Extending Maximal Completion

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Abstract

Maximal completion (Klein and Hirokawa 2011) is an elegantly simple yet powerful variant of Knuth-Bendix completion. This paper extends the approach to ordered completion and theorem proving as well as normalized completion. An implementation of the different procedures is described, and its practicality is demonstrated by various examples.

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1 Introduction

Knuth-Bendix completion [18] constitutes a milestone in the history of equational theorem proving and automated deduction in general. Given a set of input equalities $\mathcal{E}_0$, it can generate a presentation of the equational theory as a complete rewrite system $\mathcal{R}$ which may serve to decide the validity problem for the theory.

▶ **Example 1.** In order to simplify proofs found by SMT solvers, Wehrman and Stump [32] pursue an algebraic approach: proofs are represented by first-order terms, and the equivalences usable for simplification are described by 20 equations like the following ones:

$$\begin{array}{ccc} (x \cdot y) \cdot z \approx x \cdot (y \cdot z) & (\mathsf{refl} \cdot x) \approx x & (x \cdot \mathsf{refl}) \approx x \\ \mathsf{or}_1(\mathsf{refl}) \approx \mathsf{refl} & \mathsf{and}_1(\mathsf{refl}) \approx \mathsf{refl} & \mathsf{not}(\mathsf{refl}) \approx \mathsf{refl} \\ \mathsf{or}_1(x) \cdot \mathsf{or}_2^\mathsf{T} \approx \mathsf{or}_2^\mathsf{T} & \mathsf{and}_1(x) \cdot \mathsf{and}_2^\mathsf{F} \approx \mathsf{and}_2^\mathsf{F} & \mathsf{or}_2(x) \cdot \mathsf{or}_1^\mathsf{F} \approx (\mathsf{or}_1^\mathsf{F} \cdot x) \\ \mathsf{or}_1(x) \cdot \mathsf{or}_2^\mathsf{F} \approx (\mathsf{or}_2^\mathsf{F} \cdot x) & \mathsf{not}(x) \cdot \mathsf{not}(y) \approx \mathsf{not}(x \cdot y) & \mathsf{or}_2(x) \cdot \mathsf{or}_1^\mathsf{T} \approx \mathsf{or}_1^\mathsf{T} \end{array}$$

Here $\cdot$ denotes concatenation, refl is the reflexivity proof, the symbols and_i, or_i and not are used for congruence, and constants like or_1^T stand for operations with boolean constants.

A Knuth-Bendix completion procedure can transform this set of equations into a terminating and confluent rewrite system $\mathcal{R}$ consisting of 45 rules, including the following:

$$\begin{array}{rrr} (x \cdot y) \cdot z \to x \cdot (y \cdot z) & \mathsf{or}_1(x) \cdot \mathsf{or}_2^\mathsf{T} \to \mathsf{or}_2^\mathsf{T} & (\mathsf{refl} \cdot x) \to x \\ (\mathsf{or}_2(x) \cdot \mathsf{or}_1(y)) \cdot \mathsf{or}_1^\mathsf{T} \to \mathsf{or}_1(y) \cdot \mathsf{or}_1^\mathsf{T} & \mathsf{or}_2(x) \cdot \mathsf{or}_1^\mathsf{T} \to \mathsf{or}_1^\mathsf{T} & \mathsf{or}_1(\mathsf{refl}) \to \mathsf{refl} \\ (x \cdot \mathsf{and}_2(y)) \cdot \mathsf{and}_2(z) \to x \cdot \mathsf{and}_2(y \cdot z) & \mathsf{or}_2(\mathsf{refl}) \to \mathsf{refl} & \mathsf{or}_2(x) \cdot \mathsf{or}_1^\mathsf{T} \to \mathsf{or}_1^\mathsf{T} \end{array}$$

This rewrite system can be used to simplify an arbitrary proof (represented by a term) into its unique normal form. Moreover, any two proofs can be tested for equivalence simply by checking whether their normal forms are the same.

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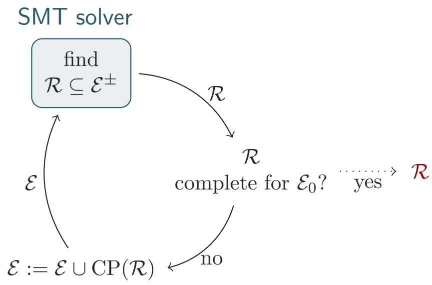

Figure 1 Maximal completion.

Knuth and Bendix presented completion as a concrete algorithm. Pioneered by Bachmair, Dershowitz, and Hsiang [5], it is nowadays more common to describe completion by an inference system, thus abstracting from concrete implementations.

More recently, Klein and Hirokawa [17] proposed a radically different approach: *Maximal completion* first approximates a complete presentation by extracting a terminating rewrite system from an equation pool. It then checks whether the candidate system is complete, and if a counterexample was found the procedure is repeated with an extended equation pool. Figure 1 illustrates the approach. Maximal completion has the advantage that the reduction order, a typically critical input parameter, need not be fixed in advance and can be changed at any point. The candidate rewrite systems are generated by means of SAT/SMT solvers; thus also advanced termination methods can be used in this setting and the search can be guided towards different objectives [25]. Despite the simple, declarative formulation of the procedure, the authors' implementation resulted in a competitive tool [17, 25].

Apart from these improvements of classical Knuth-Bendix completion, numerous variants have by now joined the family of completion calculi, aiming to make completion more versatile and powerful. One of the most prominent variants is ordered completion. It was developed by Bachmair, Dershowitz, and Plaisted to remedy the shortcoming that classical completion fails if unorientable equations like commutativity are encountered [6].

Another line of research tackled the development of dedicated completion procedures for equational systems which incorporate common algebraic theories such as associativity and commutativity [23, 12]. The latest and most generally applicable method of this kind is normalized completion, developed by Marché [22].

In this paper maximal completion is revisited (Section 3) and extended to ordered and normalized completion. More specifically, the contributions of this paper are as follows:

- Maximal ordered completion and an according equational theorem proving method are explained in detail. In particular a completeness proof is presented, showing that a ground complete system can always be found.
- The proofs for (ordered) completion require only *prime* critical pairs to be considered.
- For the case of linear input equalities, it is proven that even a complete system can be found if it exists (Section 4.2), and a bound on the number of iterations is derived.
- A maximal completion version of normalized completion (Section 5) is presented. This covers AC completion, as well as the computation of Gröbner bases [22].

Section 6 is devoted to the implementation of these procedures in the tool MædMax. Some example use cases from different application areas are demonstrated along the way. Finally, Section 7 concludes.

2 Preliminaries

In the sequel familiarity with the basics of term rewriting is assumed [2], but some key notions are recalled in this section. Let $\mathcal{T}(\mathcal{F},\mathcal{V})$ denote the set of all terms over a signature $\mathcal{F}$ and an infinite set of variables $\mathcal{V}$, and $\mathcal{T}(\mathcal{F})$ the set of all *ground* terms over $\mathcal{F}$. A *substitution* σ is a mapping from variables to terms. As usual, $t\sigma$ denotes the application of σ to the term t. A pair of terms (s,t) is sometimes considered an *equation*, which is expressed by writing $s \approx t$, and sometimes a *(rewrite) rule*, denoted $s \to t$. An equational system (ES) is a set of equations, a term rewrite system (TRS) is a set of rewrite rules. Given an ES $\mathcal{E}$, we write $\mathcal{E}^{\pm}$ to denote its *symmetric closure* $\mathcal{E} \cup \{t \approx s \mid s \approx t \in \mathcal{E}\}$. A *reduction order* is a proper and well-founded order on terms which is closed under contexts and substitutions. It is *ground total* if it is total on $\mathcal{T}(\mathcal{F})$. In the remainder most examples use the Knuth-Bendix order (KBO), written $>_{\mathsf{kbo}}$, and the lexicographic path order (LPO), written $>_{\mathsf{lpo}}$.

A TRS $\mathcal{R}$ is *terminating* if $\to_{\mathcal{R}}$ is well-founded. It is *(ground) confluent* if $s \;{}_{\mathcal{R}}^{*}\!\leftarrow \cdot \to_{\mathcal{R}}^{*} t$ implies $s \to_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow t$ for all (ground) terms s and t. It is *(ground) complete* if it is terminating and (ground) confluent. We say that $\mathcal{R}$ is a *complete presentation* of an ES $\mathcal{E}$ if $\mathcal{R}$ is complete and $\leftrightarrow_{\mathcal{R}}^{*} = \leftrightarrow_{\mathcal{E}}^{*}$. Similarly, $\mathcal{R}$ is a *ground complete presentation* of an ES $\mathcal{E}$ if $\mathcal{R}$ is ground complete and the equivalence $\leftrightarrow_{\mathcal{R}}^{*} = \leftrightarrow_{\mathcal{E}}^{*}$ holds on ground terms. For a TRS $\mathcal{R}$ and terms s and t, the notation $s \downarrow_{\mathcal{R}} t$ expresses existence of a joining sequence $s \to_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow t$. If $\mathcal{R}$ is terminating then $t{\downarrow_{\mathcal{R}}}$ denotes some fixed normal form of t, and $\mathsf{NF}(\mathcal{R})$ denotes the set of all normal forms of $\mathcal{R}$. This notation is extended to ESs $\mathcal{E}$ by writing $\mathcal{E}{\downarrow_{\mathcal{R}}}$ for the ES $\{s{\downarrow_{\mathcal{R}}} \approx t{\downarrow_{\mathcal{R}}} \mid s \approx t \in \mathcal{E} \text{ and } s{\downarrow_{\mathcal{R}}} \neq t{\downarrow_{\mathcal{R}}}\}$.

Completion procedures are based on critical pair analysis. To that end, an *overlap* of a TRS $\mathcal{R}$ is a triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2\rangle$ such that $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rules in $\mathcal{R}$ without common variables, $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$, ℓ_1 and $\ell_2|_p$ are unifiable, and if $p = \epsilon$ then $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are not variants of each other. Suppose $\langle \ell_1 \to r_1, p, \ell_2 \to r_2\rangle$ is an overlap of a TRS $\mathcal{R}$ and σ is a most general unifier of ℓ_1 and $\ell_2|_p$. Then the equation $\ell_2[r_1]_p\sigma \approx r_2\sigma$ is a *critical pair* of $\mathcal{R}$. The set of all critical pairs of $\mathcal{R}$ is denoted by $\mathrm{CP}(\mathcal{R})$. A critical pair is *prime* if no proper subterm of $\ell_1\sigma$ is reducible in $\mathcal{R}$. The set of all prime critical pairs of $\mathcal{R}$ is denoted by $\mathrm{PCP}(\mathcal{R})$. It is known that only prime critical pairs need to be considered for confluence of terminating TRSs:

▶ **Lemma 2** ([14]). *A terminating TRS $\mathcal{R}$ is confluent if and only if* $\mathrm{PCP}(\mathcal{R}) \subseteq {\downarrow_{\mathcal{R}}}$. ◀

Further preliminaries will be introduced in later sections as necessary.

3 Maximal Completion

This section recapitulates the maximal completion approach by Klein and Hirokawa [17]. A TRS $\mathcal{R}$ is said to be *over* an ES $\mathcal{E}$ if $\mathcal{R} \subseteq \mathcal{E}^{\pm}$. The set of all terminating TRSs $\mathcal{R}$ over $\mathcal{E}$ is denoted $\mathfrak{T}(\mathcal{E})$. We assume two functions $\mathfrak{R}$ and Ext such that $\mathfrak{R}(\mathcal{E}) \subseteq \mathfrak{T}(\mathcal{E})$ returns a set of terminating TRSs over $\mathcal{E}$, and the extension function Ext satisfies $\mathrm{Ext}(\mathcal{E}) \subseteq \leftrightarrow_{\mathcal{E}}^{*}$ for all ESs $\mathcal{E}$. We define maximal completion by means of the following transformation.

▶ **Definition 3.** *Given a set of input equalities $\mathcal{E}_0$ and an ES $\mathcal{E}$, let*

$$\varphi(\mathcal{E}) = \begin{cases} \mathcal{R} & \text{if } \mathcal{R} \in \mathfrak{R}(\mathcal{E}) \text{ such that } \mathrm{PCP}(\mathcal{R}) \cup \mathcal{E}_0 \subseteq {\downarrow_{\mathcal{R}}} \\ \varphi(\mathcal{E} \cup \mathrm{Ext}(\mathcal{E})) & \text{otherwise.} \end{cases}$$

Note that this definition differs from [17, Definition 2] by the use of prime critical pairs. In general φ does not need to be defined, nor is it necessarily unique. But if $\varphi(\mathcal{E}_0)$ is defined then we can assume a sequence of ESs $\mathcal{E}_1, \ldots, \mathcal{E}_k$ called *maximal completion sequence* such that $\mathcal{E}_{i+1} = \mathcal{E}_i \cup \mathrm{Ext}(\mathcal{E}_i)$ for all $0 \leqslant i < k$, and there is some $\mathcal{R} \in \mathfrak{R}(\mathcal{E}_k)$ such that $\mathrm{PCP}(\mathcal{R}) \cup \mathcal{E}_0 \subseteq \downarrow_{\mathcal{R}}$. The following theorem expresses correctness of maximal completion [17, Theorem 3]:

▶ **Lemma 4.** *If $\varphi(\mathcal{E}_0)$ is defined then it is a complete presentation of $\mathcal{E}_0$.*

Proof. Let $\varphi(\mathcal{E}_0) = \mathcal{R}$ and $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be an according maximal completion sequence. The TRS $\mathcal{R}$ must be terminating since it was returned by $\mathfrak{R}$. Because of $\mathrm{PCP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$ it is confluent by Lemma 2, and hence complete.

A simple induction argument using the global assumption $\mathrm{Ext}(\mathcal{E}) \subseteq \leftrightarrow^*_{\mathcal{E}}$ for all ESs $\mathcal{E}$ shows that $\mathcal{E}_i \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ for all $i \geqslant 0$. Since $\mathcal{R}$ is over $\mathcal{E}_k$, also $\leftrightarrow^*_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ holds. Conversely, $\mathcal{E}_0 \subseteq \downarrow_{\mathcal{R}}$ ensures $\leftrightarrow^*_{\mathcal{E}_0} \subseteq \leftrightarrow^*_{\mathcal{R}}$. So $\mathcal{R}$ is a complete presentation of $\mathcal{E}_0$. ◀

Note that maximal completion is based on just three ingredients: (1) completeness is overapproximated by termination using the function $\mathfrak{R}$, (2) a success check determines whether some TRS $\mathcal{R} \in \mathfrak{R}(\mathcal{E})$ is complete, and (3) the current set of equations $\mathcal{E}$ is extended by means of a theory-preserving function Ext.

It is natural to choose $\mathrm{Ext}(\mathcal{E})$ such that $\mathrm{Ext}(\mathcal{E}) \subseteq \bigcup_{\mathcal{R} \in \mathfrak{R}(\mathcal{E})} \mathrm{CP}(\mathcal{R})\downarrow_{\mathcal{R}}$. Klein and Hirokawa moreover proposed $\mathfrak{R}(\mathcal{E})$ to return elements of $\mathfrak{T}(\mathcal{E})$ with maximal cardinality, hence the name. The rationale for this choice is that adding rules to a complete presentation $\mathcal{R}$ of $\mathcal{E}_0$ does not hurt this property, as long as termination and the equational theory are preserved. This is formally expressed by the following lemma.

▶ **Lemma 5** ([17, Lemma 4]). *Let $\mathcal{R}$ be a complete presentation of $\mathcal{E}_0$ and $\mathcal{R}'$ a terminating TRS such that $\mathcal{R} \subseteq \mathcal{R}' \subseteq \leftrightarrow^*_{\mathcal{E}_0}$. Then also $\mathcal{R}'$ is a complete presentation of $\mathcal{E}_0$.* ◀

Nevertheless a maximal terminating TRS may constitute an unfortunate choice in maximal completion, as illustrated by the next example.

▶ **Example 6.** Let $\mathcal{E}_0$ consist of the following four equations:

$$x + 0 \approx x \qquad \mathsf{s}(x+y) \approx x + \mathsf{s}(y) \qquad \mathsf{z}(x) \approx 0 \qquad \mathsf{z}(\mathsf{s}(x+y)) \approx \mathsf{z}(x + \mathsf{s}(0))$$

Let $\mathcal{R}_1$ be the TRS obtained by orienting all equations from left to right:

$$x + 0 \to x \qquad \mathsf{s}(x+y) \to x + \mathsf{s}(y) \qquad \mathsf{z}(x) \to 0 \qquad \mathsf{z}(\mathsf{s}(x+y)) \to \mathsf{z}(x + \mathsf{s}(0))$$

Termination of $\mathcal{R}_1$ can e.g. be verified using a KBO with $\mathsf{s} > +$ and $w_0 = w(f) = 1$ for all function symbols f. Thus $\mathfrak{R}(\mathcal{E}_0) = \{\mathcal{R}_1\}$ is a valid choice for maximal completion. Now the first two rules admit the overlap $\mathsf{s}(x) \leftarrow \mathsf{s}(x+0) \to x + \mathsf{s}(0)$ which creates an irreducible critical pair $\mathsf{s}(x) \approx x + \mathsf{s}(0)$. There are also three critical pairs involving the last rule, but they are all joinable. Let thus $\mathcal{E}_1$ be $\mathcal{E}_0 \cup \{\mathsf{s}(x) \approx x + \mathsf{s}(0)\}$. Using the same reduction order, all equations can be oriented into the TRS $\mathcal{R}_2 = \mathcal{R}_1 \cup \{x + \mathsf{s}(0) \to \mathsf{s}(x)\}$. Suppose $\mathfrak{R}(\mathcal{E}_1)$ is $\{\mathcal{R}_2\}$. There is only one new non-joinable overlap: $\mathsf{s}(\mathsf{s}(x)) \leftarrow \mathsf{s}(x + \mathsf{s}(0)) \to x + \mathsf{s}(\mathsf{s}(0))$, so let $\mathcal{E}_2 = \mathcal{E}_1 \cup \{\mathsf{s}(\mathsf{s}(x)) \approx x + \mathsf{s}(\mathsf{s}(0))\}$. Repeating this strategy will fail to produce a finite complete system, as it gives rise to infinitely many equations $\mathsf{s}^n(x) \approx x + \mathsf{s}^n(0)$.

So this reduction order does not lead to a finite complete presentation of $\mathcal{E}_0$. But in fact $\mathcal{R}_1$ is the only terminating TRS over $\mathcal{E}_0$ which has four rules: This is because the last equation can only be oriented from left to right, and the second cannot be oriented from right to left in combination with the last without violating termination.

Suppose that $\mathfrak{R}(\mathcal{E}_0)$ contains instead the following TRS $\mathcal{R}_1'$ which has only three rules:

$$x + \mathsf{0} \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x+y) \qquad \mathsf{z}(x) \to \mathsf{0}$$

Termination of $\mathcal{R}_1'$ can be shown by changing the precedence in the above KBO to $+ > \mathsf{s}$. There are no critical pairs, and $\mathcal{R}_1'$ joins the input equalities $\mathcal{E}_0$. So maximal completion can succeed immediately by returning $\mathcal{R}_1'$.

In the implementation in the tool MædMax the function $\mathfrak{R}$ chooses rewrite systems $\mathcal{R}$ over $\mathcal{E}$ which can *reduce* rather than *orient* a maximal number of equations in $\mathcal{E}$. Note that the TRS $\mathcal{R}_1'$ in Example 6 is optimal in this sense, since it reduces all equations in $\mathcal{E}_0$.

▶ **Example 7.** In nine iterations of maximal completion, that is within nine recursive calls of the procedure φ, the proof reduction system described in Example 1 can be transformed into a complete rewrite system $\mathcal{R}$. The maximal completion run produces 150 equations and takes about 10 seconds. It is worth noting that to complete this system, LPO or KBO alone do not suffice; advanced termination techniques like dependency pairs are required, see [25].

4 Ordered Completion and Theorem Proving

This section is devoted to the extension of maximal completion to ordered completion and equational theorem proving. The basic procedure was already outlined in [36].

First some concepts specific to this setting are introduced. In this section a ground total reduction order $>$ is considered, unless stated otherwise. Given a reduction order $>$ and an ES $\mathcal{E}$, the *ordered rewrite system* $\mathcal{E}_>$ consists of all rules $s\sigma \to t\sigma$ such that $s \approx t \in \mathcal{E}$ and $s\sigma > t\sigma$. A triple $(\mathcal{R}, \mathcal{E}, >)$ of a TRS $\mathcal{R}$, an ES $\mathcal{E}$, and a reduction order $>$ is called *ground complete* if the (possibly infinite) TRS $\mathcal{R} \cup \mathcal{E}_>$ is. An equation $s \approx t$ is *ground joinable* over a TRS $\mathcal{R}$ if $s\sigma \downarrow_{\mathcal{R}} t\sigma$ for all grounding substitutions σ. Ordered completion uses a relaxed definition of critical pairs. Given a reduction order $>$ and an ES $\mathcal{E}$, an *extended overlap* consists of two variable-disjoint variants $\ell_1 \approx r_1$ and $\ell_2 \approx r_2$ of equations in $\mathcal{E}^{\pm}$ such that $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$ and ℓ_1 and $\ell_2|_p$ are unifiable with most general unifier σ. An extended overlap which satisfies $r_1\sigma \not> \ell_1\sigma$ and $r_2\sigma \not> \ell_2\sigma$ gives rise to the *extended critical pair* $\ell_2[r_1]_p\sigma \approx r_2\sigma$. The set $\mathrm{CP}_>(\mathcal{E})$ consists of all extended critical pairs between equations in $\mathcal{E}$. An extended critical pair is *prime* if all proper subterms of $\ell_1\sigma$ are $\mathcal{E}_>$-normal forms. The set of prime extended critical pairs among equations in $\mathcal{E}$ is denoted by $\mathrm{PCP}_>(\mathcal{E})$.

Next, an ordered version of maximal completion gets defined. Let $\mathfrak{R}_{\mathsf{o}}$ be a function such that $\mathfrak{R}_{\mathsf{o}}(\mathcal{E}) \subseteq \mathfrak{T}(\mathcal{E})$ returns a set of *totally terminating* TRSs over $\mathcal{E}$, that is TRSs $\mathcal{R}$ which are contained in a ground total reduction order $>$. Moreover, the extension function $\mathrm{Ext}_{\mathsf{o}}$ is supposed to satisfy $\mathrm{Ext}_{\mathsf{o}}(\mathcal{E}) \subseteq \leftrightarrow_{\mathcal{E}}^*$ for all ESs $\mathcal{E}$.

▶ **Definition 8.** *Given a set of input equalities $\mathcal{E}_0$ and an ES $\mathcal{E}$, let*

$$\varphi_{\mathsf{o}}(\mathcal{E}) = \begin{cases} (\mathcal{R}, \mathcal{E}{\downarrow_{\mathcal{R}}}, >) & \text{if } \mathcal{R} \in \mathfrak{R}_{\mathsf{o}}(\mathcal{E}) \text{ and all equations in } \mathcal{E}_0 \cup \mathrm{PCP}_>(\mathcal{E}{\downarrow_{\mathcal{R}}} \cup \mathcal{R}) \\ & \text{are ground joinable in } \mathcal{R} \cup (\mathcal{E}{\downarrow_{\mathcal{R}}})_> \\ \varphi_{\mathsf{o}}(\mathcal{E} \cup \mathrm{Ext}_{\mathsf{o}}(\mathcal{E})) & \text{otherwise.} \end{cases}$$

In order to show correctness of this procedure, the following auxiliary result is useful:

▶ **Lemma 9.** *Suppose $\mathcal{R} \subseteq >$, $\mathcal{R} \cup \mathcal{E} \subseteq \leftrightarrow_{\mathcal{E}_0}^*$ and all equations in $\mathcal{E}_0 \cup \mathrm{PCP}_>(\mathcal{E} \cup \mathcal{R})$ are ground joinable in $\mathcal{R} \cup \mathcal{E}_>$. Then $(\mathcal{R}, \mathcal{E}, >)$ is a ground complete presentation of $\mathcal{E}_0$.*

Proof. Let $\mathcal{S}$ denote the TRS $\mathcal{R} \cup \mathcal{E}_>$, which terminates because it is contained in $>$. We can thus show ground confluence of $\mathcal{S}$ via local ground confluence. The inclusion

$$\xleftrightarrow[\mathrm{PCP}(\mathcal{S})]{} \subseteq \xleftrightarrow[\mathrm{PCP}_>(\mathcal{R}\cup\mathcal{E})]{} \cup \downarrow_{\mathcal{S}} \tag{1}$$

holds on ground terms according to [11, Lemma 26]. By assumption we have $\mathcal{S}$-ground joinability of $\mathrm{PCP}_>(\mathcal{R} \cup \mathcal{E})$, and hence $\leftrightarrow_{\mathrm{PCP}(\mathcal{S})} \subseteq \downarrow_{\mathcal{S}}$ on ground terms. So by Lemma 2 the TRS $\mathcal{S}$ is confluent on ground terms.

Since $\mathcal{R} \cup \mathcal{E} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ was assumed, also $\leftrightarrow^*_{\mathcal{S}} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ holds. Moreover $\mathcal{E}_0$ is $\mathcal{S}$-ground joinable by assumption. Hence the equivalence $\leftrightarrow^*_{\mathcal{S}} = \leftrightarrow^*_{\mathcal{E}_0}$ is satisfied on ground terms, so $\mathcal{S}$ is a ground complete presentation of $\mathcal{E}_0$. ◀

Now correctness of the transformation $\varphi_\circ$ is obvious:

▶ **Lemma 10.** *If $\varphi_\circ(\mathcal{E}_0)$ is defined then it is a ground complete presentation of $\mathcal{E}_0$.* ◀

Note that Definition 8 uses the idea of Definition 3 in the setting of ground completeness but suffers the major drawback of an undecidable success check since ground joinability of ordered rewriting is undecidable [20]. An implementation thus has to rely on sufficient ground joinability criteria, an example of which is stated next. Its correctness follows from the more sophisticated test presented in [33].

▶ **Lemma 11.** *An equation $s \approx t$ is ground joinable in $\mathcal{R} \cup \mathcal{E}_>$ if $s \downarrow_{\mathcal{R}} t$ or $s{\downarrow_{\mathcal{R}}} \approx t{\downarrow_{\mathcal{R}}} \in \mathcal{E}$.* ◀

In our implementation $\mathrm{Ext}_\circ(\mathcal{E})$ is chosen as a subset of $\bigcup_{\mathcal{R}\in\mathfrak{R}_\circ(\mathcal{E})} \mathrm{PCP}_>(\mathcal{R} \cup \mathcal{E}{\downarrow_{\mathcal{R}}}){\downarrow_{\mathcal{R}}}$.

Bachmair, Dershowitz, and Plaisted showed that their ordered completion procedures always succeed in producing a ground complete system (though possibly in the limit) [6]. Next, we derive a similar property for maximal ordered completion, under the assumption that all prime critical pairs are considered. To this end, we consider an infinite maximal ordered completion sequence $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \ldots$ such that $\mathcal{E}_{i+1} = \mathcal{E}_i \cup \mathrm{Ext}_\circ(\mathcal{E}_i)$ for all $i \geqslant 0$. Let moreover $\mathcal{E}^\infty$ denote the limit $\bigcup_i \mathcal{E}_i$. The following statement holds by the global assumption on $\mathrm{Ext}_\circ$.

▶ **Lemma 12.** *The conversion equivalence $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{E}_i}$ holds for all $i \geqslant 0$.* ◀

It is known that ground complete systems remain ground complete when they get (moderately) reduced, the following result follows from Lemma 5 and [11, Theorem 43].

▶ **Lemma 13.** *If $\mathcal{R} \cup \mathcal{E}_>$ is ground complete then so is $\mathcal{R} \cup (\mathcal{E}{\downarrow_{\mathcal{R}}})_>$.* ◀

Next we show the main completeness result for maximal ordered completion:

▶ **Theorem 14.** *Suppose $\mathrm{Ext}_\circ(\mathcal{E}) \supseteq \bigcup_{\mathcal{R}\in\mathfrak{R}_\circ(\mathcal{E})} \mathrm{PCP}_>(\mathcal{R} \cup \mathcal{E}){\downarrow_{\mathcal{R}}}$ for all ESs $\mathcal{E}$. For any $\mathcal{R} \in \mathfrak{R}_\circ(\mathcal{E}^\infty)$ the system $\mathcal{R} \cup (\mathcal{E}^\infty{\downarrow_{\mathcal{R}}})_>$ is a ground complete presentation of $\mathcal{E}_0$.*

Proof. Let $\mathcal{R} \in \mathfrak{R}_\circ(\mathcal{E}^\infty)$. The following arguments show that $\mathcal{S} = \mathcal{R} \cup (\mathcal{E}^\infty)_>$ is ground complete. The claim then follows from Lemma 13.

The TRS $\mathcal{S}$ is terminating because $\mathcal{S} \subseteq >$. In order to show that $\mathcal{S}$ considered as a TRS on ground terms is also confluent, according to Lemma 2 applied to $\mathcal{S}$ it suffices to show that all prime critical pairs of $\mathcal{S}$ are joinable. So consider an equation $s \approx t \in \mathrm{PCP}(\mathcal{S})$. Like in the proof of Lemma 9, we can use [11, Lemma 26] to obtain inclusion (1). So we have $s \downarrow_{\mathcal{S}} t$, or there is some equation $u \approx v \in \mathrm{PCP}_>(\mathcal{R} \cup \mathcal{E}^\infty)$ such that $s \leftrightarrow_{u\approx v} t$. In the former case, there is nothing to show. Otherwise, we have $u{\downarrow_{\mathcal{R}}} \approx v{\downarrow_{\mathcal{R}}} \in \mathrm{Ext}_\circ(\mathcal{E}^\infty) \subseteq \mathcal{E}^\infty$ by assumption. But then $u{\downarrow_{\mathcal{R}}} \approx v{\downarrow_{\mathcal{R}}}$ is $\mathcal{S}$-ground joinable by Lemma 11, and hence $s \approx t$ is joinable.

By Lemma 12 and the definition of $\mathcal{E}^\infty$, the inclusion $\mathcal{E}^\infty \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ holds. The equivalence $\leftrightarrow^*_{\mathcal{E}^\infty} = \leftrightarrow^*_{\mathcal{E}_0}$ thus follows from $\mathcal{E}_0 \subseteq \mathcal{E}^\infty$. Because $>$ is ground complete, $\leftrightarrow_{\mathcal{S}} = \leftrightarrow_{\mathcal{E}^\infty}$ holds on ground terms, which implies $\leftrightarrow^*_{\mathcal{S}} = \leftrightarrow^*_{\mathcal{E}_0}$. ◀

▶ **Example 15.** Consider the following ES $\mathcal{E}_0$ axiomatizing a Boolean ring, where multiplication is denoted by concatenation.

(1)	$(x+y)+z \approx x+(y+z)$	(2)	$x+y \approx y+x$	(3)	$\mathsf{0}+x \approx x$
(4)	$x(y+z) \approx xy+xz$	(5)	$(xy)z \approx x(yz)$	(6)	$xy \approx yx$
(7)	$(x+y)z \approx xz+yz$	(8)	$xx \approx x$	(9)	$x+x \approx \mathsf{0}$
(10)	$\mathsf{1}x \approx x$				

Let (i) denote equation (i) oriented from left to right, and $(\bar{\text{i}})$ the reverse orientation. Suppose $\mathcal{R}_1$ is the TRS $\{(1),(3),(\bar{4}),(5),(\bar{7}),(8),(9),(10)\}$, and $\mathfrak{R}_\mathsf{o}(\mathcal{E}_0) = \{\mathcal{R}_1\}$. Now the set $\mathrm{Ext}_\mathsf{o}(\mathcal{E}_0)$ may consist of the following extended critical pairs of rules among $\mathcal{R}_1$ and the unorientable commutativity equation:

(11)	$x+(y+z) \approx y+(x+z)$	(12)	$x(yz) \approx y(xz)$	(13)	$x+\mathsf{0} \approx x$
(14)	$y+(x+y) \approx x$	(15)	$x(yx) \approx xy$	(16)	$x\mathsf{1} \approx x$
(17)	$y+(y+x) \approx x$	(18)	$x(xy) \approx xy$	(19)	$\mathsf{0}x \approx \mathsf{0}$

(where all $\mathcal{R}_1$-joinable critical pairs, like $x+(x+\mathsf{0}) \approx \mathsf{0}$ or $x\mathsf{0} \approx y\mathsf{0}$, are omitted). We obtain $\mathcal{E}_1 = \mathcal{E}_0 \cup \mathrm{Ext}_\mathsf{o}(\mathcal{E}_0)$. Now $\mathfrak{R}_\mathsf{o}(\mathcal{E}_1)$ may contain the TRS $\mathcal{R}_2$ consisting of the rules $(1),(3),(4),(5),(7),\ldots,(10),(13),\ldots,(19)$. This TRS is LPO-terminating, so there is a ground-total reduction order $>$ that contains $\to_{\mathcal{R}_2}$. We have $\mathcal{E}_1{\downarrow}_{\mathcal{R}_2} = \{(2),(6),(11),(12)\}$, and it can be shown that for $\mathcal{E} = \mathcal{E}_1{\downarrow}_{\mathcal{R}_2}$ the system $\mathcal{R}_2 \cup \mathcal{E}_>$ is ground complete. Despite its simplicity, neither WM [1] nor E [28] or Vampire [19] succeed on this example.

4.1 Theorem Proving

Next the approach is extended to purely equational theorem proving: Given a set of equations $\mathcal{E}_0$ and a goal equation $s \approx t$ as input, the aim is to decide whether $s \leftrightarrow^*_{\mathcal{E}_0} t$ holds. Let Ext_g be a binary function on ESs such that $\mathrm{Ext}_\mathsf{g}(\mathcal{G},\mathcal{E}) \subseteq \leftrightarrow^*_{\mathcal{E}\cup\mathcal{G}} \setminus \leftrightarrow^*_{\mathcal{E}}$ for all ESs $\mathcal{E}$ and $\mathcal{G}$. In our implementation, $\mathrm{Ext}_\mathsf{g}(\mathcal{G},\mathcal{E})$ consists of extended critical pairs between an equation in $\mathcal{G}$ and an equation in $\mathcal{E}$. The following relation φ_g maps a pair of ESs $\mathcal{E}$ and $\mathcal{G}$ to YES or NO.

▶ **Definition 16.** *Given an ES $\mathcal{E}_0$, an initial ground goal $s_0 \approx t_0$ and ESs $\mathcal{E}$ and $\mathcal{G}$, let*

$$\varphi_\mathsf{g}(\mathcal{E},\mathcal{G}) = \begin{cases} \text{YES} & \text{if } s \downarrow_{\mathcal{R}\cup\mathcal{E}_>} t \text{ for some } s \approx t \in \mathcal{G} \text{ and } \mathcal{R} \in \mathfrak{R}_\mathsf{o}(\mathcal{E}), \\ \text{NO} & \text{if } \mathcal{R} \cup (\mathcal{E}{\downarrow}_\mathcal{R})_> \text{ is a ground complete presentation of } \mathcal{E}_0 \\ & \text{but } s_0 \not\downarrow_{\mathcal{R}\cup\mathcal{E}_>} t_0, \text{ for some } \mathcal{R} \in \mathfrak{R}_\mathsf{o}(\mathcal{E}), \text{ and} \\ \varphi_\mathsf{g}(\mathcal{E}\cup\mathcal{E}',\mathcal{G}\cup\mathcal{G}') & \text{for } \mathcal{G}' = \mathrm{Ext}_\mathsf{g}(\mathcal{G},\mathcal{R}\cup\mathcal{E}) \text{ and } \mathcal{E}' = \mathrm{Ext}_\mathsf{o}(\mathcal{E}). \end{cases}$$

For a set of input equations $\mathcal{E}_0$ and an initial goal $s_0 \approx t_0$, a maximal ordered completion procedure can then be run on the tuple $(\mathcal{E}_0, \{s_0 \approx t_0\})$. Note that the parameter $\mathcal{G}$ of φ_g denotes a disjunction of goals, not a conjunction. Due to the declarative nature of φ_g the following correctness result is straightforward.

▶ **Lemma 17.** *Let $\mathcal{E}_0$ be an ES and $s_0 \approx t_0$ be a ground goal. If $\varphi_\mathsf{g}(\mathcal{E}_0,\{s_0 \approx t_0\})$ is defined then $\varphi_\mathsf{g}(\mathcal{E}_0,\{s_0 \approx t_0\}) = \text{YES}$ if and only if $s_0 \leftrightarrow^*_{\mathcal{E}_0} t_0$.* ◀

▶ **Example 18.** The conditional confluence tool ConCon [29] interfaces equational theorem provers like MædMax to show infeasibility of conditional critical pairs, which can be used to prove confluence of conditional TRSs. Here a conditional critical pair is called *infeasible* if the involved conditions $s_1 \approx t_1, \ldots, s_n \approx t_n$ do not admit a substitution σ such that $s_i\sigma \leftrightarrow^*_{\mathcal{E}_0} t_i\sigma$ for all i. For example, ConCon encounters for Cops #340 the axioms $\mathcal{E}_0$:

$$\mathsf{f}(x_1, y_1) \approx \mathsf{g}(z_1) \qquad \mathsf{f}(x_1, \mathsf{h}(y_1)) \approx \mathsf{g}(z_1)$$

and the conditions $x_1 \approx y_1$, $\mathsf{h}(x_2) \approx y_1$, $x_1 \approx y_2$, $x_2 \approx y_2$. Let $s \approx t$ be the goal equation $\mathsf{conds}(x_1, \mathsf{h}(x_2), x_1, x_2) \approx \mathsf{conds}(y_1, y_1, y_2, y_2)$, using a fresh symbol conds. In order to decide whether there is a substitution σ such that $s\sigma \leftrightarrow^*_{\mathcal{E}_0} t\sigma$ holds, a common trick is used [6]: existence of such a σ can be refuted if the (ground) goal $\mathsf{true} \approx \mathsf{false}$ is not entailed by $\mathcal{E}_0$ extended with the following two equations:

$$\mathsf{eq}(x, x) \approx \mathsf{true} \quad (1) \qquad \mathsf{eq}(\mathsf{conds}(x_1, h(x_2), x_1, x_2), \mathsf{conds}(y_1, y_1, y_2, y_2)) \approx \mathsf{false} \quad (2)$$

For this extended ES $\mathcal{E}_0'$ a maximal ordered procedure call $\varphi_\mathsf{g}(\mathcal{E}_0', \{\mathsf{true} \approx \mathsf{false}\})$ can result in the answer NO immediately because a ground complete system exists, consisting of the two rewrite rules obtained when orienting (1) and (2) from left to right plus the following (unoriented) equations:

$$\mathsf{f}(x_1, y_1) \approx \mathsf{g}(z_1) \qquad \mathsf{f}(x_1, y_1) \approx \mathsf{f}(x_2, y_2) \qquad \mathsf{g}(x_1) \approx \mathsf{g}(y_1)$$

Both true and false are in normal form with respect to this system, so no suitable σ exists.

4.2 Completeness for Linear Systems

We conclude this section with a completeness result. A natural question in the context of completion is whether a complete system can be found by a completion procedure whenever it exists. For standard completion, it is well known that this is not the case: for example, the ES consisting of the equations $\mathsf{f}(x) \approx \mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b}) \approx \mathsf{b}$ cannot be completed by Knuth-Bendix completion, or (standard) maximal completion if $\mathrm{Ext}(\mathcal{E}) \subseteq \bigcup_{\mathcal{R} \in \mathfrak{R}(\mathcal{E})} \mathrm{CP}(\mathcal{R})$. Nevertheless a complete presentation is given by the TRS $\{\mathsf{f}(x) \to \mathsf{b}\}$ [16].

For ordered completion, two sufficient conditions are known to answer this question in the positive: Bachmair, Dershowitz, and Plaisted showed that a complete system can always be found if the reduction order is ground total [6], and Devie proved that complete representations are invariably found for linear systems, irrespective of the order's totality [8].

Next, a completeness result for linear systems in the spirit of the result by Devie [8] is presented. To that end, the reduction order $>$ does not need to be ground total. In order to express that the reduction order leading to the presupposed completion system must be considered by the procedure, the function $\mathfrak{R}$ is said to *support* a reduction order $>$ if $\mathfrak{R}(\mathcal{E})$ contains a maximal TRS $\mathcal{R}$ such that $\mathcal{R} \subseteq >$, for all ESs $\mathcal{E}$.

Devie's notion of *linear overlaps* refers to extended overlaps which satisfy $\ell_1 > r_1$ and $r_2 \not> \ell_2$, or $\ell_2 > r_2$ and $r_1 \not> \ell_1$. Critical pairs originating from such overlaps are called *linear critical pairs*, and the set of all linear critical pairs formed using equations in $\mathcal{E}$ is denoted by $\mathsf{LCP}_>(\mathcal{E})$. A TRS $\mathcal{R}$ is called *reduced* if for all rules $\ell \to r$ in $\mathcal{R}$ both $r \in \mathsf{NF}(\mathcal{R})$ and $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ hold.

▶ **Theorem 19.** *Let $\mathcal{E}_0$ be a linear ES which admits a complete and reduced presentation as the TRS $\mathcal{C}$ such that $\mathcal{C} \subseteq >$. Suppose moreover that $\mathfrak{R}$ supports $>$, $\mathrm{Ext}_\mathsf{o}(\mathcal{E})$ is linear whenever $\mathcal{E}$ is linear, and $\mathrm{Ext}_\mathsf{o}(\mathcal{E}) \supseteq \bigcup_{\mathcal{R} \in \mathfrak{R}(\mathcal{E})} \mathsf{LCP}_>(\mathcal{R} \cup \mathcal{E})$ for all ESs $\mathcal{E}$.*

Then $\varphi_\mathsf{o}(\mathcal{E}_0)$ is defined and constitutes a complete TRS.

Proof. Let $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \ldots$ be a maximal ordered completion sequence, and $\mathcal{S}_i$ denote the TRS $\mathcal{E}_{i>}$. It can be assumed that $\mathcal{E}_i$ is linear for all $i \geqslant 0$, because $\mathcal{E}_0$ is linear and $\mathrm{Ext}_{\mathsf{o}}$ is supposed to preserve linearity.

Consider a cost function c defined on equation steps as follows: for $\ell \approx r \in \mathcal{E}_i$, let $c(s = C[\ell\sigma] \leftrightarrow_{\ell\approx r} C[r\sigma] = t)$ be $\{t\}$ if $\ell\sigma > r\sigma$, $\{s\}$ if $r\sigma > \ell\sigma$, and $\{s,t\}$ otherwise. This measure is extended to conversions $P\colon t_0 \leftrightarrow t_1 \leftrightarrow \ldots \leftrightarrow t_n$ by defining $c(P)$ as the multiset union $\bigcup_{0\leqslant i<n} c(t_i \leftrightarrow t_{i+1})$. In the sequel $P \gg Q$ is written to abbreviate $c(P) >_{\mathsf{mul}} c(Q)$.

Consider a rule $\ell \to r$ in $\mathcal{C}$. As $\mathcal{C}$ is assumed to be a complete presentation of $\mathcal{E}_0$, there is a conversion $\ell \leftrightarrow^*_{\mathcal{E}_0} r$. According to Lemma 12, also $\ell \leftrightarrow^*_{\mathcal{E}_i} r$ holds for all $i \geqslant 0$. Let $P^i_{\ell\to r}$ be fixed conversions $\ell \leftrightarrow^*_{\mathcal{E}_i} r$ which are minimal with respect to $\gg$, for all $i \geqslant 0$.

We show that for every i, if $P^i_{\ell\to r}$ is not of the form $\ell \to^*_{\mathcal{S}_i} r$ then there is a conversion $P^{i+1}_{\ell\to r}$ which has fewer steps and satisfies $P^i_{\ell\to r} \gg P^{i+1}_{\ell\to r}$. Note that all conversions $\ell \leftrightarrow^*_{\mathcal{E}_i} r$ must have at least one step: otherwise, we would have $\ell = r$, which contradicts $\mathcal{C} \subseteq >$ because $>$ is well-founded.

Let $P^i_{\ell\to r}$ be a minimal conversion $\ell \leftrightarrow^*_{\mathcal{E}_i} r$. Since it has at least one step, we can assume some term r' such that $P^i_{\ell\to r}$ has the form $\ell \leftrightarrow^*_{\mathcal{E}_i} r' \leftrightarrow_{\mathcal{E}_i} r$, and $r' \neq r$. Note that the last step $r' \leftrightarrow_{\mathcal{E}_i} r$ must satisfy $r' > r$: By conversion equivalence, we must have $r' \leftrightarrow^*_{\mathcal{C}} r$. Since $\mathcal{C}$ is complete, $r' \downarrow_{\mathcal{C}} r$ holds. Because $\mathcal{C}$ is also reduced, the term r is irreducible, so we have $r' \to^*_{\mathcal{C}} r$. By the above assumption that $r' \neq r$ this means that $r' \to^+_{\mathcal{C}} r$, which implies $r' > r$ because $\mathcal{C} \subseteq >$. So the equation step $r' \leftrightarrow_{\mathcal{E}_i} r$ is an ordered rewrite step $r' \to_{\mathcal{S}_i} r$.

If $\ell \leftrightarrow^*_{\mathcal{E}_i} r$ is not of the form $\ell \to^*_{\mathcal{S}_i} r$ then it must therefore contain a peak involving non-$\mathcal{S}_i$ step followed by an $\mathcal{S}_i$ step, that is, a peak of the form

$$Q\colon s \xleftrightarrow[\ell_1\approx r_1,\sigma]{} u \xrightarrow[\ell_2\approx r_2,\sigma]{} t$$

for some terms s, t, and u, equations $\ell_1 \approx r_1$, $\ell_2 \approx r_2 \in \mathcal{E}_i$, and a substitution σ such that $\ell_1\sigma \not> r_1\sigma$ but $\ell_2\sigma > r_2\sigma$, so $\ell_2\sigma \to r_2\sigma \in \mathcal{S}_i$. Note that $c(Q) = \{s,u,t\}$.

(a) If $\ell_1 \approx r_1$ and $\ell_2 \approx r_2$ form a proper overlap then $s \leftrightarrow_{\mathsf{LCP}_>(\mathcal{E}_i)} t$ because $\ell_1\sigma \not> r_1\sigma$ and $\ell_2\sigma > r_2\sigma$. By assumption $\mathsf{LCP}_>(\mathcal{E}_i) \subseteq \mathcal{E}_{i+1}$. Hence there is a conversion $P^{i+1}_{\ell\to r}\colon \ell \leftrightarrow^*_{\mathcal{E}_{i+1}} r$ where Q is replaced by $Q'\colon s \leftrightarrow_{\mathcal{E}_{i+1}} t$ and $c(Q) >_{\mathsf{mul}} \{s,t\} \geqslant_{\mathsf{mul}} c(Q')$. Moreover, $P^{i+1}_{\ell\to r}$ has fewer steps than $P^i_{\ell\to r}$.

(b) Suppose $\ell_1 \approx r_1$ and $\ell_2 \approx r_2$ occur in parallel. Then the two steps can be swapped, so there is a term v which allows for the conversion $Q'\colon s \to_{\ell_2\sigma\to r_2\sigma} v \leftrightarrow_{\ell_1\sigma\approx r_1\sigma} t$. This contradicts the assumption that $P^i_{\ell\to r}$ was minimal: we have $c(Q) >_{\mathsf{mul}} \{v,v,t\} = c(Q')$ because $s > v$.

(c) Similarly, if $\ell_1 \approx r_1$ and $\ell_2 \approx r_2$ form a variable overlap then because $\mathcal{E}_i$ is linear there is a term v such that there is a conversion $Q\colon s \to^=_{\ell_2\sigma\to r_2\sigma} v \leftrightarrow^=_{\ell_1\sigma\approx r_1\sigma} t$. But this again contradicts minimality of $P^i_{\ell\to r}$ because $C(Q) >_{\mathsf{mul}} \{v,v,t\} \geqslant_{\mathsf{mul}} c(Q')$ due to $s > v$.

Let k be the maximal number of steps of $P^0_{\ell\to r}$, for $\ell \to r \in \mathcal{C}$. The above argument shows that $\ell \to^*_{\mathcal{S}_k} r$ holds for all $\ell \to r \in \mathcal{C}$. Hence we have $\mathsf{NF}(\mathcal{S}_k) \subseteq \mathsf{NF}(\mathcal{C})$.

Let $\mathcal{S}$ be the TRS $\mathcal{R} \cup (\mathcal{E}_k{\downarrow_{\mathcal{R}}})_>$. Any term reducible by $\mathcal{S}_k$ must also be reducible in $\mathcal{S}$, which implies $\mathsf{NF}(\mathcal{S}) \subseteq \mathsf{NF}(\mathcal{S}_k)$ and hence $\mathsf{NF}(\mathcal{S}) \subseteq \mathsf{NF}(\mathcal{C})$. Since moreover $\mathcal{R} \cup \mathcal{E}_k \subseteq \leftrightarrow^*_{\mathcal{C}}$ implies $\to_{\mathcal{S}_k} \subseteq \leftrightarrow^*_{\mathcal{C}}$ and $\mathcal{S}$ is terminating because $\mathcal{S} \subseteq >$, the TRS $\mathcal{S}$ is complete according to [11, Lemma 31]. ◀

Note that the above proof implies a bound on the number of iterations needed to derive a complete system, namely the number of $\mathcal{E}_0$-steps required for conversions of the rules in the complete system $\mathcal{C}$. Naturally, due to incompleteness of implementations, this bound cannot be kept up in practice.

▶ **Example 20.** Consider the linear ES $\mathcal{E}$ consisting of the following three equations:

$$\mathsf{f}(\mathsf{a},\mathsf{i}(x)) \approx \mathsf{f}(\mathsf{b},\mathsf{b}) \qquad \mathsf{g}(\mathsf{b},x) \approx \mathsf{g}(\mathsf{a},\mathsf{a}) \qquad \mathsf{f}(\mathsf{a},x) \approx \mathsf{f}(\mathsf{a},y)$$

The TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{a},x) \to \mathsf{f}(\mathsf{b},\mathsf{b}),\ \mathsf{g}(\mathsf{b},x) \to \mathsf{g}(\mathsf{a},\mathsf{a})\}$ is terminating and confluent, as is easily checked by state-of the art tools. We also have $\mathcal{E}_0 \subseteq {\downarrow_{\mathcal{R}}}$, and from the conversion

$$\mathsf{f}(\mathsf{a},x) \leftrightarrow \mathsf{f}(\mathsf{a},\mathsf{i}(x)) \leftrightarrow \mathsf{f}(\mathsf{b},\mathsf{b}) \tag{2}$$

we can conclude $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R}}$, so $\mathcal{R}$ is a complete presentation of $\mathcal{E}_0$. By Theorem 19, maximal ordered completion supporting $> = \to^+_{\mathcal{R}}$ will succeed with a complete system, and according to the bound derived in the proof, this takes at most two iterations since (2) has two steps.

5 Normalized Completion

Many algebraic theories like groups and rings feature associative and commutative operators. However, since the commutativity equation cannot be oriented into a terminating rewrite rule, such theories cannot be handled by standard Knuth-Bendix completion. This triggered the development of dedicated completion calculi that can deal with such cases [23, 12].

Various generalizations have been proposed to extend completion to different algebraic theories, apart from plain AC. A version for general theories $\mathcal{T}$ has been proposed in [12, 4], provided that $\mathcal{T}$ admits finitary unification and the subterm ordering modulo $\mathcal{T}$ is well-founded. Constrained completion [13] constitutes an attempt to overcome these restrictions on the theory, it admits for instance completion modulo AC with a unit element (ACU). However, it excludes other theories such as Abelian groups.

Normalized completion [21, 22, 34] can be seen as the last result in this line of research. It has three advantages over earlier methods. (1) It allows completion modulo any theory $\mathcal{T}$ that can be represented as an AC-complete rewrite system $\mathcal{S}$. (2) Critical pairs need not be computed for the theory $\mathcal{T}$, which may not be finitary or even have a decidable unification problem. Instead, any theory between AC and $\mathcal{T}$ can be used. (3) The AC-compatible reduction order used to establish termination need not be compatible with $\mathcal{T}$. This is beneficial for theories like ACU where no $\mathcal{T}$-compatible simplification order exists.

▶ **Example 21.** Consider an Abelian group with AC operator $\cdot$ and an endomorphism f as described by the following three equations:

$$\mathsf{e}\cdot x \approx x \qquad \mathsf{i}(x)\cdot x \approx \mathsf{e} \qquad \mathsf{f}(x\cdot y) \approx \mathsf{f}(x)\cdot\mathsf{f}(y)$$

together with ACRPO [24] with precedence $\mathsf{f} > \mathsf{i} > \cdot > \mathsf{e}$. Using AC completion, or equivalently normalized completion with respect to $\mathcal{S} = \varnothing$, one obtains the following AC complete TRS $\mathcal{R}_{\mathrm{AC}}$:

$$\begin{array}{ccc} \mathsf{e}\cdot x \to x & \mathsf{i}(x)\cdot x \to \mathsf{e} & \mathsf{i}(\mathsf{e}) \to \mathsf{e} \\ \mathsf{i}(\mathsf{i}(x)) \to x & \mathsf{i}(x\cdot y) \to \mathsf{i}(x)\cdot\mathsf{i}(y) & \mathsf{f}(x\cdot y) \to \mathsf{f}(x)\cdot\mathsf{f}(y) \\ \mathsf{f}(\mathsf{e}) \to \mathsf{e} & \mathsf{f}(\mathsf{i}(x)) \to \mathsf{i}(\mathsf{f}(x)) & \end{array}$$

Alternatively, one can perform normalized completion with respect to an AC complete representation of Abelian groups, like for example the following TRS $\mathcal{S}_{\mathsf{G}}$ [3]:

$$\mathsf{e}\cdot x \to x \qquad \mathsf{i}(x)\cdot x \to \mathsf{e} \qquad \mathsf{i}(\mathsf{e}) \to \mathsf{e} \qquad \mathsf{i}(\mathsf{i}(x)) \to x \qquad \mathsf{i}(x\cdot y) \to \mathsf{i}(x)\cdot\mathsf{i}(y)$$

Note that $\mathcal{S}_{\mathsf{G}} \subseteq {>}$. Normalized completion with respect to $\mathcal{S}_{\mathsf{G}}$ results in the TRS $\mathcal{R}_{\mathsf{G}}$:

$$\mathsf{f}(x\cdot y) \to \mathsf{f}(x)\cdot\mathsf{f}(y) \qquad \mathsf{f}(\mathsf{e}) \to \mathsf{e} \qquad \mathsf{f}(\mathsf{i}(x)) \to \mathsf{i}(\mathsf{f}(x))$$

Before proposing a maximal normalized completion procedure, we recall some concepts and notations related to AC rewriting and normalized rewriting.

AC Rewriting and Unification. A TRS $\mathcal{R}$ *terminates modulo* AC whenever the relation $\to_{\mathcal{R}/\mathrm{AC}}$ is well-founded. To establish AC termination we will consider AC-compatible reduction orders $>$, i.e., reduction orders that satisfy $\leftrightarrow^*_{\mathrm{AC}} \cdot > \cdot \leftrightarrow^*_{\mathrm{AC}} \subseteq >$. The TRS $\mathcal{R}$ is *complete modulo* AC if it terminates modulo AC and the relation $\leftrightarrow^*_{\mathrm{AC}\cup\mathcal{R}}$ coincides with $\to^*_{\mathcal{R}/\mathrm{AC}} \cdot \leftrightarrow^*_{\mathrm{AC}} \cdot \leftarrow^*_{\mathcal{R}/\mathrm{AC}}$. It is an AC-*complete presentation* of an ES $\mathcal{E}$ if $\mathcal{R}$ is AC complete and $\leftrightarrow^*_{\mathcal{E}\cup\mathrm{AC}} = \leftrightarrow^*_{\mathcal{R}\cup\mathrm{AC}}$.

Let $\mathcal{L}$ be a theory with finitary and decidable unification problem. A substitution σ constitutes an $\mathcal{L}$-*unifier* of two terms s and t if $s\sigma \leftrightarrow^*_{\mathcal{L}} t\sigma$ holds. An $\mathcal{L}$-*overlap* is a quadruple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2\rangle_\Sigma$ consisting of rewrite rules $\ell_1 \to r_1$, $\ell_2 \to r_2$, a position $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$, and a complete set Σ of $\mathcal{L}$-unifiers of $\ell_2|_p$ and ℓ_1. Then $\ell_2[r_1]_p\sigma \approx r_2\sigma$ constitutes an $\mathcal{L}$-*critical pair* for every $\sigma \in \Sigma$. For two sets of rewrite rules $\mathcal{R}_1$ and $\mathcal{R}_2$, we also write $\mathrm{CP}_{\mathcal{L}}(\mathcal{R}_1, \mathcal{R}_2)$ for the set of all $\mathcal{L}$-critical pairs emerging from an overlap where $\ell_1 \to r_1 \in \mathcal{R}_1$ and $\ell_2 \to r_2 \in \mathcal{R}_2$, and $\mathrm{CP}_{\mathcal{L}}(\mathcal{R}_1)$ for the set of all $\mathcal{L}$-critical pairs such that $\ell_1 \to r_1$, $\ell_2 \to r_2 \in \mathcal{R}_1$.

We assume there is a fixed set of AC symbols $\mathcal{F}_{\mathrm{AC}} \subseteq \mathcal{F}$. For a rewrite rule $\ell \to r$ with $+ \in \mathcal{F}_{\mathrm{AC}}$ the notation $(\ell \to r)^e$ refers to the *extended rule* $\ell + x \to r + x$, where $x \in \mathcal{V}$ is fresh. The TRS $\mathcal{R}^e$ contains all rules in $\mathcal{R}$ plus all extended rules $\ell + x \to r + x$ such that $\ell \to r \in \mathcal{R}$ [3].

Normalized Rewriting. We define normalized rewriting as in [22] but use a different notation to distinguish it from the common notation for rewriting modulo. Let $\mathcal{T}$ be a theory which has an AC-complete presentation as a TRS $\mathcal{S}$.

Two terms s and t admit an $\mathcal{S}$-*normalized* $\mathcal{R}$-*rewrite step* if

$$s \xrightarrow[\mathcal{S}/\mathrm{AC}]{!} \cdot \xleftrightarrow[\mathrm{AC}]{*} \cdot \xrightarrow[\ell\to r]{p} \cdot \xleftrightarrow[\mathrm{AC}]{*} t \tag{3}$$

for some rule $\ell \to r$ in $\mathcal{R}$ and position p. We abbreviate (3) by $s \to^p_{\ell\to r\backslash\mathcal{S}} t$ and write $s \to_{\mathcal{R}\backslash\mathcal{S}} t$ if $s \to^p_{\ell\to r\backslash\mathcal{S}} t$ for a rule $\ell \to r$ in $\mathcal{R}$ and position p. Let $>$ be an AC-compatible reduction order such that $\mathcal{S} \subseteq >$. For any set of rewrite rules $\mathcal{R}$ satisfying $\mathcal{R} \subseteq >$ the normalized rewrite relation $\to_{\mathcal{R}\backslash\mathcal{S}}$ is well-founded [21, 22], so we can consider equational proofs of the form $s \to^!_{\mathcal{R}\backslash\mathcal{S}} \cdot \leftrightarrow^*_{\mathcal{T}} \cdot \leftarrow^!_{\mathcal{R}\backslash\mathcal{S}} t$. These normal form proofs play a special role and are called *normalized rewrite proofs*. Because $\mathcal{S}$ is AC-complete for $\mathcal{T}$, any such proof can be transformed into a proof $s \Downarrow_{\mathcal{R}\backslash\mathcal{S}} t$, where $\Downarrow_{\mathcal{R}\backslash\mathcal{S}}$ abbreviates the relation

$$\xrightarrow[\mathcal{R}\backslash\mathcal{S}]{!} \cdot \xrightarrow[\mathcal{S}/\mathrm{AC}]{!} \cdot \xleftrightarrow[\mathrm{AC}]{*} \cdot \xleftarrow[\mathcal{S}/\mathrm{AC}]{!} \cdot \xleftarrow[\mathcal{R}\backslash\mathcal{S}]{!}$$

A TRS $\mathcal{R}$ is an $\mathcal{S}$-*complete presentation* of a set of equations $\mathcal{E}$ if $\to_{\mathcal{R}\backslash\mathcal{S}}$ is terminating and the relations $\leftrightarrow^*_{\mathcal{E}\cup\mathcal{T}}$ and $\to^!_{\mathcal{R}\backslash\mathcal{S}} \cdot \leftrightarrow^*_{\mathcal{T}} \cdot \leftarrow^!_{\mathcal{R}\backslash\mathcal{S}}$, hence $\Downarrow_{\mathcal{R}\backslash\mathcal{S}}$, coincide.

In the remainder of this section we assume that $\mathfrak{R}_{\mathcal{S}}(\mathcal{E})$ is a finite set of rewrite systems $\mathcal{R}$ such that $\mathcal{R} \cup \mathcal{S}$ is AC terminating, for all ESs $\mathcal{E}$. Moreover, let the function $\mathrm{Ext}_{\mathcal{S}}$ satisfy $\mathrm{Ext}_{\mathcal{S}}(\mathcal{E}) \subseteq \leftrightarrow^*_{\mathrm{AC}\cup\mathcal{S}\cup\mathcal{E}}$ for all ESs $\mathcal{E}$. We write $\mathrm{CP}_{\mathcal{S}}(\mathcal{R})$ for the set of critical pairs

$$\mathrm{CP}_{\mathcal{L}}(\mathcal{R}^e) \cup \mathrm{CP}_{\mathrm{AC}}(\mathcal{S}^e, \mathcal{R}^e) \cup \mathrm{CP}_{\mathrm{AC}}(\mathcal{R}^e, \mathcal{S}^e)$$

▶ **Definition 22.** *Given a set of input equalities $\mathcal{E}_0$ and an ES $\mathcal{E}$, let*

$$\varphi_{\mathcal{S}}(\mathcal{E}) = \begin{cases} \mathcal{R} & \text{if } \mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{E}) \text{ such that } \mathrm{CP}_{\mathcal{S}}(\mathcal{R}) \cup \mathcal{E}_0 \subseteq \Downarrow_{\mathcal{R}\backslash\mathcal{S}} \\ \varphi_{\mathcal{S}}(\mathcal{E} \cup \mathrm{Ext}_{\mathcal{S}}(\mathcal{E})) & \text{otherwise.} \end{cases}$$

The proof of the following correctness statement is a straightforward adaptation of the respective result for standard completion (Lemma 4).

▶ **Lemma 23.** *If $\varphi_{\mathcal{S}}(\mathcal{E})$ is defined then it is an $\mathcal{S}$-complete presentation of $\mathcal{E}_0$.*

Proof. Suppose $\varphi_{\mathcal{S}}(\mathcal{E}_0) = \mathcal{R}$, so $\mathcal{R} \cup \mathcal{S}$ is AC terminating since it was returned by $\mathfrak{R}_{\mathcal{S}}$. Because of $\mathrm{CP}_{\mathcal{S}}(\mathcal{R}) \subseteq \Downarrow_{\mathcal{R}\backslash\mathcal{S}}$ the TRS $\mathcal{R}$ is $\mathcal{S}$-complete according to the results by Marché [22].

Let $\mathcal{E}_1, \dots, \mathcal{E}_k$ be a sequence of normalized maximal completion, that is $\mathcal{E}_{i+1} = \mathcal{E}_i \cup \mathrm{Ext}_{\mathcal{S}}(\mathcal{E}_i)$ for all $1 \leqslant i < k$ and there is some $\mathcal{R} \in \mathfrak{R}_{\mathcal{S}}(\mathcal{E}_k)$ such that $\mathrm{CP}_{\mathcal{S}}(\mathcal{R}) \cup \mathcal{E}_0 \subseteq \Downarrow_{\mathcal{R}\backslash\mathcal{S}}$. A simple induction argument using the global assumption that $\mathrm{Ext}_{\mathcal{S}}(\mathcal{E}) \subseteq \leftrightarrow^*_{\mathrm{AC}\,\cup\,\mathcal{S}\,\cup\,\mathcal{E}}$ for all ESs $\mathcal{E}$ shows that $\mathcal{E}_k \subseteq \leftrightarrow^*_{\mathrm{AC}\,\cup\,\mathcal{S}\,\cup\,\mathcal{E}_0}$. Since $\mathcal{R}$ is over $\mathcal{E}_k$, also $\leftrightarrow^*_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathrm{AC}\,\cup\,\mathcal{S}\,\cup\,\mathcal{E}_0}$ holds. Conversely, $\mathcal{E}_0 \subseteq \Downarrow_{\mathcal{R}\backslash\mathcal{S}}$ is assumed. So $\mathcal{R}$ is an $\mathcal{S}$-complete presentation of $\mathcal{E}_0$. ◀

The maximal normalized completion implementation in MædMax can for instance complete the ES in Example 21 with respect to both AC (so $\mathcal{S} = \varnothing$) or group theory (using $\mathcal{S}_{\mathsf{G}}$).

6 Implementation

In this section we briefly summarize an implementation of the discussed variants of maximal completion in the tool MædMax [36]. MædMax is implemented in OCaml and available as a command-line tool as well as via a web interface, on the accompanying website also example input can be found.[1] Input problems can be submitted in the TPTP [31] as well as the trs format.[2] The tool supports standard maximal completion, maximal ordered completion and theorem proving, as well as normalized completion. However, many modules are used for all of these modes. For the former, MædMax incorporates the extended Maxcomp version [25] which supports advanced termination techniques like dependency pairs.

In the following paragraphs we comment on the implementation of the three components corresponding to the main steps in maximal completion: (1) finding (AC) terminating TRSs, (2) success checks, and (3) selection of new equations and goals.

Finding rewrite systems. In order to find (AC) terminating rewrite systems that play the role of $\mathfrak{R}(\mathcal{E})$ and $\mathfrak{R}_{\mathcal{S}}(\mathcal{E})$, respectively, MædMax adheres to the basic approach of Maxcomp [17] in that it solves optimization problems by means of a maxSAT/maxSMT solver. The objective of this optimization can be to (a) maximize the number of oriented equations as done in Maxcomp, or (b) the equations in $\mathcal{E}$ that are reducible, or to (c) minimize the number of rules or (d) the number of critical pairs. These optimization targets can also be combined, and completeness requirements as described in [25] can be added. Strategy (b) in combination with (c) has proved to be particularly useful, because it prefers small TRSs which can simplify many equations. This is especially beneficial in presence of AC symbols, where many rewrite rules and hence many critical pairs can drastically impact performance.

In order to guarantee termination of the resulting system, SMT encodings of termination techniques are used. These are LPO, KBO, and linear polynomials for ordered completion, where a ground-total reduction order is desired. For standard completion, MædMax additionally supports dependency pairs, a dependency graph approximation, and argument filterings for LPO and KBO, as described earlier [25]. These techniques can also be combined in a strategy involving sequential composition and choice. As a means to ensure AC termination, ACRPO is encoded [24]. The supported SMT solvers are Yices 1.0 [10] and Z3 [7].

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[1] `http://cl-informatik.uibk.ac.at/software/maedmax/`

[2] `https://www.lri.fr/~marche/tpdb/format.html`

Success checks. For standard and normalized completion, it is straightforward to check whether all critical pairs are joinable. In the latter case, MædMax only supports AC critical pairs. To conclude ground confluence, our tool supports the criterion of [33].

Selection. The extension functions Ext, $\mathrm{Ext}_{\mathrm{g}}$, and $\mathrm{Ext}_{\mathcal{S}}$ are implemented to add a subset of (extended, AC) critical pairs among rules in $\mathcal{R}$, and equations/goal for the case of ordered completion. In any case the selected equations get reduced to $\mathcal{R}$-normal form before they are added. MædMax severely limits the number of critical pairs that are added in every iteration to confine the exponential blowup. The selection heuristic prefers small equations and old, but not yet reducible equations.

Furthermore, MædMax can output equational (dis)proofs and ground completion proofs in a format that can be validated by the proof checker CeTA [30]. Further implementation details and evaluations on standard benchmark sets can be found in [36, 25].

We conclude with a final example illustrating a practical application. The tool AQL[3] performs functorial data integration by means of a category-theoretic approach [27], taking advantage of (ground) completion. The following problem was communicated by the authors.

▶ **Example 24.** Consider two database tables yIsAL and yIsAW relating amphibians to land and water animals, respectively. The relationship between their entries are described by 400 ground equations over symbols yIsAL, yIsALL, yIsAW, yIsAWW (which correspond to fields in the schemas) and 449 constants of the form $\mathsf{a}_i, \mathsf{w}_i, \mathsf{l}_i$ representing data items. The following six example equations may convey an impression:

$$\mathsf{yIsAW}(\mathsf{a}_1) \approx \mathsf{w}_{29} \qquad \mathsf{yIsAW}(\mathsf{a}_{78}) \approx \mathsf{w}_{16} \qquad \mathsf{yIsAW}(\mathsf{a}_{61}) \approx \mathsf{w}_{30}$$
$$\mathsf{yIsAL}(\mathsf{a}_{37}) \approx \mathsf{l}_{80} \qquad \mathsf{yIsAL}(\mathsf{a}_{84}) \approx \mathsf{l}_{6} \qquad \mathsf{yIsAL}(\mathsf{a}_{29}) \approx \mathsf{l}_{47}$$

In addition, the equation $\mathsf{yIsALL}(\mathsf{yIsAL}(x)) \approx \mathsf{yIsAWW}(\mathsf{yIsAW}(x))$ describes a mapping to a second database schema. A ground complete presentation of the entire system thus constitutes a representation of the data, translated to the second schema. MædMax discovers a complete presentation of 889 rules in less then 20 seconds, while AQL's internal completion prover fails. MædMax' automatic mode switches to linear polynomials for such systems with many symbols, which turned out to be faster than LPO or KBO in this situation.

7 Conclusion

This paper explored variants of maximal completion, corresponding to ordered and normalized completion. These methods have multiple advantages over earlier approaches:

- The reduction order, a notoriously critical parameter, need not be fixed in advance. This also holds for tools with an automatic mode such as RRL [15], but there it is unsound to change the order once it was fixed [26]. In contrast, no such problem occurs in maximal completion.
- Using maxSMT encodings, the choice of an ordering can be "steered" towards beneficial properties of the resulting system (e.g. to orient a maximal number of equations, to reduce a maximal number of equations, or to stimulate complete systems [25]).
- Maximal completion exploits the advantage of parallelization in that multiple reduction orders can be considered (by choosing multiple rewrite systems in every iteration). Theorem 19 shows that in the linear case any complete system for a supported ordering

[3] `http://categoricaldata.net/aql.html`

will be found. But at the same time rewriting and critical pair computation are shared among the processes corresponding to the different choices of an ordering.

- Efficiency is gained by orienting multiple equations at the same time. Theorem 19 shows that this also admits a (theoretical) bound on the number of required iterations.
- Finally, the definitions and the corresponding proofs are concise and simple: neither proof orders [5] nor notions like peak or source decreasingness [11] are required.

Several directions for future work arise. First, we believe that also the completeness result for ground-total reduction orders carries over to maximal ordered completion [6]. The general case of completeness is still an open problem. Another interesting because practically relevant variant of completion operates on logically constrained rewrite systems (LCTRSs) [35]. Supporting maximal completion procedure for this setting might thus be a useful addition to MædMax. Maximal completion can be considered an approximation- and conflict-based approach: complete TRSs are overapproximated by terminating TRSs, and if a conflict (that is a non-joinable critical pair) is encountered, the approximation is refined. It would be interesting to investigate connections to other conflict-driven learning approaches such as lazy SMT solving or DPLL [9].

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