsf{e}^{-1} \to \mathsf{e}$$
$$(x^{-1})^{-1} \to x \qquad (x \cdot y)^{-1} \to x^{-1} \cdot y^{-1}$$

How to decide theory? . . . e.g., check that $\phi(\mathsf{g}(x), y) \not\approx \phi(x, \mathsf{g}(y))$?

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How to decide theory? ... e.g., check that $\phi(\mathsf{g}(x), y) \not\approx \phi(x, \mathsf{g}(y))$?

Solve with term rewriting?

- Knuth-Bendix completion ☹ unorientable equation
- ordered completion ☹ inefficient in presence of AC
- AC completion ☹ many CPs
- normalized completion ☺ e.g. modulo group theory

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Content

- Preliminaries
- Normalized Completion
- Normalized Completion with Termination Tools
- Implementation
- Conclusion

Consider signature $\mathcal{F}$ containing AC-symbols $\mathcal{F}_{\mathsf{AC}} \subseteq \mathcal{F}$.

$$\mathsf{AC} = \{\, f(x, f(y, z)) \approx f(f(x, y), z),\ f(x, y) \approx f(y, x) \ \mid f \in \mathcal{F}_{\mathsf{AC}} \}$$

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Definition (AC Rewriting)

- $u \xrightarrow[\ell \to r/\mathsf{AC}]{p} t$ if $u \leftrightarrow^*_{\mathsf{AC}} \cdot \xrightarrow[\ell \to r]{p} \cdot \leftrightarrow^*_{\mathsf{AC}} t$
- $u \to_{R/\mathsf{AC}} t$ if $u \xrightarrow[\ell \to r/\mathsf{AC}]{p} t$ for some $\ell \to r \in R$ and position p

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Definition

- TRS R is AC terminating if $\nexists\ t_0 \to_{R/\mathsf{AC}} t_1 \to_{R/\mathsf{AC}} t_2 \to_{R/\mathsf{AC}} \cdots$

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- TRS R is AC terminating if $\nexists\ t_0 \to_{R/\mathsf{AC}} t_1 \to_{R/\mathsf{AC}} t_2 \to_{R/\mathsf{AC}} \cdots$
- TRS R is AC convergent if AC terminating and $\forall u, t$

$$u \xleftrightarrow[R \cup \mathsf{AC}]{*} t \qquad \text{iff} \qquad u \xrightarrow[R/\mathsf{AC}]{*} \cdot \xleftrightarrow[\mathsf{AC}]{*} \cdot \xleftarrow[R/\mathsf{AC}]{*} t$$

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$$u \xleftrightarrow[R \cup \mathsf{AC}]{*} t \qquad \text{iff} \qquad u \xrightarrow[R/\mathsf{AC}]{*} \cdot \xleftrightarrow[\mathsf{AC}]{*} \cdot \xleftarrow[R/\mathsf{AC}]{*} t$$

Fact

R is AC terminating iff compatible with AC reduction order $\succ$

Normalized Completion

Fix theory T with AC convergent TRS S with $\leftrightarrow^*_{S \cup \mathrm{AC}} = \leftrightarrow^*_T$ and $S \subseteq \succ$.

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Example

- associativity, commutativity & identity

$$T: \quad x \cdot y \approx y \cdot x \qquad\qquad S: \quad x \cdot \mathsf{e} \to x$$
$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$
$$x \cdot \mathsf{e} \approx x$$

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- Abelian group theory

$$T: \quad \begin{aligned} x \cdot y &\approx y \cdot x \\ (x \cdot y) \cdot z &\approx x \cdot (y \cdot z) \\ x \cdot \mathrm{e} &\approx x \\ x^{-1} \cdot x &\approx \mathrm{e} \end{aligned} \qquad S: \quad \begin{aligned} x \cdot \mathrm{e} &\to x \\ x^{-1} \cdot x &\to \mathrm{e} \\ \mathrm{e}^{-1} &\to \mathrm{e} \\ (x^{-1})^{-1} &\to x \\ (x \cdot y)^{-1} &\to x^{-1} \cdot y^{-1} \end{aligned}$$

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- commutative ring theory, theory of finite rings, ...

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$u \xrightarrow[R \backslash S]{} t$ if $u' = u{\downarrow_S}$

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Definition (Normalized Rewriting)

$u \xrightarrow[R\backslash S]{} t$ if $u' = u{\downarrow_S}$

$u{\downarrow_S}$ is $\rightarrow_{S/\mathsf{AC}}$-normal form of u

Normalized Completion

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Definition (Normalized Rewriting)

$u \xrightarrow[R\backslash S]{} t$ if $u' = u{\downarrow_S}$ and $u' \xrightarrow[R/\mathsf{AC}]{} t$

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Example

associativity, commutativity & identity

$$T: \quad x \cdot y \approx y \cdot x \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot \mathsf{e} \approx x$$

$$S: \quad x \cdot \mathsf{e} \to x \qquad (x \cdot y)^{-1} \to x^{-1} \cdot y^{-1}$$

for $R = \{(x \cdot y)^{-1} \to x^{-1} \cdot y^{-1}\}$ have

$$(\mathsf{a} \cdot \mathsf{b})^{-1} \xrightarrow[R \backslash S]{} \mathsf{a}^{-1} \cdot \mathsf{b}^{-1} \qquad (\mathsf{e} \cdot \mathsf{b})^{-1} \not\xrightarrow[R \backslash S]{} \mathsf{e}^{-1} \cdot \mathsf{b}^{-1}$$

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$u \xrightarrow[R \backslash S]{} t$ if $u' = u{\downarrow_S}$ and $u' \xrightarrow[R/\mathsf{AC}]{} t$

Definition (S-convergence)

R is S-convergent for set of equations E if $\to_{R \backslash S}$ is well-founded and

$$t \xleftrightarrow[E \cup S \cup \mathsf{AC}]{*} u$$

implies existence of rewrite proof

$$t \xrightarrow[R \backslash S]{!} \cdot \xleftrightarrow[T]{*} \cdot \xleftarrow[R \backslash S]{!} u$$

Definition ($\mathcal{N}$)

E: equations R: rewrite rules $\succ$: AC-reduction order, $S \subseteq \succ$

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deduce $\quad \dfrac{E, R}{}$

if $u \leftrightarrow^*_{R \cup T} v$

Definition ($\mathcal{N}$)

E: equations R: rewrite rules $\succ$: AC-reduction order, $S \subseteq \succ$

deduce $\dfrac{E, R}{E \cup \{u \approx v\}, R}$ if $u \leftrightarrow^*_{R \cup T} v$

Definition ($\mathcal{N}$)

E: equations R: rewrite rules $\succ$: AC-reduction order, $S \subseteq \succ$

deduce $$\frac{E, R}{E \cup \{u \approx v\}, R}$$ if $u \leftrightarrow^*_{R \cup T} v$

simplify $$\frac{E \uplus \{u \simeq v\}, R}{}$$ if $u \to_{R \setminus S} t$

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deduce
$$\frac{E, R}{E \cup \{u \approx v\}, R}$$
if $u \leftrightarrow^*_{R \cup T} v$

simplify
$$\frac{E \uplus \{u \simeq v\}, R}{E \cup \{t \simeq v\}, R}$$
if $u \to_{R \setminus S} t$

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collapse $$\frac{E, R \uplus \{u \rightarrow v\}}{}$$ if $u \rightarrow_{R \setminus S} t$

Definition ($\mathcal{N}$)

E: equations R: rewrite rules $\succ$: AC-reduction order, $S \subseteq \succ$

deduce
$$\frac{E, R}{E \cup \{u \approx v\}, R} \quad \text{if } u \leftrightarrow^*_{R \cup T} v$$

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$$\frac{E \uplus \{u \simeq v\}, R}{E \cup \{t \simeq v\}, R} \quad \text{if } u \rightarrow_{R \setminus S} t$$

collapse
$$\frac{E, R \uplus \{u \rightarrow v\}}{E \cup \{t \approx v\}, R} \quad \text{if } u \rightarrow_{R \setminus S} t$$

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collapse $$\frac{E, R \uplus \{u \rightarrow v\}}{E \cup \{t \approx v\}, R}$$ if $u \rightarrow_{R \setminus S} t$

slightly simpler than in Marché 1996

Definition ($\mathcal{N}$)

E: equations R: rewrite rules $\succ$: AC-reduction order, $S \subseteq \succ$

deduce
$$\frac{E, R}{E \cup \{u \approx v\}, R}$$
if $u \leftrightarrow^*_{R \cup T} v$

simplify
$$\frac{E \uplus \{u \simeq v\}, R}{E \cup \{t \simeq v\}, R}$$
if $u \to_{R \setminus S} t$

collapse
$$\frac{E, R \uplus \{u \to v\}}{E \cup \{t \approx v\}, R}$$
if $u \to_{R \setminus S} t$

compose
$$\frac{E, R \uplus \{v \to u\}}{}$$

Definition ($\mathcal{N}$)

E: equations R: rewrite rules $\succ$: AC-reduction order, $S \subseteq \succ$

deduce
$$\frac{E, R}{E \cup \{u \approx v\}, R} \quad \text{if } u \leftrightarrow^*_{R \cup T} v$$

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$$\frac{E \uplus \{u \simeq v\}, R}{E \cup \{t \simeq v\}, R} \quad \text{if } u \rightarrow_{R \setminus S} t$$

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$$\frac{E, R \uplus \{u \rightarrow v\}}{E \cup \{t \approx v\}, R} \quad \text{if } u \rightarrow_{R \setminus S} t$$

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$$\frac{E, R \uplus \{v \rightarrow u\}}{} \quad \text{if } u \rightarrow_{R \setminus S} t$$

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$$\frac{E, R}{E \cup \{u \approx v\}, R}$$
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simplify
$$\frac{E \uplus \{u \simeq v\}, R}{E \cup \{t \simeq v\}, R}$$
if $u \rightarrow_{R \setminus S} t$

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$$\frac{E, R \uplus \{u \rightarrow v\}}{E \cup \{t \approx v\}, R}$$
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$\Theta_g(u, v) = \mathrm{CP}_{\mathrm{AC}}(\{u \to v\}, S^e)$
$\Psi_g(u, v) = \{u \to v\}$

(Θ, Ψ) are S-normalizing pair

Definition (Fairness)

run $(E_0, \varnothing) \vdash_{\mathcal{N}} (E_1, R_1) \vdash_{\mathcal{N}} \cdots \vdash_{\mathcal{N}} (E_k, R_k)$

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L-critical pairs
among (extended) rules of R_k

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$\mathrm{AC} \subseteq L \subseteq T$
L can be chosen to have
good properties wrt unification

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Theorem (Correctness) Marché 1996

If $(E, \varnothing) \vdash_{\mathcal{N}}^{*} (\varnothing, R)$ is fair then R is S-convergent for E.

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with respect to proof reduction order $\succ$

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Then for any fair run $(E, \varnothing) \vdash_{\mathcal{N}}^* (\varnothing, R')$ *applying* $\succ$ *and full inter-reduction* R' *is equal to* R *up to variable renaming and AC equivalence.*

Definition (S-Normalizing Pair) Marché 1996

for set of equations E containing $u \simeq v$ and set of rewrite rules R, (Θ, Ψ) constitutes S-normalizing pair if $u \succ v$ and

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ensures progress
e.g. by orienting equations

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ensures that equational proofs decrease even if rules get composed

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Issue

- does not preserve equational theory

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Issue

- does not guarantee S-convergence: for $S = \{\mathsf{b} + x \to \mathsf{b}\}$ run

$$(\{\mathsf{a} + x \approx \mathsf{a}\}, \varnothing) \vdash (\varnothing, \{\mathsf{a} + x \to \mathsf{a}\})$$

using $(\Theta, \Psi) = (\varnothing, \{\mathsf{a} + x \to \mathsf{a}\})$ is fair, but AC-critical pair $\mathsf{a} \approx \mathsf{b}$ is not joinable

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2. for all $\ell \to r$ in $\Psi(u,v)$, proof $P\colon\ s\ {}_S{\leftarrow}\ w \leftrightarrow^*_{\mathsf{AC}} \cdot \to_{\ell \to r} \cdot \to^*_{R \setminus S} t$ with TRS R there is proof Q in $S, \Theta(u,v), \Psi(u,v) \cup R$ such that $P \Rightarrow Q$ and terms in Q are smaller than w
3. $\Theta(u,v)$, $\Psi(u,v)$ are contained in $\leftrightarrow^*_{E \cup R \cup \mathcal{T}}$ and $\Psi(u,v) \subseteq \succ$

Critical Pair Criteria

Aim: filter out superfluous critical pairs

Critical Pair Criteria

Definition

peak $P:\ s \underset{p}{\leftarrow} u \leftrightarrow_L^* u' \underset{\epsilon}{\rightarrow} t$ is composite if there are

- terms $u_0, \ldots, u_{n+1}$ such that $s = u_0$, $t = u_{n+1}$, and $u \succ u_i$
- proofs P_i proving $u_i \simeq u_{i+1}$ such that $P \ggcurly P_i$ for all $1 \leqslant i \leqslant n$

proof ordering
for normalized completion

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Lemma

Bachmair & Dershowitz 94

Composite critical pairs can be omitted in standard completion

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Compositeness in normalized completion

peak P is composite

- if $u \neq u\downarrow_S$ — S-reducibility

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- if u is reducible strictly below p — primality (Kapur *et al* 88)

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- if $u \rightarrow v$

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peak P is composite

- if $u \neq u\downarrow_S$ — S-reducibility
- if u is reducible strictly below p — primality (Kapur *et al* 88)
- if $u \rightarrow v$ and $s \simeq v$ and $t \simeq v$ were already considered — connectedness (Küchlin 85)

Example (Abelian Group + Endomorphisms + Group Action)

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot y \approx y \cdot x$$
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... but remaining equations cannot all be oriented

Normalized Completion with Termination Tools

Definition (orient in $\mathcal{N}_{TT}$)

E: set of equations R: set of rewrite rules C: set of rewrite rules

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Corollary

If $(E, \varnothing, \varnothing) \vdash^*_{\mathcal{N}_{TT}} (\varnothing, R, C)$ *is fair then* R *is* S*-convergent for* E

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yields G-convergent TRS:

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Normalized completion with termination tools modulo *GE*:

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yields *GE*-convergent TRS (much faster):

$$\phi(\mathsf{e}, x) \to x \qquad \phi(\mathsf{f}(x), \mathsf{e}) \to \mathsf{f}(x) \qquad \phi(x, \mathsf{f}(y)) \to \phi(\mathsf{f}(y) \cdot x, \mathsf{e})$$
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Example (Binary Arithmetic)

$$x + y \approx y + x$$
$$(x + y) + z \approx x + (y + z)$$
$$x + \# \approx x$$

$$(x + y)0 \approx x0 + y0$$
$$(x + y)1 \approx x0 + y1$$
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$$\mathsf{triple}(x) \approx (x0 + x)$$

cannot be completed with AC-RPO or AC-KBO.

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Normalized completion with termination tools modulo $S = \{x + \# \to x\}$ produces S-convergent TRS:

$$x0 + y0 \to (x + y)0$$
$$x0 + y1 \to (x + y)1$$
$$x1 + y1 \to (x + y + \#1)0$$
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- fully automatic in that
  - no reduction order required as input
  - applicable theory detected automatically
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- term indexing with AC-discrimination trees
- source code, binary and web interface available on-line:

http://cl-informatik.uibk.ac.at/software/mkbtt

Experiments

20 problems collected from the literature.

theory S	mkbtt AC	mkbtt AG	mkbtt auto	C*i*ME
G94-abelian groups (AG)	1.6	0.1	0.1	0.05
AG + homomorphism	181.7	4.8	4.8	0.05
LS96-G0	1.9	0.1	0.1	?
LS96-G1	∞	12.4	12.5	?
G94-arithmetic	14.9	–	13.8	?
G94-AC-ring with unit	22.9	7.2	0.1	0.1
MU04-binary arithmetic	2.9	–	3.0	?
MU04-ternary arithmetic	18.1	–	17.3	?
CGA	∞	15.4	15.2	?
CRE	∞	216.7	145.1	?
#successes	10	7	13	4

- completion time in seconds, ∞ is timeout (600 seconds)
- ?: no suitable reduction order for C*i*ME
- -: theory not applicable

Conclusion

- simpler collapse rule due to new proof order
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Challenge: Tarski's High School Algebra Problem*

$$((1+x)^y + (1+x+x^2)^y)^x \cdot ((1+x^3)^x + (1+x^2+x^4)^x)^y$$
$$\approx ((1+x)^x + (1+x+x^2)^x)^y \cdot ((1+x^3)^y + (1+x^2+x^4)^y)^x$$

does not follow from "high school algebra":

$$x + y \approx y + x \qquad (x+y)+z \approx x+(y+z) \qquad x \cdot (y+z) \approx x \cdot y + x \cdot z$$
$$x \cdot y \approx y \cdot x \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad x \cdot 1 \approx x$$
$$1^x \approx 1 \qquad x^1 \approx x \qquad x^{y+z} \approx x^y \cdot x^z$$
$$(x \cdot y)^z \approx x^z \cdot y^z \qquad (x^y)^z \approx x^{y \cdot z}$$

*thanks to Johannes Waldmann for communicating this example

Definition (Extended Rules)

$R^e = R \cup \{f(\ell, x) \to f(r, x) \mid \ell \to r \in R, \text{root}(\ell) = f \in \mathcal{F}_{\text{AC}},\ x \in \mathcal{V} \text{ fresh}\}$

back

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Example

for $\mathcal{F}_{\mathrm{AC}} = \{\cdot\}$ and $R = \{ \mathrm{e}^{-1} \to \mathrm{e},\ x \cdot x^{-1} \to \mathrm{e} \}$.

$$R^e = \{ \mathrm{e}^{-1} \to \mathrm{e},\ x \cdot x^{-1} \to \mathrm{e},\ x \cdot x^{-1} \cdot y \to \mathrm{e} \cdot y \}$$

▸ back

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Definition (L-Overlap)

Let L have decidable and finite unification problem.
$\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ is L-overlap if

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Example

$x \cdot x^{-1} \to \mathrm{e}$ and $z \cdot z^{-1} \cdot y \to \mathrm{e} \cdot y$ create $\mathrm{CP}_{\mathrm{AC}}$ $\mathrm{e} \cdot (z \cdot z^{-1})^{-1} \approx \mathrm{e}$

back